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Upper and Lower Weak (τ_1, τ_2)-continuity

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Abstract. This paper is concerned with the concepts of upper and lower weakly (τ_1, τ_2)-continuous multifunctions. Moreover, some characterizations of upper and lower weakly (τ_1, τ_2)-continuous multifunctions are investigated.

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1. Introduction

The concept of weakly continuous functions was introduced by Levine [26]. Husain [23] introduced and studied the notion of almost continuous functions. Janković [24] introduced almost weak continuity as a generalization of both weak continuity and almost continuity. Noiri [27] investigated several characterizations of almost weakly continuous functions. Rose [34] introduced the notion of subweakly continuous functions and investigated the relationships between subweak continuity and weak continuity. Popa and Noiri [32] introduced the concept of weakly (τ, m)-continuous functions as functions from a topological space into a set satisfying some minimal conditions and investigated several characterizations of weakly (τ, m)-continuous functions. Ekici et al. [22] introduced and studied the concept of weakly λ-continuous functions. Duangphui et al. [21] introduced and investigated the notion of weakly $(\mu, \mu')^{(m,n)}$-continuous functions. Moreover, some characterizations of almost (Λ, p)-continuous functions, strongly $\theta(\Lambda, p)$-continuous functions, almost strongly $\theta(\Lambda, p)$-continuous functions, $\theta(\Lambda, p)$-continuous functions, weakly (Λ, b)-continuous functions, $\theta(\star)$-precontinuous functions, $\star$-continuous functions, θ-$\mathscr{I}$-continuous functions, almost (g, m)-continuous functions, (Λ, sp)-continuous functions, $\delta p(\Lambda, s)$-continuous functions, $(\Lambda, p(\star))$-continuous functions, pairwise weakly M-continuous

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functions, (τ_1, τ_2)-continuous functions, almost (τ_1, τ_2)-continuous functions and weakly (τ_1, τ_2)-continuous functions were presented in [36], [38], [10], [33], [16], [9], [8], [5], [2], [40], [37], [7], [3], [17], [15] and [11], respectively.

Popa [29] and Smithson [35] independently introduced the notion of weakly continuous multifunctions. Popa and Noiri [31] introduced a class of multifunctions called weakly α-continuous multifunctions. Furthermore, Popa and Noiri [30] investigated some characterizations of upper and lower weakly β-continuous multifunctions. Noiri and Popa [28] introduced and investigated the notion of weakly m-continuous multifunctions as a multifunction from a set satisfying certain minimal condition into a topological space. Boonpok and Viriyapong [19] introduced and studied the concepts upper and lower almost weakly (τ_1, τ_2)-continuous multifunctions. Laprom et al. [25] introduced and investigated the notions of upper and lower almost $\beta(\tau_1, \tau_2)$-continuous multifunctions. Viriyapong and Boonpok [39] introduced and studied the concepts of upper and lower weakly $(\tau_1, \tau_2)\alpha$-continuous multifunctions. Moreover, several characterizations of weakly $(\tau_1, \tau_2)\delta$-semicontinuous multifunctions, almost weakly $\star$-continuous multifunctions, weakly $\star$-continuous multifunctions, weakly α-$\star$-continuous multifunctions, weakly $\imath^\star$-continuous multifunctions, weakly quasi (Λ, sp)-continuous multifunctions and weakly (Λ, sp)-continuous multifunctions were established in [6], [18], [4], [13], [12], [41] and [14], respectively. In this paper, we introduce the concepts of upper and lower weakly (τ_1, τ_2)-continuous multifunctions. In particular, some characterizations of upper and lower weakly (τ_1, τ_2)-continuous multifunctions are discussed.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [20] if $A = \tau_1\text{-Cl}(\tau_2\text{-Cl}(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. Let A be a subset of a bitopological space (X, τ_1, τ_2). The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [20] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [20] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 1. [20] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2\text{-Cl}(A)$ and $\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Cl}(A)) = \tau_1\tau_2\text{-Cl}(A)$.

