On Certain Sufficient Conditions for Analytic Univalent Functions

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Abstract. In this paper, we introduce a new class $B_{m}^{l}(\alpha, \delta)$ of functions which is defined by hypergeometric function and obtain its relations with some well-known subclasses of analytic univalent functions. Furthermore, as a special case, we show that convex functions of order $1/2$ are also members of the family $B_{m}^{l}(\alpha, \delta)$.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic and univalent in the open disc $U = \{z : |z| < 1\}$ and normalized by $f(0) = 0 = f'(0) - 1$. We denote by $S^*(\alpha)$ and $K(\alpha)$ the subclasses of $\mathcal{A}$ consisting of all functions which are, respectively starlike and convex of order $\alpha$. Thus,

$$S^*(\alpha) = \left\{ f \in \mathcal{A} : \Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha, \ 0 \leq \alpha < 1, \ z \in U \right\}$$

and

$$K(\alpha) = \left\{ f \in \mathcal{A} : \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, \ 0 \leq \alpha < 1, \ z \in U \right\}.$$

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We notice that \( K(\alpha) \subset S^*(\alpha) \subset \mathcal{A} \). Further,
\[
R(\alpha) = \{ f \in \mathcal{A} : \text{Re} (f(z)) > \alpha, \; 0 \le \alpha < 1, \; z \in U \}.
\]

If \( f \) and \( g \) are analytic functions in \( U \), we say that \( f \) is subordinate to \( g \), written \( f \prec g \), if there is a function \( w \) analytic in \( U \), with \( w(0) = 0, \; |w(z)| < 1 \), for all \( z \in U \) such that \( f(z) = g(w(z)) \) for all \( z \in U \). If \( g \) is univalent, then \( f \prec g \) if and only if \( f(0) = g(0) \) and \( f(U) \subseteq g(U) \).

For functions \( \Phi \in \mathcal{A} \) given by \( \Phi(z) = z + \sum_{n=2}^{\infty} \phi_n z^n \) and \( \Psi \in \mathcal{A} \) given by \( \Psi(z) = z + \sum_{n=2}^{\infty} \psi_n z^n \), we define the Hadamard product (or Convolution) of \( \Phi \) and \( \Psi \) by
\[
(\Phi \ast \Psi)(z) = z + \sum_{n=2}^{\infty} \phi_n \psi_n z^n, \; z \in U.
\]

For complex parameters \( \alpha_1, \ldots, \alpha_l \) and \( \beta_1, \ldots, \beta_m \) \((\beta_j \neq 0, -1, \ldots; j = 1, 2, \ldots, m)\) the generalized hypergeometric function \( iF_m(z) \) is defined by
\[
iF_m(z) \equiv iF_m(\alpha_1, \ldots; \alpha_l; \beta_1, \ldots; \beta_m; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_l)_n z^n}{(\beta_1)_n \cdots (\beta_m)_n n!}
\]
\( (l \leq m + 1; \; l, m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; z \in U) \)

where \( N \) denotes the set of all positive integers and \((\alpha)_n\) is the Pochhammer symbol defined by
\[
(\alpha)_n = \begin{cases} 
1, & n = 0 \\
\alpha(\alpha + 1)(\alpha + 2)\cdots(\alpha + n - 1), & n \in \mathbb{N}.
\end{cases}
\]

Let \( H(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) : \mathcal{A} \to \mathcal{A} \) be a linear operator defined by
\[
[(H(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)(f))(z) := z \cdot iF_m(\alpha_1, \alpha_2, \ldots; \alpha_l; \beta_1, \beta_2, \ldots, \beta_m; z) \ast f(z) = z + \sum_{n=2}^{\infty} \Gamma_n a_n z^n
\]
\( (\alpha) = \frac{(\alpha_1)_{n-1} \cdots (\alpha_l)_{n-1}}{(n-1)! (\beta_1)_{n-1} \cdots (\beta_m)_{n-1}}. \)

For notational simplicity, we can use a shorter notation \( H^l_m[\alpha_1] \) for \( H(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) \) in the sequel. The linear operator \( H^l_m[\alpha_1] \) is called Dziok-Srivastava operator (see [3]), includes (as its special cases) various other linear operators introduced and studied by Bernardi [1], Carlson and Shaffer [2], Libera [6], Livingston [7], Ruscheweyh [8] and Srivastava-Owa [9].

