EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 17, No. 3, 2024, 2092-2105
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Decomposition of the Unitary Representation of $\mathrm{SL}_2(\mathbb{R})$ on the Upper Half Plane into Irreducible Components

Fatimah Abdullah Alabbad

Department of Mathematics and Statistics, Faculty of science, King Faisal University, Saudi Arabia

Abstract. The main purpose of this paper is to find the inversion formula for the covariant transform $\mathcal{W}^{\rho_k}_{\varphi_0}$. This formula is equivalent to the decomposition of the unitary representation ρ_k into irreducible components. We consider an eigenvalue $1+s^2$ of the Casimir operator:

$$d\rho_k(C) = -4v^2\left(\partial_u^2 + \partial_v^2\right), \quad \text{where } k=0.$$

To find the inversion formula, first we study the representations of $\mathrm{SL}_2(\mathbb{R})$, ρ_k and ρ_τ, induced from the complex characters of K and N respectively. Then, we find the induced covariant transform $\mathcal{W}^{\rho_k}_{\varphi_0}$ with N-eigenvector to obtain a transform in the space $L_2(\mathrm{SL}_2(\mathbb{R})/N)$. Thereafter, we compute the contravariant transform with K-eigenvector

$$\mathcal{M}^{\rho_\tau}_{\phi_0} : L_2(\mathrm{SL}_2(\mathbb{R})/N) \to L_2(\mathrm{SL}_2(\mathbb{R})/K).$$

2020 Mathematics Subject Classifications: 22D10,22D30,42A38

Key Words and Phrases: Unitary representation, $\mathrm{SL}_2(\mathbb{R})$ group, Covariant transform, Inversion formula

1. Introduction

Integral transforms establishes a correspondence between functions on a manifold X and functions on some manifold M of submanifolds of X. The main problems are in the description of the images and kernels of these transforms and in the construction of explicit inversion formulas recovering the original objects from their images. The first book devoted to this area was by I. M. Gelfand,M. I. Graev and N. Ya. Vilenkin [4]. From the 1940s, one of the main problems in mathematics was to develop an analog of the Fourier transform for noncommutative Lie group. For the group $SL_2(\mathbb{C})$, I. M. Gelfand and M. A. Naimark constructed a theory in which the role of exponential functions was played by irreducible infinite-dimensional unitary representations of the $SL_2(\mathbb{C})$ group. Obtaining analogs of the inversion formula and the Plancherel formula for the Fourier transform was

DOI: https://doi.org/10.29020/nybg.ejpam.v17i3.5245

Email address: falabbad@kfu.edu.sa (F. A. Alabbad)

the most important result of this theory. Our contribution is to use a new method starting with the covariant transform to obtain the inversion formula.

Action of $\mathrm{SL}_2(\mathbb{R})$ by linear-fractional transformation on complex numbers produces isometrical motions of the Lobachevsky geometry. It is less known that there are related actions of $\mathrm{SL}_2(\mathbb{R})$ on dual and double numbers which have the form $z = x + \iota y$, $\iota^2 = 0$ or $\iota^2 = 1$, correspondingly. We write ε and j instead of ι within dual and double numbers, respectively.
Three possible values $-1, 0$ and 1 of $\sigma := \iota^2$ will be referred to elliptic, parabolic and hyperbolic cases, respectively.

A generic cycle [9], § 4.2 is the set of points $(u, v) \in \mathbb{R}^2$ defined for all values of σ by the equation

$$k(u^2 - \sigma v^2) - 2lu - 2nv + m = 0. \tag{1}$$

This equation is represented by a point (k, l, n, m) from a projective space $\mathbb{P}^3$, since for a scaling factor $\lambda \neq 0$, the point $(\lambda k, \lambda l, \lambda n, \lambda m)$ defines an equation equivalent to (1). We call $\mathbb{P}^3$ the cycle space and refer to the initial $\mathbb{R}^2$ as the point space.
In order to obtain a connection with the Möbius action, we arrange numbers (k, l, n, m) into the matrix [9], Definition 4.11

$$C_{\breve{\sigma}} = \begin{pmatrix} l + \breve{\iota} n & -m \\ k & -l + \breve{\iota} n \end{pmatrix}. \tag{2}$$

The values of $\breve{\sigma} := \breve{\iota}^2$ are $-1, 0$ or 1 may be chosen to be independent of the values of σ.