(2) If $A \subseteq B$, then $\tau_1\tau_2\text{-Cl}(A) \subseteq \tau_1\tau_2\text{-Cl}(B)$.

(3) $\tau_1\tau_2\text{-Cl}(A)$ is $\tau_1\tau_2$-closed.

(4) A is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2\text{-Cl}(A)$.

(5) $\tau_1\tau_2$-*Cl*$(X - A) = X - \tau_1\tau_2$-*Int*$(A)$.

A subset A of a bitopological space (X, τ_1, τ_2) is called $(\tau_1, \tau_2)r$*-open* [39] (resp. $(\tau_1, \tau_2)s$*-open* [6], $(\tau_1, \tau_2)p$*-open* [6], $(\tau_1, \tau_2)\beta$*-open* [6]) if $A = \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$ (resp. $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A))$, $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$, $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A)))$). The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open) set is called $(\tau_1, \tau_2)r$*-closed*, $(\tau_1, \tau_2)s$*-closed*, $(\tau_1, \tau_2)p$*-closed*, $(\tau_1, \tau_2)\beta$*-closed*. A subset A of a bitopological space (X, τ_1, τ_2) is called $\alpha(\tau_1, \tau_2)$*-open* [42] if $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A)))$. The complement of an $\alpha(\tau_1, \tau_2)$-open set is called $\alpha(\tau_1, \tau_2)$*-closed*.

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, following [1] we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and

$$F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}.$$

In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$.

3. Upper and lower weakly (τ_1, τ_2)-continuous multifunctions

In this section, we introduce the notions of upper and lower weakly (τ_1, τ_2)-continuous multifunctions. Moreover, some characterizations of upper and lower weakly (τ_1, τ_2)-continuous multifunctions are discussed.

Definition 1. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be upper weakly (τ_1, τ_2)-continuous if for each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y containing $F(x)$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2$-Cl$(V)$.*

Theorem 1. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F *is upper weakly* (τ_1, τ_2)*-continuous;*

(2) $F^+(V) \subseteq \tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$ *for every* $\sigma_1\sigma_2$*-open set* V *of* Y*;*

(3) $\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(K))) \subseteq F^-(K)$ *for every* $\sigma_1\sigma_2$*-closed set* K *of* Y*;*

(4) $\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B)))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(B))$ *for every subset* B *of* Y*;*

(5) $F^+(\sigma_1\sigma_2\text{-Int}(B)) \subseteq \tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B))))$ *for every subset* B *of* Y*;*

(6) $\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V))$ *for every* $\sigma_1\sigma_2$*-open set* V *of* Y*;*

(7) $\tau_1\tau_2\text{-Cl}(F^-(V)) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V))$ *for every* $\sigma_1\sigma_2$*-open set* V *of* Y*;*

(8) $\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(K))) \subseteq F^-(K)$ *for every* $(\sigma_1, \sigma_2)r$*-closed set* K *of* Y.

Proof. (1) $\Rightarrow$ (2): Let V be any $\sigma_1\sigma_2$-open set of Y such that $x \in F^+(V)$. Then, $F(x) \subseteq V$. There exists a $\tau_1\tau_2$-open set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. Thus, $U \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V))$. Since U is $\tau_1\tau_2$-open, we have $x \in \tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$ and hence $F^+(V) \subseteq \tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$.

(2) $\Rightarrow$ (3): Let K be any $\sigma_1\sigma_2$-closed set of Y. Then, $Y - K$ is $\sigma_1\sigma_2$-open in Y and by (2),

$$\begin{aligned} X - F^-(K) &= F^+(Y-K) \\ &\subseteq \tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-Cl}(Y-K))) \\ &= X - \tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(K))). \end{aligned}$$

Thus, $\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(K))) \subseteq F^-(K)$.

(3) $\Rightarrow$ (4): Let B be any subset of Y. Then, $\sigma_1\sigma_2\text{-Cl}(B)$ is a $\sigma_1\sigma_2$-closed set of Y and by (3), $\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(B)))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(B))$.