For \( 0 \leq \alpha < 1 \) and \( \delta \geq 0 \), let \( B^l_m(\alpha, \delta) \) consisting of functions of the form (1) and satisfying the condition
\[
\left| \frac{H^l_m[\alpha_1 + 1]f(z)}{z} \left( \frac{z^{\delta}}{H^l_m[\alpha_1]f(z)} \right)^{\delta} - 1 \right| < 1 - \alpha, \; z \in U.
\]

\( (\alpha) = \frac{(\alpha_1)_{n-1} \cdots (\alpha_l)_{n-1}}{(n-1)! (\beta_1)_{n-1} \cdots (\beta_m)_{n-1}}. \)
The class $B_m^l(\alpha, \delta)$ is a unified class of analytic functions which includes various new subclasses of analytic univalent functions. We observe that

**Example 1.** If $l = 2$ and $m = 1$ with $\alpha_1 = 1, \alpha_2 = 1, \beta_1 = 1$ then

$$B_1^2(\alpha, \delta) := \left\{ f \in \mathcal{A} : \left| \frac{f^{(l)}(z)}{f(z)} \right| \frac{1}{\delta} - 1 < 1 - \alpha, \, \delta \geq 0, \, 0 \leq \alpha < 1, \, z \in U \right\}.$$  

The class $B_1^2(\alpha, \delta)$ has been studied by Frasin and Jahangiri [5]. Further $B_2^2(\alpha, 2)$ has been studied by Frasin and Darus [4]. Also we note that $B_2^2(\alpha, 1) \equiv S^*(\alpha)$ and $B_2^2(\alpha, 0) \equiv R(\alpha)$.

**Example 2.** If $l = 2$ and $m = 1$ with $\alpha_1 = \eta + 1 (\eta > -1), \alpha_2 = 1, \beta_1 = 1$, then

$$B(\eta, \alpha, \delta) := \left\{ f \in \mathcal{A} : \left| \frac{D^{\eta+1}f(z)}{z} \left( \frac{z}{D^{\eta}f(z)} \right) \frac{1}{\delta} - 1 \right| < 1 - \alpha, \, \eta > -1, \, \delta \geq 0, \, 0 \leq \alpha < 1, \, z \in U \right\},$$

where $D^\eta f(z)$ is called Ruscheweyh derivative operator [8] defined by

$$D^{\eta}f(z) := \frac{z}{(1-z)^{\eta+1}} \ast f(z) \equiv H_1^2(\eta + 1, 1; 1)f(z).$$

Also we observe that $B(0, \alpha, 1) \equiv K(\alpha)$.

**Example 3.** If $l = 2$ and $m = 1$ with $\alpha_1 = \mu + 1 (\mu > -1), \alpha_2 = 1, \beta_1 = \mu + 2$, then

$$B(\mu, \alpha, \delta) := \left\{ f \in \mathcal{A} : \left| \frac{J_{\mu+1}f(z)}{z} \left( \frac{z}{J_{\mu}f(z)} \right) \frac{1}{\delta} - 1 \right| < 1 - \alpha, \, \mu > -1, \, \delta \geq 0, \, 0 \leq \alpha < 1, \, z \in U \right\},$$

where $J_{\mu}$ is a Bernardi operator [1] defined by

$$J_{\mu}f(z) := \frac{\mu + 1}{z^\mu} \int_0^z t^{\mu-1}f(t)dt \equiv H_2^2(\mu + 1, 1; \mu + 2)f(z).$$

Note that the operator $J_1$ was studied earlier by Libera [6] and Livingston [7].

**Example 4.** If $l = 2$ and $m = 1$ with $\alpha_1 = a (a > 0), \alpha_2 = 1, \beta_1 = c (c > 0)$, then

$$B(a, c, \alpha, \delta) := \left\{ f \in \mathcal{A} : \left| \frac{L(a + 1, c)f(z)}{z} \left( \frac{z}{L(a, c)f(z)} \right) \frac{1}{\delta} - 1 \right| < 1 - \alpha, \, \delta \geq 0, \, 0 \leq \alpha < 1, \, z \in U \right\},$$

where $L(a, c)$ is a well-known Carlson-Shaffer linear operator [2] defined by

$$L(a, c)f(z) := \left( \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{k+1} \right) \ast f(z) \equiv H_2^2(a, 1; c)f(z).$$

The object of the present paper is to investigate the sufficient condition for functions to be in the class $B_m^l(\alpha, \delta)$. Furthermore, as a special case, we show that convex functions of order 1/2 are also members of the family $B_m^l(\alpha, \delta)$. 
2. Main Results

To prove our results we need the following lemma.