Theorem 1. *[9], Theorem 4.13 Let a matrix* $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ *defines a Möbius transformation*

$$g : (u + \iota v) \to \frac{a(u + \iota v) + b}{c(u + \iota v) + d}. \tag{3}$$

Then the image $\tilde{C}_{\breve{\sigma}}$ *of a cycle* $C_{\breve{\sigma}}$ *under transformation with* $g \in \mathrm{SL}_2(\mathbb{R})$ *is given by similarity of the matrix* (2)*:*

$$\tilde{C}_{\breve{\sigma}} = g C_{\breve{\sigma}} g^{-1}. \tag{4}$$

Definition 1. *[9], Definition 5.11 For two cycles* C *and* C_1*, define the cycles product by:*

$$\langle C, C_1 \rangle = -\operatorname{tr}(C \bar{C}_1), \tag{5}$$

where tr *denotes the trace of a matrix.*

We can find the explicit expression of the cycle product (5) with $\sigma = -1, 0$ and 1:

$$\langle C, C_1 \rangle = k m_1 + k_1 m - 2 l l_1 + 2\sigma n n_1, \tag{6}$$

where $C = (k, l, n, m)$ and $C_1 = (k_1, l_1, n_1, m_1)$.

Definition 2. *[3], Chap. 3, § 1.2 On the hyperbolic plane one can define circles of infinitely large radius (horocycles), which are the limits of non-Euclidean circles as the center and the radius of these circles consistently tend to infinity. In the Lobachevsky model, the horocycles are represented either as Euclidean circles tangent to the real axis or as lines parallel to the real axis.*

The horizontal line $v-1=0$ as a cycle is represented by the matrix $\begin{pmatrix} -\frac{i}{2} & -1 \\ 0 & -\frac{i}{2} \end{pmatrix}$. This line is invariant under the subgroup $N=\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, that is

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -\frac{i}{2} & -1 \\ 0 & -\frac{i}{2} \end{pmatrix}\begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} -\frac{i}{2} & -1 \\ 0 & -\frac{i}{2} \end{pmatrix}.$$

Thus all horocycles obtained by $\mathrm{SL}_2(\mathbb{R})$ action are parametrized by points of the homogeneous space $\mathrm{SL}_2(\mathbb{R})/N$.
The image of $v-1=0$ under the lower triangular matrix $\begin{pmatrix} \xi_1 & 0 \\ \xi_2 & \frac{1}{\xi_1} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ is

$$\begin{pmatrix} \xi_1 & 0 \\ \xi_2 & \frac{1}{\xi_1} \end{pmatrix}\begin{pmatrix} -\frac{i}{2} & -1 \\ 0 & -\frac{i}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{\xi_1} & 0 \\ -\xi_2 & \xi_1 \end{pmatrix}=\begin{pmatrix} \xi_1\xi_2-\frac{i}{2} & -\xi_1^2 \\ \xi_2^2 & -\xi_1\xi_2-\frac{i}{2}, \end{pmatrix} \tag{7}$$

that is, cycle $(\xi_2^2, \xi_1\xi_2, \frac{1}{2}, \xi_1^2)$ with the equation

$$\begin{aligned} &\xi_2^2u^2+\xi_2^2v^2-2\xi_1\xi_2u-v+\xi_1^2=0 \\ \Leftrightarrow\ &(\xi_2^2u^2-2\xi_1\xi_2u+\xi_1^2)+\xi_2^2v^2=v \\ \Leftrightarrow\ &(\xi_2u-\xi_1)^2+(\xi_2v)^2=v \\ \Leftrightarrow\ &|(\xi_2u-\xi_1)+i\xi_2v|^2=v \\ \Leftrightarrow\ &|\xi_2(u+iv)-\xi_1|^2=v \\ \Leftrightarrow\ &|\xi_2z-\xi_1|^2=v, \quad z=u+iv, \quad (\xi_1,\xi_2)\in\mathbb{R}^2\setminus\{0\}. \end{aligned} \tag{8}$$

Therefore, the point (ξ_1, ξ_2) of the parabolic upper half plane $\mathrm{SL}_2(\mathbb{R})/N$ parametrizes the space of horocycles. Denote by $h(\xi)=h(\xi_1,\xi_2)$ the horocycle given by (8).

Every horocycle has a unique common point with the real axis, which is called the center of the horocycle. Horocycles with common center are said to be parallel. Note that a horocycle $h(\xi_1,\xi_2)$ is tangent to the real axis at the point $\frac{\xi_1}{\xi_2}$, hence every parallel horocycle is of the form $\{h(\lambda\xi_1,\lambda\xi_2) : 0<\lambda<\infty\}$ for some chosen (ξ_1,ξ_2) [3], Chap. 3, § 1.2.
Now, in order to find the invariant distance of a point z in the upper half plane to the horocycle $h(\xi)$, first we calculate the distance from a point $z_1=(u,v)\in h(\lambda\xi_1,\lambda\xi_2)$ to a horocycle $h(\xi_1,\xi_2)$. The point $z_2=(u,\lambda^{-2}v)$ is in the horocycle $h(\xi_1,\xi_2)$:

$$|\lambda\xi_2z-\lambda\xi_1|^2=v \quad \Rightarrow \quad |\xi_2z-\xi_1|^2=\lambda^{-2}v.$$