(4) $\Rightarrow$ (5): Let B be any subset of Y. By (4), we have

$$\begin{aligned} X - \tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B)))) &= \tau_1\tau_2\text{-Cl}(X - F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B)))) \\ &= \tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y-B)))) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(Y-B)) \\ &= X - F^+(\sigma_1\sigma_2\text{-Int}(B)) \end{aligned}$$

and hence $F^+(\sigma_1\sigma_2\text{-Int}(B)) \subseteq \tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B))))$.

(5) $\Rightarrow$ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y such that $F(x) \subseteq V$. Then, $x \in F^+(V) \subseteq \tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$ and there exists a $\tau_1\tau_2$-open set U of X containing x such that $U \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V))$. Thus, $F(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$ and hence F is upper weakly (τ_1, τ_2)-continuous.

(4) $\Rightarrow$ (6) and (6) $\Rightarrow$ (7): The proofs are obvious.

(7) $\Rightarrow$ (8): Let K be any $(\sigma_1, \sigma_2)r$-closed set of Y. Thus by (7),

$$\begin{aligned} \tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(K))) &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) \\ &= F^-(K). \end{aligned}$$

(8) $\Rightarrow$ (3): Let K be any $\sigma_1\sigma_2$-closed set of Y. Then, $\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))$ is $(\sigma_1, \sigma_2)r$-closed in Y and $\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) = \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(K)) = \sigma_1\sigma_2\text{-Int}(K)$. By (8),

$$\begin{aligned} \tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(K))) &= \tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))))) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) \\ &\subseteq F^-(K). \end{aligned}$$

Definition 2. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be lower weakly (τ_1, τ_2)-continuous if for each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $\sigma_1\sigma_2$-Cl$(V) \cap F(z) \neq \emptyset$ for each $z \in U$.*

Theorem 2. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F *is lower weakly* (τ_1, τ_2)*-continuous;*

(2) $F^-(V) \subseteq \tau_1\tau_2\text{-}Int(F^-(\sigma_1\sigma_2\text{-}Cl(V)))$ *for every* $\sigma_1\sigma_2$*-open set* V *of* Y*;*

(3) $\tau_1\tau_2\text{-}Cl(F^+(\sigma_1\sigma_2\text{-}Int(K))) \subseteq F^+(K)$ *for every* $\sigma_1\sigma_2$*-closed set* K *of* Y*;*

(4) $\tau_1\tau_2\text{-}Cl(F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B)))) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(B))$ *for every subset* B *of* Y*;*

(5) $F^-(\sigma_1\sigma_2\text{-}Int(B)) \subseteq \tau_1\tau_2\text{-}Int(F^-(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(B))))$ *for every subset* B *of* Y*;*

(6) $\tau_1\tau_2\text{-}Cl(F^+(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(V))$ *for every* $\sigma_1\sigma_2$*-open set* V *of* Y*;*

(7) $\tau_1\tau_2\text{-}Cl(F^+(V)) \subseteq F^+(\sigma_1\sigma_2\text{-}Cl(V))$ *for every* $\sigma_1\sigma_2$*-open set* V *of* Y*;*

(8) $\tau_1\tau_2\text{-}Cl(F^+(\sigma_1\sigma_2\text{-}Int(K))) \subseteq F^+(K)$ *for every* $(\sigma_1, \sigma_2)r$*-closed set* K *of* Y*.*

Proof. The proof is similar to that of Theorem 1.

Definition 3. [11] *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be weakly (τ_1, τ_2)-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2\text{-}Cl(V)$. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be weakly (τ_1, τ_2)-continuous if f has this property at each point of X.*

Corollary 1. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f *is weakly* (τ_1, τ_2)*-continuous;*

(2) $f^{-1}(V) \subseteq \tau_1\tau_2\text{-}Int(f^{-1}(\sigma_1\sigma_2\text{-}Cl(V)))$ *for every* $\sigma_1\sigma_2$*-open set* V *of* Y*;*