Lemma 1. [5] Let \( p \) be analytic in \( U \) with \( p(0) = 1 \) and suppose that

\[
\text{Re} \left\{ 1 + \frac{zp'(z)}{p(z)} \right\} > \frac{3\alpha - 1}{2\alpha}.
\]  

Then \( \text{Re} \{ p(z) \} > \alpha \) for \( z \in U \) and \( \frac{1}{2} \leq \alpha < 1 \).

Using Lemma 1, we first prove the following theorem.

Theorem 1. Let \( f(z) \) be the functions of the form (1), \( \delta \geq 0 \) and \( \frac{1}{2} \leq \alpha < 1 \). If

\[
(\alpha_1 + 1) \frac{H^l_m[\alpha_1 + 2]f(z)}{H^l_m[\alpha_1 + 1]f(z)} - \delta \alpha_1 \frac{H^l_m[\alpha_1 + 1]f(z)}{H^l_m[\alpha_1]f(z)} + \alpha_1(\delta - 1) < 1 + \beta z,
\]  

where \( \beta = \frac{3\alpha - 1}{2\alpha} \), then \( f(z) \in B^l_m(\alpha, \delta) \).

Proof. Define the function \( p(z) \) by

\[
p(z) := \frac{H^l_m[\alpha_1 + 1]f(z)}{z} \left( \frac{z}{H^l_m[\alpha_1]f(z)} \right)^\delta.
\]  

Then the function \( p(z) \) is analytic in \( U \) and \( p(0) = 1 \). Therefore, differentiating (10) logarithmically and the simple computation yields

\[
\frac{zp'(z)}{p(z)} = (\alpha_1 + 1) \frac{H^l_m[\alpha_1 + 2]f(z)}{H^l_m[\alpha_1 + 1]f(z)} - \delta \alpha_1 \frac{H^l_m[\alpha_1 + 1]f(z)}{H^l_m[\alpha_1]f(z)} + \alpha_1(\delta - 1) - 1.
\]  

By the hypothesis of the theorem, we have

\[
\text{Re} \left\{ 1 + \frac{zp'(z)}{p(z)} \right\} > \frac{3\alpha - 1}{2\alpha}.
\]  

Hence by Lemma 1, we have

\[
\text{Re} \left\{ \frac{H^l_m[\alpha_1 + 1]f(z)}{z} \left( \frac{z}{H^l_m[\alpha_1]f(z)} \right)^\delta \right\} > \alpha, \ z \in U.
\]  

Therefore in view of definition \( f(z) \in B^l_m(\alpha, \delta) \).

For \( l = 2 \) and \( m = 1 \) with \( \alpha_1 = a (a > 0), \alpha_2 = 1, \beta_1 = c (c > 0) \), we obtain the following corollary.
Corollary 1. Let $\frac{1}{2} \leq \alpha < 1$. If $f \in \mathcal{A}$ and

$$Re \left\{ (a+1) \frac{L[a+2,c]f(z)}{L[a+1,c]f(z)} - \delta a \frac{L[a+1,c]f(z)}{L[a,c]f(z)} + a(\delta - 1) \right\} > \frac{3a-1}{2a}, \; z \in U,$$

then

$$Re \left\{ \frac{L[a+1,c]f(z)}{z} \left( \frac{z}{L[a,c]f(z)} \right)^{\delta} \right\} > \alpha, \; z \in U.$$

Therefore $f(z) \in B(a,c,\alpha,\delta)$.