Note that the points z_1 and z_2 are on the same vertical line, thus the distance between them is

$$\begin{aligned}\left|\int_{v\lambda^{-2}}^{v} \frac{1}{y}\,dy\right| &= \left|\log v - \log v\lambda^{-2}\right| \\ &= 2\left|\log \lambda\right|.\end{aligned} \tag{9}$$

Thus, all points of the horocycle $h(\lambda\xi_1, \lambda\xi_2)$ are placed at the same distance $2|\log\lambda|$ from the parallel horocycle $h(\xi_1, \xi_2)$. Then, the signed distance from a point $z \in h(\lambda\xi)$ to a horocycle $h(\xi)$ is

$$\begin{aligned}\varrho(z;\xi) &= -2\log\lambda \\ &= \log\left(v^{-1}|\xi_2 z - \xi_1|^2\right),\end{aligned} \tag{10}$$

because $\lambda^{-2} = v^{-1}|\xi_2 z - \xi_1|^2$.

2. Induced representations of the group $\mathrm{SL}_2(\mathbb{R})$

(i) For the subgroup $K = \left\{\begin{pmatrix}\cos t & \sin t \\ -\sin t & \cos t\end{pmatrix} : t \in \mathbb{R}\right\}$, the homogeneous space $\mathrm{SL}_2(\mathbb{R})/K$ are parametrised by points of the upper half-plane $\mathbb{H}^+$. The respective maps are:

$$\begin{aligned}\mathbf{p}\begin{pmatrix}a & b \\ c & d\end{pmatrix} &= \left(\frac{bd+ac}{c^2+d^2}, \frac{1}{c^2+d^2}\right), \\ \mathbf{s}(u,v) &= \frac{1}{\sqrt{v}}\begin{pmatrix}v & u \\ 0 & 1\end{pmatrix}, \\ \mathbf{r}\begin{pmatrix}a & b \\ c & d\end{pmatrix} &= \frac{1}{\sqrt{c^2+d^2}}\begin{pmatrix}d & -c \\ c & d\end{pmatrix}.\end{aligned} \tag{11}$$

The decomposition defined by the formula $g = \mathbf{s}(\mathbf{p}(g))\mathbf{r}(g)$ takes the form:

$$\begin{pmatrix}a & b \\ c & d\end{pmatrix} = \frac{1}{c^2+d^2}\begin{pmatrix}1 & bd+ac \\ 0 & c^2+d^2\end{pmatrix}\begin{pmatrix}d & -c \\ c & d\end{pmatrix}. \tag{12}$$

The $\mathrm{SL}_2(\mathbb{R})$-action defined by the formula $g \cdot x = \mathbf{p}(g * \mathbf{s}(x))$ takes the form:

$$\begin{pmatrix}a & b \\ c & d\end{pmatrix} : (u,v) \mapsto \left(\frac{(au+b)(cu+d)+cav^2}{(cu+d)^2+(cv)^2}, \frac{v}{(cu+d)^2+(cv)^2}\right). \tag{13}$$

This map preserves the upper half plane $v > 0$. We can simplify this map as a linear-fractional transformation with the complex number unit $i^2 = -1$:

$$\begin{pmatrix}a & b \\ c & d\end{pmatrix} : w \mapsto \frac{aw+b}{cw+d}, \quad \text{where } w = u + iv. \tag{14}$$

The left invariant measure on the upper half plane $\mathbb{H}^+$ is equal to

$$d\mu(w) = \frac{dudv}{v^2}, \quad w = u + iv. \tag{15}$$

The character $\chi_k \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} = e^{-ikt}$, $k \in \mathbb{Z}$ of K, induces a linear representation ρ_k on the space of square integrable functions, which is given by the formula:

$$[\rho_k(g)f](w) = \overline{\chi_\tau(\mathbf{r}(g^{-1} * \mathbf{s}(w)))} f(g^{-1} \cdot w), \tag{16}$$

where $g \in \mathrm{SL}_2(\mathbb{R})$ and $w \in \mathrm{SL}_2(\mathbb{R})/K$.
By simple calculation we obtain [7], § 8:

$$[\rho_k(g)f](w) = \frac{|a - cw|^k}{(a - cw)^k} f\left(\frac{dw - b}{a - cw}\right), \quad \text{where}\, w = u + iv. \tag{17}$$