(3) $\tau_1\tau_2\text{-}Cl(f^{-1}(\sigma_1\sigma_2\text{-}Int(K))) \subseteq f^{-1}(K)$ *for every* $\sigma_1\sigma_2$*-closed set* K *of* Y*;*

(4) $\tau_1\tau_2\text{-}Cl(f^{-1}(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(B))$ *for every subset* B *of* Y*;*

(5) $f^{-1}(\sigma_1\sigma_2\text{-}Int(B)) \subseteq \tau_1\tau_2\text{-}Int(f^{-1}(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(B))))$ *for every subset* B *of* Y*;*

(6) $\tau_1\tau_2\text{-}Cl(f^{-1}(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ *for every* $\sigma_1\sigma_2$*-open set* V *of* Y*;*

(7) $\tau_1\tau_2\text{-}Cl(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ *for every* $\sigma_1\sigma_2$*-open set* V *of* Y*;*

(8) $\tau_1\tau_2\text{-}Cl(f^{-1}(\sigma_1\sigma_2\text{-}Int(K))) \subseteq f^{-1}(K)$ *for every* $(\sigma_1, \sigma_2)r$*-closed set* K *of* Y*.*

Theorem 3. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F *is upper weakly* (τ_1, τ_2)*-continuous;*

(2) $\tau_1\tau_2$*-Cl*$(F^-(\sigma_1\sigma_2$*-Int*$(\sigma_1\sigma_2$*-Cl*$(V)))) \subseteq F^-(\sigma_1\sigma_2$*-Cl*$(V))$ *for every* $(\sigma_1, \sigma_2)\beta$*-open set* V *of* Y*;*

(3) $\tau_1\tau_2$*-Cl*$(F^-(\sigma_1\sigma_2$*-Int*$(\sigma_1\sigma_2$*-Cl*$(V)))) \subseteq F^-(\sigma_1\sigma_2$*-Cl*$(V))$ *for every* $(\sigma_1, \sigma_2)s$*-open set* V *of* Y*.*

Proof. (1) ⇒ (2): This follows from (4) of Theorem 1.

(2) ⇒ (3): The proof is obvious since every $(\sigma_1, \sigma_2)s$-open set is $(\sigma_1, \sigma_2)\beta$-open.

(3) ⇒ (1): Since every $\sigma_1\sigma_2$-open set is $(\sigma_1, \sigma_2)s$-open, the proof is obvious by (7) of Theorem 1.

Theorem 4. *For a multifunction* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) F *is lower weakly* (τ_1, τ_2)*-continuous;*

(2) $\tau_1\tau_2$*-Cl*$(F^+(\sigma_1\sigma_2$*-Int*$(\sigma_1\sigma_2$*-Cl*$(V)))) \subseteq F^+(\sigma_1\sigma_2$*-Cl*$(V))$ *for every* $(\sigma_1, \sigma_2)\beta$*-open set* V *of* Y*;*

(3) $\tau_1\tau_2$*-Cl*$(F^+(\sigma_1\sigma_2$*-Int*$(\sigma_1\sigma_2$*-Cl*$(V)))) \subseteq F^+(\sigma_1\sigma_2$*-Cl*$(V))$ *for every* $(\sigma_1, \sigma_2)s$*-open set* V *of* Y*.*

Proof. The proof is similar to that of Theorem 3.

Corollary 2. *For a function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) f *is weakly* (τ_1, τ_2)*-continuous;*

(2) $\tau_1\tau_2$*-Cl*$(f^{-1}(\sigma_1\sigma_2$*-Int*$(\sigma_1\sigma_2$*-Cl*$(V)))) \subseteq f^{-1}(\sigma_1\sigma_2$*-Cl*$(V))$ *for every* $(\sigma_1, \sigma_2)\beta$*-open set* V *of* Y*;*

(3) $\tau_1\tau_2$*-Cl*$(f^{-1}(\sigma_1\sigma_2$*-Int*$(\sigma_1\sigma_2$*-Cl*$(V)))) \subseteq f^{-1}(\sigma_1\sigma_2$*-Cl*$(V))$ *for every* $(\sigma_1, \sigma_2)s$*-open set* V *of* Y*.*