Taking $l = 2$ and $m = 1$ with $\alpha_1 = \mu + 1(\mu > -1)$, $\alpha_2 = 1$, $\beta_1 = \mu + 2$, we get

Corollary 2. Let $\frac{1}{2} \leq \alpha < 1$. If $f \in \mathcal{A}$ and

$$Re \left\{ (\mu + 2) \frac{J_{\mu+2}f(z)}{J_{\mu+1}f(z)} - \delta (\mu + 1) \frac{J_{\mu+1}f(z)}{J_{\mu}f(z)} + (\mu + 1)(\delta - 1) \right\} > \frac{3a-1}{2a}, \; z \in U,$$

then

$$Re \left\{ \frac{J_{\mu+1}f(z)}{z} \left( \frac{z}{J_{\mu}f(z)} \right)^{\delta} \right\} > \alpha, \; z \in U.$$

Therefore $f(z) \in B(\mu,\alpha,\delta)$.

Choosing $l = 2$ and $m = 1$ with $\alpha_1 = \eta + 1(\eta > -1)$, $\alpha_2 = 1$, $\beta_1 = 1$, we have

Corollary 3. Let $\frac{1}{2} \leq \alpha < 1$. If $f \in \mathcal{A}$ and

$$Re \left\{ (\eta + 2) \frac{D^{\eta+2}f(z)}{D^{\eta+1}f(z)} - \delta (\eta + 1) \frac{D^{\eta+1}f(z)}{D^{\eta}f(z)} + \eta(\eta + 1)(\delta - 1) \right\} > \frac{3a-1}{2a}, \; z \in U,$$

then

$$Re \left\{ \frac{D^{\eta+1}f(z)}{z} \left( \frac{z}{D^{\eta}f(z)} \right)^{\delta} \right\} > \alpha, \; z \in U.$$

Therefore $f(z) \in B(\eta,\alpha,\delta)$.

Choosing $l = 2$ and $m = 1$ with $\alpha_1 = 1$, $\alpha_2 = 1$ and $\beta_1 = 1$, we have

Corollary 4. [5] If $f \in \mathcal{A}$ and

$$Re \left\{ 1 + \frac{zf^{(l)}(z)}{f(z)} + \delta \left( 1 - \frac{zf^{(l)}(z)}{f(z)} \right) \right\} > \frac{3a-1}{2a}, \; z \in U,$$

then

$$Re \left\{ f^{(l)}(z) \left( \frac{z}{f(z)} \right)^{\delta} \right\} > \alpha, \; z \in U.$$

Therefore $f(z) \in B^2_1(\alpha,\delta)$. 
Choosing $l = 2$ and $m = 1$ with $\alpha_1 = 2, \alpha_2 = 1, \beta_1 = 1, \delta = 1$ and $\alpha = \frac{1}{2}$ we have

**Corollary 5.** If $f \in \mathcal{A}$ and

$$\Re \left\{ \frac{z^2 f'''' + 6zf''(z) + 6f'(z)}{2f'(z) + zf''} - \frac{zf''(z)}{f'(z)} \right\} > \frac{3}{2}, z \in U,$$

then

$$\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0, z \in U.$$

That is, $f(z) \in K$.

Choosing $l = 2$ and $m = 1$ with $\alpha_1 = 2, \alpha_2 = 1, \beta_1 = 1, \delta = 0$ and $\alpha = \frac{1}{2}$ we have

**Corollary 6.** If $f \in \mathcal{A}$ and

$$\Re \left\{ \frac{z^2 f'''' + 4zf''(z) + 2f'(z)}{2f'(z) + zf''} \right\} > \frac{1}{2}, z \in U,$$

then

$$\Re \left\{ f'(z) + \frac{zf''(z)}{2} \right\} > \frac{1}{2}, z \in U.$$

Choosing $l = 2$ and $m = 1$ with $\alpha_1 = 1, \alpha_2 = 1, \beta_1 = 1, \delta = 1$ and $\alpha = \frac{1}{2}$ we have

**Corollary 7.** If $f \in \mathcal{A}$ and

$$\Re \left\{ \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right\} > -\frac{3}{2}, z \in U,$$

then

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \frac{1}{2}, z \in U.$$

That is, $f(z)$ is starlike of order $1/2$.

Choosing $l = 2$ and $m = 1$ with $\alpha_1 = 1, \alpha_2 = 1, \beta_1 = 1, \delta = 0$ and $\alpha = \frac{1}{2}$ we have

**Corollary 8.** If $f \in \mathcal{A}$ and

$$\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \frac{1}{2}, z \in U$$

then

$$\Re \left\{ f'(z) \right\} > \frac{1}{2}, z \in U.$$

That is $f(z) \in B(0, 1/2) = R_{1/2}$.

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