We consider the basis in the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$:

$$\mathcal{A} = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{18}$$

They generate one-parameter subgroup of $\mathrm{SL}_2(\mathbb{R})$:

$$e^{t\mathcal{A}} = \begin{pmatrix} e^{-\frac{t}{2}} & 0 \\ 0 & e^{\frac{t}{2}} \end{pmatrix}, \; e^{tB} = \begin{pmatrix} \cosh\frac{t}{2} & \sinh\frac{t}{2} \\ \sinh\frac{t}{2} & \cosh\frac{t}{2} \end{pmatrix}, \; e^{tZ} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

The derived representations are:

$$\begin{aligned}
d\rho_k^{\mathcal{A}} &= w\partial_w + \bar{w}\partial_{\bar{w}} && (19)\\
&= u\partial_u + v\partial_v, && (20)\\
d\rho_k^{B} &= \frac{1}{4}k(w - \bar{w}) \cdot I - \frac{1}{2}(1 - w^2)\partial_w - \frac{1}{2}(1 - \bar{w}^2)\partial_{\bar{w}} && (21)\\
&= \frac{1}{2}kvi \cdot I - \frac{1}{2}(1 - u^2 + v^2)\partial_u + uv\partial_v, && (22)\\
d\rho_k^{Z} &= -\frac{1}{2}k(w - \bar{w}) \cdot I - (1 + w^2)\partial_w - (1 + \bar{w}^2)\partial_{\bar{w}} && (23)\\
&= -\mathrm{i}kv \cdot I - (1 + u^2 - v^2)\partial_u - 2uv\partial_v, && (24)
\end{aligned}$$

where $w = u + \mathrm{i}v$, $\partial_w = \frac{1}{2}(\partial_u - \mathrm{i}\partial_v)$ and $\partial_{\bar{w}} = \frac{1}{2}(\partial_u + \mathrm{i}\partial_v)$.
The Casimir operator is:

$$\begin{aligned}
d\rho_k(C) &= d\rho_\tau^{Z^2 - 4\mathcal{A}^2 - 4B^2} \\
&= 4\mathrm{i}kv\partial_u - 4v^2(\partial_u^2 + \partial_v^2).
\end{aligned} \tag{25}$$

(ii) For the subgroup $N' = \left\{ \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} : n \in \mathbb{R} \right\}$, the homogeneous space $\mathrm{SL}_2(\mathbb{R})\,/N'$ can be identified with the upper half plane. The respective maps are:

$$\begin{aligned} \mathbf{p}\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \left(\frac{b}{d}, \frac{1}{d^2}\right), \\ \mathbf{s}(u,v) &= \frac{1}{\sqrt{v}}\begin{pmatrix} v & u \\ 0 & 1 \end{pmatrix}, \\ \mathbf{r}\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ \frac{c}{d} & 1 \end{pmatrix}. \end{aligned} \tag{26}$$

The maps $\mathbf{p}$ and $\mathbf{s}$ produce the following decomposition $g = \mathbf{s}(\mathbf{p}(g))\mathbf{r}(g)$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{d^2}\begin{pmatrix} 1 & bd \\ 0 & d^2 \end{pmatrix}\begin{pmatrix} d & 0 \\ c & d \end{pmatrix}, \quad \text{where } d \neq 0. \tag{27}$$

The action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathrm{SL}_2(\mathbb{R})\,/\acute{N}$ defined by the formula $g \cdot x = \mathbf{p}(g * \mathbf{s}(x))$ takes the form:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (u,v) \mapsto \left(\frac{au+b}{cu+d}, \frac{v}{(cu+d)^2}\right). \tag{28}$$

It preserves the upper half plane $v > 0$. We can rewrite this map as a linear-fractional transformation with the dual number unit $\varepsilon^2 = 0$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : w \mapsto \frac{aw+b}{cw+d}, \quad \text{where } w = u + \varepsilon v. \tag{29}$$

The complex character χ_τ of N' is:

$$\chi_\tau \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} = e^{-2\pi \mathrm{i} \tau n}, \quad \text{where } \tau \in \mathbb{R}.$$

This character induces a linear representation ρ_τ on the space of square integrable functions, which is given by the formula:

$$[\rho_\tau(g)f](w) = \overline{\chi_\tau(\mathbf{r}(g^{-1} * \mathbf{s}(w)))} f(g^{-1} \cdot w), \tag{30}$$

where $g \in \mathrm{SL}_2(\mathbb{R})$ and $w \in \mathrm{SL}_2(\mathbb{R})/\acute{N}$.
A direct calculation shows that [7], § 8:

$$[\rho_\tau(g)f](w) = \exp\left(-2\pi \mathrm{i} \frac{\tau c v}{a - cu}\right) f\left(\frac{dw - b}{a - cw}\right), \tag{31}$$

where $w = u + \varepsilon v$ and $f \in L_2(\mathbb{H}^+, d\mu)$.
This representation is unitary on the space of functions on the upper half plane of

dual numbers.