Theorem 5. *For a multifunction* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) F *is upper weakly* (τ_1, τ_2)*-continuous;*

(2) $\tau_1\tau_2$*-Cl*$(F^-(\sigma_1\sigma_2$*-Int*$(\sigma_1\sigma_2$*-Cl*$(V)))) \subseteq F^-(\sigma_1\sigma_2$*-Cl*$(V))$ *for every* $(\sigma_1, \sigma_2)p$*-open set* V *of* Y*;*

(3) $\tau_1\tau_2$*-Cl*$(F^-(V)) \subseteq F^-(\sigma_1\sigma_2$*-Cl*$(V))$ *for every* $(\sigma_1, \sigma_2)p$*-open set* V *of* Y*;*

(4) $F^+(V) \subseteq \tau_1\tau_2$*-Int*$(F^+(\sigma_1\sigma_2$*-Cl*$(V)))$ *for every* $(\sigma_1, \sigma_2)p$*-open set* V *of* Y*.*

Proof. (1) ⇒ (2): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. Since $\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))$ is $\sigma_1\sigma_2$-open, by Theorem 1(7)

$$\begin{aligned}\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))\\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V)).\end{aligned}$$

(2) ⇒ (3): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. By (2), we have

$$\begin{aligned}\tau_1\tau_2\text{-Cl}(F^-(V)) &\subseteq \tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))\\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V)).\end{aligned}$$

(3) ⇒ (4): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. Thus by (3),

$$\begin{aligned}X - \tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-Cl}(V)) &= \tau_1\tau_2\text{-Cl}(X - F^+(\sigma_1\sigma_2\text{-Cl}(V)))\\ &= \tau_1\tau_2\text{-Cl}(F^-(Y - \sigma_1\sigma_2\text{-Cl}(V)))\\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V)))\\ &= X - F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))\\ &\subseteq X - F^+(V)\end{aligned}$$

and hence $F^+(V) \subseteq \tau_1\tau_2$-Int$(F^+(\sigma_1\sigma_2$-Cl$(V)))$.

(4) ⇒ (1): Let V be any $\sigma_1\sigma_2$-open set of Y. Then, V is $(\sigma_1, \sigma_2)p$-open and by (4), $F^+(V) \subseteq \tau_1\tau_2$-Int$(F^+(\sigma_1\sigma_2$-Cl$(V)))$. By Theorem 1(2), F is upper weakly (τ_1, τ_2)-continuous.

Theorem 6. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower weakly (τ_1, τ_2)-continuous;

(2) $\tau_1\tau_2$-Cl$(F^+(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))) \subseteq F^+(\sigma_1\sigma_2$-Cl$(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;

(3) $\tau_1\tau_2$-Cl$(F^+(V)) \subseteq F^+(\sigma_1\sigma_2$-Cl$(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;

(4) $F^-(V) \subseteq \tau_1\tau_2$-Int$(F^-(\sigma_1\sigma_2$-Cl$(V)))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y.

Proof. The proof is similar to that of Theorem 5.

Corollary 3. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is weakly (τ_1, τ_2)-continuous;

(2) $\tau_1\tau_2$-Cl$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;

(3) $\tau_1\tau_2$-Cl$(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;

(4) $f^{-1}(V) \subseteq \tau_1\tau_2$-Int$(f^{-1}(\sigma_1\sigma_2$-Cl$(V)))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y.

4. Several characterizations

Recall that a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-*compact* [20] if every cover of X by $\tau_1\tau_2$-open sets of X has a finite subcover.