The derived representations of the elements $\mathcal{A}$, B and Z (18) of $\mathfrak{sl}_2(\mathbb{R})$ are:

$$d\rho_\tau^{\mathcal{A}} = w\partial_w + \bar{w}\partial_{\bar{w}} \tag{32}$$

$$= u\partial_u + v\partial_v, \tag{33}$$

$$d\rho_\tau^{B} = -\pi \mathrm{i} v\tau \cdot I - \frac{1}{2}(1-w^2)\partial_w - \frac{1}{2}(1-\bar{w}^2)\partial_{\bar{w}} \tag{34}$$

$$= -\pi \mathrm{i} v\tau \cdot I - \frac{1}{2}(1-u^2)\partial_u + uv\partial_v, \tag{35}$$

$$d\rho_\tau^{Z} = 2\pi \mathrm{i} v\tau \cdot I - (1+w^2)\partial_w - (1+\bar{w}^2)\partial_{\bar{w}} \tag{36}$$

$$= 2\pi \mathrm{i} v\tau \cdot I - (1+u^2)\partial_u - 2uv\partial_v, \tag{37}$$

where $w = u + \varepsilon v$, $\partial_w = \frac{1}{2}(\partial_u + \frac{1}{\varepsilon}\partial_v)$ and $\partial_{\bar{w}} = \frac{1}{2}(\partial_u - \frac{1}{\varepsilon}\partial_v)$.

The Casimir operator is:

$$\begin{aligned} d\rho_\tau(C) &= d\rho_\tau^{Z^2-4\mathcal{A}^2-4B^2} \\ &= -8\pi \mathrm{i} v\tau\partial_u - 4v^2\partial_v^2. \end{aligned} \tag{38}$$

In the following we will find some eigenfunctions, and the special role of them will become obvious later.

2.1. Joint eigenvector of $d\rho_k(C)$ with $d\rho_k^N$

First, we calculate the eigenvector of the derived representation $d\rho_k^N$:

$$\left[d\rho_k^N f\right](w,\bar{w}) = -\partial_u f(w,\bar{w}) = -(\partial_w + \partial_{\bar{w}})f(w,\bar{w}) = 0. \tag{39}$$

The solution is $f(w,\bar{w}) = \phi(v)$, where ϕ is an arbitrary function.

Then, we solve the differential equation

$$d\rho_k(C)\phi(v) = (1+s^2)\phi(v), \quad s \in \mathbb{R},$$

where $d\rho_k(C)$ is the Casimir operator(25), and $k = 0$ for simplicity.

This equation becomes

$$-4v^2\frac{d^2\phi}{dv^2}(v) - (1+s^2)\phi(v) = 0. \tag{40}$$

It is a Cauchy-Euler equation, therefore the solution takes the form $\phi(v) = v^m$. Differentiating gives $\frac{d^2\phi}{dv^2}(v) = m(m-1)v^{m-2}$, and substituting into (40) leeds to

$$\begin{aligned} -4m(m-1)v^m - (1+s^2)v^m &= 0 \\ \Rightarrow -4m(m-1) - (1+s^2) &= 0 \\ \Rightarrow m &= \frac{1 \pm is}{2}. \end{aligned}$$

Hence, the set of fundamental solution is

$$\left\{ v^{\frac{1+is}{2}}, v^{\frac{1-is}{2}} \right\}. \tag{41}$$

2.2. Joint eigenvector of $d\rho_\tau(C)$ with $d\rho_\tau^Z$

To begin, we look for an eigenvector of the derived representation $d\rho_\tau^Z$ (37) with $\tau = 0$. To do that, we solve the equation $d\rho_\tau^Z f(w, \bar{w}) = 0$ using the method of characteristics:

$$\frac{du}{1+u^2} = \frac{dv}{2uv} = \frac{df}{2\pi \mathrm{i}\tau v f}.$$

$$\frac{du}{1+u^2} = \frac{dv}{2uv} \qquad \Rightarrow \qquad \frac{2udu}{1+u^2} = \frac{dv}{v} \qquad \Rightarrow \qquad 2C_1 = \frac{v}{1+u^2}.$$

We need to obtain another integral curve which involves f. Since $\tau = 0$, then

$$\frac{dv}{2uv} = \frac{df}{2\pi \mathrm{i}\tau v f} \qquad \Rightarrow \qquad \frac{df}{f} = 0 \qquad \Rightarrow \qquad C_2 = f.$$

Hence, the general solution is of the form $C_2 = \psi(C_1)$, that is

$$f(w, \bar{w}) = \psi\left(\frac{v}{2(1+u^2)}\right), \quad w = u + \varepsilon v, \tag{42}$$

where ψ is an arbitrary function. To specify this function, we solve the equation

$$d\rho_\tau(C)\psi\left(\frac{v}{2(1+u^2)}\right) = (1+s^2)\psi\left(\frac{v}{2(1+u^2)}\right), \tag{43}$$

where $d\rho_\tau(C)$ is the Casimir operator(38). This equation turns into

$$-4\frac{v^2}{(1+u^2)^2}\frac{d^2\psi}{dv^2}\left(\frac{v}{2(1+u^2)}\right) - (1+s^2)\psi\left(\frac{v}{2(1+u^2)}\right) = 0. \tag{44}$$

Using the substitution $t = \frac{v}{2(1+u^2)}$, then we obtain

$$-4t^2\frac{d^2\psi}{dt^2}(t) - (1+s^2)\psi(t) = 0. \tag{45}$$

It is a Cauchy-Euler equation, so let $\psi(t) = t^m$ and substitute in the differential equation(45), then $m = \frac{1+is}{2}$ and $m = \frac{1-is}{2}$ are two distinct possible values of m. Therefore, the set of fundamental solution is $\left\{t^{\frac{1+is}{2}}, t^{\frac{1-is}{2}}\right\}$.

Finally, the set of fundamental solution for the equation(44) is

$$\left\{\left(\frac{v}{2(1+u^2)}\right)^{\frac{1+is}{2}}, \left(\frac{v}{2(1+u^2)}\right)^{\frac{1-is}{2}}\right\}. \tag{46}$$

3. Induced Covariant transform

Definition 3. *[8], § 5.1 Let H be a closed subgroup of G and $f \in \mathcal{H}$ such that*

$$\rho(h)f = \chi(h)f \tag{47}$$

for some character χ of H where $h \in H$ and ρ is a unitary representation of a Lie group G in a Hilbert space $\mathcal{H}$. For a section $\mathbf{s}$ from G/H to G, the induced covariant transform $\mathcal{W}_f^\rho$ is a map from the Hilbert space $\mathcal{H}$ to a space of function on G/H given as follows:

$$\mathcal{W}_f : v \mapsto \tilde{v}(x) = \langle v, \rho(\mathbf{s}(x))f\rangle, \qquad x \in G/H. \tag{48}$$

The map $v \mapsto \tilde{v}(x) = \tilde{v}(\mathbf{s}(x))$ intertwines ρ with the representation ρ_χ in a certain function space on G/H induced by the character χ of H. That is,

$$\rho_\chi \circ \mathcal{W}_f^\rho = \mathcal{W}_f^\rho \circ \rho. \tag{49}$$

Example 1. *We will find the induced wavelet transform with N-eigenvector that intertwines respectively the representation ρ_k (17) where $k = 0$ with the representation ρ_τ (31). We take the fiducial vector $\varphi_0(w, \bar{w}) = v^{\frac{1+is}{2}}$ (41) which would be the eigenvector for the representation $\rho_k \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. That is*

$$\rho_k \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \varphi_0 = \chi_\tau \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \varphi_0. \tag{50}$$

Then, the corresponding induced covariant transform is:

$$\begin{aligned} \left[\mathcal{W}_{\varphi_0}^{\rho_k} f\right](\xi) &= \langle f, \rho_k(\mathbf{s}(\xi_1, \xi_2))\varphi_0\rangle \\ &= \left\langle f, \rho_k \begin{pmatrix} \xi_1 & 0 \\ \xi_2 & \frac{1}{\xi_1} \end{pmatrix} v^{\frac{1+is}{2}} \right\rangle \\ &= \int_{\mathbb{H}^+} f(w) \left(\frac{v}{|\xi_1 - \xi_2 w|^2} \right)^{\frac{1-is}{2}} \frac{dudv}{v^2} \\ &= \int_{\mathbb{H}^+} f(w) \exp\left\{ -\frac{1-is}{2} \varrho(w; \xi) \right\} \frac{dudv}{v^2}, \quad w = u + iv, \end{aligned} \tag{51}$$

where $\varrho(w; \xi)$ (10) is the signed distance from the point w to the horocycle $h(\xi)$, $\xi = (\xi_1, \xi_2)$.

4. Contravariant transform

Definition 4. *[6], § 5 Let ρ be a unitary square integrable representation of the group $\mathrm{SL}_2(\mathbb{R})$ on a Hilbert space $\mathcal{H}$ and H be a closed subgroup of $\mathrm{SL}_2(\mathbb{R})$. Let $X = \mathrm{SL}_2(\mathbb{R})/H$ be a homogeneous space with an invariant measure dx. We define the function $w_{\mathbf{s}(x)} =$*

$\rho(\mathbf{s}(x))w_0$, where $w_0 \in \mathcal{H}$ and $\mathbf{s}$ is a section map. The contravariant transform $\mathcal{M}^{\rho}_{w_0}$ is a map $L_2(X) \to \mathcal{H}$ defined by

$$\mathcal{M}^{\rho}_{w_0} f = \int_X f(x) w_{\mathbf{s}(x)}\, dx, \qquad x \in X. \tag{52}$$

For an admissible vector w_0 [1], Definition 8.1.1, the contravariant transform in this setup is known as a reconstruction formula.