Definition 4. *A bitopological space (X, τ_1, τ_2) is said to be quasi (τ_1, τ_2)-$\mathscr{H}$-closed if every $\tau_1\tau_2$-open cover $\{U_\gamma \mid \gamma \in \Gamma\}$, there exists a finite subset Γ_0 of Γ such that*

$$X = \cup\{\tau_1\tau_2\text{-}Cl(U_\gamma) \mid \gamma \in \Gamma_0\}.$$

Theorem 7. *Let $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be an upper weakly (τ_1, τ_2)-continuous surjective multifunction such that $F(x)$ is $\sigma_1\sigma_2$-compact for each $x \in X$. If (X, τ_1, τ_2) is $\tau_1\tau_2$-compact, then (Y, σ_1, σ_2) is quasi (σ_1, σ_2)-$\mathscr{H}$-closed.*

Proof. Let $\{V_\gamma \mid \gamma \in \Gamma\}$ be any $\sigma_1\sigma_2$-open cover of Y. For each $x \in X$, $F(x)$ is $\sigma_1\sigma_2$-compact and there exists a finite subset $\Gamma(x)$ of Γ such that $F(x) \subseteq \cup\{V_\gamma \mid \gamma \in \Gamma(x)\}$. Now, set $V(x) = \cup\{V_\gamma \mid \gamma \in \Gamma(x)\}$. Since F is upper weakly (τ_1, τ_2)-continuous, there exists a $\tau_1\tau_2$-open set $U(x)$ of X containing x such that $F(U(x)) \subseteq \sigma_1\sigma_2\text{-Cl}(V(x))$. The family $\{U(x) \mid x \in X\}$ is a $\tau_1\tau_2$-open cover of X by $\tau_1\tau_2$-open sets. Since (X, τ_1, τ_2) is $\tau_1\tau_2$-compact, there exists a finite number of points, say, $x_1, x_2, ..., x_n$ in X such that $X = \cup\{U(x_i) \mid 1 \leq i \leq n\}$. Thus,

$$\begin{aligned} Y = F(X) &= \cup\{F(U(x_i)) \mid 1 \leq i \leq n\} \\ &\subseteq \cup\{\sigma_1\sigma_2\text{-Cl}(V(x_i)) \mid 1 \leq i \leq n\} \\ &\subseteq \cup\{\sigma_1\sigma_2\text{-Cl}(V_\gamma) \mid \gamma \in \Gamma(x_i), 1 \leq i \leq n\}. \end{aligned}$$

This shows that (Y, σ_1, σ_2) is quasi (σ_1, σ_2)-$\mathscr{H}$-closed.

The $\tau_1\tau_2$-*frontier* [17] of a subset A of a bitopological space (X, τ_1, τ_2), denoted by $\tau_1\tau_2\text{-fr}(A)$, is defined by

$$\tau_1\tau_2\text{-fr}(A) = \tau_1\tau_2\text{-Cl}(A) \cap \tau_1\tau_2\text{-Cl}(X - A) = \tau_1\tau_2\text{-Cl}(A) - \tau_1\tau_2\text{-Int}(A).$$

Theorem 8. *The set of all points x of X at which a multifunction*

$$F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$$

is not upper weakly (τ_1, τ_2)-continuous is identical with the union of the $\tau_1\tau_2$-frontier of the upper inverse images of the $\sigma_1\sigma_2$-closures of $\sigma_1\sigma_2$-open sets containing $F(x)$.

Proof. Let x be a point of X at which F is not upper weakly (τ_1, τ_2)-continuous. Then, there exists a $\sigma_1\sigma_2$-open set V containing $F(x)$ such that $U \cap (X - F^+(\sigma_1\sigma_2\text{-Cl}(V))) \neq \emptyset$ for every $\tau_1\tau_2$-open set U containing x. Then, we have $x \in \tau_1\tau_2\text{-Cl}(X - F^+(\sigma_1\sigma_2\text{-Cl}(V)))$. Since $x \in F^+(V)$, $x \in \tau_1\tau_2\text{-Cl}(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$ and hence $x \in \tau_1\tau_2\text{-fr}(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$.

Conversely, suppose that V is a $\sigma_1\sigma_2$-open set of Y containing $F(x)$ such that

$$x \in \tau_1\tau_2\text{-fr}(F^+(\sigma_1\sigma_2\text{-Cl}(V))).$$

If F is upper weakly (τ_1,τ_2)-continuous at x, there exists a $\tau_1\tau_2$-open set U of X containing x such that $U \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V))$; hence $x \in \tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$. This is a contradiction and hence F is not upper weakly (τ_1,τ_2)-continuous at x.