Example 2. *For the representation ρ_τ (31) with $\tau = 0$, we take the K-eigenvector $\phi_0(w, \bar{w}) = \left(\frac{v}{2+2u^2}\right)^{\frac{1+is}{2}}$ (46). Then, the corresponding contravariant transform is:*

$$\begin{aligned} \left[\mathcal{M}^{\rho_\tau}_{\phi_0} f\right](w) &= \int_{h(\xi)} f(\xi)\rho_\tau(\mathbf{s}(\xi_1, \xi_2))\phi_0\, d\xi \\ &= \int_{\mathbb{R}^2} f(\xi)\rho_\tau\left(\begin{pmatrix} \xi_1 & 0 \\ \xi_2 & \frac{1}{\xi_1} \end{pmatrix}\right)\left(\frac{v}{2(1+u^2)}\right)^{\frac{1+is}{2}} d\xi_1 d\xi_2 \\ &= \int_{\mathbb{R}^2} f(\xi)\left(\frac{v\xi_1^2}{2\xi_1^2(\xi_1 - \xi_2 u)^2 + 2u^2}\right)^{\frac{1+is}{2}} d\xi_1 d\xi_2, \quad w = u + \varepsilon v. \end{aligned} \tag{53}$$

Proposition 1. *[6], Prop. 6.4 Contravariant transform $\mathcal{M}_{w_0}$ intertwines left regular representation Λ on $L_2(\mathrm{SL}_2(\mathbb{R}))$ and ρ:*

$$\mathcal{M}_{w_0}\Lambda(g) = \rho(g)\mathcal{M}_{w_0}. \tag{54}$$

Let ρ be an irreducible square integrable representation and φ_0 and w_0 be admissible vectors. The covariant transform intertwines ρ and the left regular representation Λ:

$$\mathcal{W}_{\varphi_0}\rho(g) = \Lambda(g)\mathcal{W}_{\varphi_0}.$$

Combining with (54), we see that the composition $\mathcal{M}_{w_0} \circ \mathcal{W}_F$ intertwines ρ with itself. That is,

$$(\mathcal{M}_{w_0} \circ \mathcal{W}_{\varphi_0}) \circ \rho(g) = \rho(g) \circ (\mathcal{M}_{w_0} \circ \mathcal{W}_{\varphi_0}). \tag{55}$$

Thus, from the Schur's lemma we have the relation

$$\mathcal{M}_{w_0} \circ \mathcal{W}_{\varphi_0} = kI, \tag{56}$$

for some constant $k \in \mathbb{C}$.

On the other hand, and from the orthogonality relations [1], § 8.2:

$$\langle \mathcal{W}_{\varphi_1} f_1, \mathcal{W}_{\varphi_2} f_2\rangle = \langle f_1, f_2\rangle\langle C\varphi_2, C\varphi_1\rangle, \tag{57}$$

where C is a unique positive, self adjoint and invertible operator in the Hilbert space. This operator is known as Duflo-Moore operator.

If $f_1, f_2 \in \mathcal{H}$, we have

$$\begin{aligned} \langle \mathcal{M}_{w_0} \circ \mathcal{W}_{\varphi_0} f_1, f_2\rangle &= \langle \mathcal{W}_{\varphi_0} f_1, \mathcal{W}_{w_0} f_2\rangle \\ &= \langle f_1, f_2\rangle\langle Cw_0, C\varphi_0\rangle \\ &= \langle\langle C\varphi_0, Cw_0\rangle f_1, f_2\rangle. \end{aligned} \tag{58}$$

Thus

$$\mathcal{M}_{w_0} \circ \mathcal{W}_{\varphi_0} = \langle C\varphi_0, Cw_0 \rangle I. \tag{59}$$

And for non-orthogonal vectors w_0 and φ_0, we get $\langle C\varphi_0, Cw_0 \rangle = k \neq 0$.