Theorem 9. *The set of all points of X at which a multifunction*

$$F : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$$

is not lower weakly (τ_1,τ_2)-continuous is identical with the union of the $\tau_1\tau_2$-frontier of the lower inverse images of the $\sigma_1\sigma_2$-closures of $\sigma_1\sigma_2$-open sets meeting $F(x)$.

Proof. The proof is similar to that of Theorem 8.

Definition 5. [20] *A bitopological space (X,τ_1,τ_2) is said to be $\tau_1\tau_2$-connected if X cannot be written as the union of two nonempty disjoint $\tau_1\tau_2$-open sets.*

Recall that a subset A of a bitopological space (X,τ_1,τ_2) is said to be $\tau_1\tau_2$-*clopen* [20] if A is both $\tau_1\tau_2$-open and $\tau_1\tau_2$-closed.

Theorem 10. *If $F : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is an upper or lower weakly (τ_1,τ_2)-continuous surjective multifunction such that $F(x)$ is $\sigma_1\sigma_2$-connected for each $x \in X$ and (X,τ_1,τ_2) is $\tau_1\tau_2$-connected, then (Y,σ_1,σ_2) is $\sigma_1\sigma_2$-connected.*

Proof. Suppose that (Y,σ_1,σ_2) is not $\sigma_1\sigma_2$-connected. There exist non-empty $\sigma_1\sigma_2$-open sets U and V of Y such that $U \cap V = \emptyset$ and $U \cup V = Y$. Since $F(x)$ is $\sigma_1\sigma_2$-connected for each $x \in X$, either $F(x) \subseteq U$ or $F(x) \subseteq V$. If $x \in F^+(U \cup V)$, then $F(x) \subseteq U \cup V$ and hence $x \in F^+(U) \cup F^+(V)$. Moreover, since F is surjective, there exist x and y in X such that $F(x) \subseteq U$ and $F(y) \subseteq V$; hence $x \in F^+(U)$ and $y \in F^+(V)$. Therefore, we obtain the following:

(1) $F^+(U) \cup F^+(V) = F^+(U \cup V) = X$;

(2) $F^+(U) \cap F^+(V) = F^+(U \cap V) = \emptyset$;

(3) $F^+(U) \neq \emptyset$ and $F^+(V) \neq \emptyset$.

Next, we show that $F^+(U)$ and $F^+(V)$ are $\tau_1\tau_2$-open in X. (i) Let F be upper weakly (τ_1,τ_2)-continuous. By Theorem 1, $F^+(V) \subseteq \tau_1\tau_2\text{-Int}(F^+(\sigma_1\sigma_2\text{-Cl}(V))) = \tau_1\tau_2\text{-Int}(F^+(V))$ since V is $\sigma_1\sigma_2$-clopen. Thus, $F^+(V) = \tau_1\tau_2\text{-Int}(F^+(V))$ and hence $F^+(V)$ is $\tau_1\tau_2$-open in X. Similarly, we obtain $F^+(U)$ is $\tau_1\tau_2$-open in X. Consequently, this shows that (X,τ_1,τ_2) is not $\tau_1\tau_2$-connected. (ii) Let F be lower weakly (τ_1,τ_2)-continuous. By Theorem 2, $\tau_1\tau_2\text{-Cl}(F^+(V)) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V)) = F^+(V)$ since V is $\sigma_1\sigma_2$-clopen. Therefore, $F^+(V) = \tau_1\tau_2\text{-Cl}(F^+(V))$ and so $F^+(V)$ is $\tau_1\tau_2$-closed in X. Thus, we have $F^+(U)$ is $\tau_1\tau_2$-open in X. Similarly, we obtain $F^+(V)$ is $\tau_1\tau_2$-open in X. Consequently, this shows that (X,τ_1,τ_2) is not $\tau_1\tau_2$-connected. This completes the proof.

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