4.1. Inversion formula

We will find the inversion formula for the covariant transform (51) from the relation (59) with the contravariant transform $\mathcal{M}^{\rho_\tau}_{\phi_0}$ (53):

$$\begin{aligned} f(w) &= \frac{1}{\langle C\varphi_0, C\phi_0 \rangle} \left[\mathcal{M}^{\rho_\tau}_{\phi_0} \left(\mathcal{W}^{\rho_k}_{\varphi_0} f\right)\right](w) \\ &= \frac{1}{\langle C\varphi_0, C\phi_0 \rangle} \int_{\mathbb{R}^2} \mathcal{W}^{\rho_k}_{\varphi_0} f(\xi) \left(\frac{v\xi_1^2}{2\xi_1^2(\xi_1 - \xi_2 u)^2 + 2u^2}\right)^{\frac{1+is}{2}} d\xi_1 d\xi_2. \end{aligned} \tag{60}$$

The function $h(s) = \langle C\varphi_0, C\phi_0 \rangle$ must be explicitly identified.
The following result is an inversion formula similar to that in Gelfand's book [3], Chap. 3, Theorem 3.2, but with a difference in the eigenvector and with a different method.

Theorem 2. *For $f \in L_2(\mathbb{R}^2_+, d\mu)$, we have the inversion formula*

$$f(w) = \frac{1}{2\pi^2} \int_{\mathbb{R}} s \tanh \frac{\pi s}{2} \left(\int_{\mathbb{R}^2} \mathcal{W}^{\rho_k}_{\varphi_0} f(\xi) \rho_\tau(\mathbf{s}(\xi_1, \xi_2)) \phi_0(w) \, d\xi_1 d\xi_2 \right) ds, \tag{61}$$

where

$$\rho_\tau(\mathbf{s}(\xi_1, \xi_2))\phi_0(w) = \left(\frac{v\xi_1^2}{2\xi_1^2(\xi_1 - \xi_2 u)^2 + 2u^2}\right)^{\frac{1+is}{2}},$$

and $\mathcal{W}^{\rho_k}_{\varphi_0} f$ is the covariant transform (51).

Proof. To find the inversion formula for the covariant transform (51), we need to identify $\langle C\varphi_0, Cw_0 \rangle$ in (60):

$$\begin{aligned} \langle C\varphi_0, Cw_0 \rangle &= \frac{1}{f(w)} \left[\mathcal{M}^{\rho_\tau}_{\phi_0} \left(\mathcal{W}^{\rho_k}_{\varphi_0} f\right)\right](w) \\ &= \frac{1}{f(w)} \int_{\mathbb{R}^2} \mathcal{W}^{\rho_k}_{\varphi_0} f(\xi) \left(\frac{v\xi_1^2}{2\xi_1^2(\xi_1 - \xi_2 u)^2 + 2u^2}\right)^{\frac{1+is}{2}} d\xi_1 d\xi_2. \end{aligned} \tag{62}$$

To identify this function, it is enough to compute the composition of the covariant transform and the contravariant transform for one particular function. Let

$$f_0(w) = \frac{v^{\mathrm{i}s+\frac{1}{2}}(1+v)^{-\frac{\mathrm{i}s+1}{2}}}{1+u^2} \in L_2(\mathbb{R}^2_+, d\mu(w)), \quad 0 < \Re(\mathrm{i}s) < 1. \tag{63}$$

We compute the covariant transform for the function f_0:

$$\mathcal{W}^{\rho_k}_{\varphi_0} f_0(\xi) = \int_0^{+\infty} \int_{-\infty}^{+\infty} \frac{v^{\mathrm{i}s+\frac{1}{2}}(1+v)^{-\frac{\mathrm{i}s+1}{2}}}{1+u^2} v^{\frac{1-\mathrm{i}s}{2}} \left(\frac{1}{|\xi_1 - \xi_2 w|^2}\right)^{\frac{1-\mathrm{i}s}{2}} \frac{du dv}{v^2}. \tag{64}$$

And for $\xi = (\xi_1, 0)$, this value becomes

$$\begin{aligned} \mathcal{W}^{\rho_k}_{\varphi_0} f_0(\xi) &= \int_0^{+\infty} \int_{-\infty}^{+\infty} \frac{v^{\mathrm{i}s+\frac{1}{2}}(1+v)^{-\frac{\mathrm{i}s+1}{2}}}{1+u^2} v^{\frac{1-\mathrm{i}s}{2}-2} \left(\xi_1^2\right)^{\frac{\mathrm{i}s-1}{2}} \, du dv \\ &= \left(\xi_1^2\right)^{\frac{is-1}{2}} \int_0^{+\infty} v^{\frac{is}{2}-1}(1+v)^{-\frac{is+1}{2}} \left(\int_{-\infty}^{+\infty} \frac{1}{1+u^2} \, du \right) dv \\ &= \left(\xi_1^2\right)^{\frac{is-1}{2}} \int_0^{+\infty} v^{\frac{is}{2}-1}(1+v)^{-\frac{is+1}{2}} \pi \, dv \\ &= \pi e^{\frac{\mathrm{i}s-1}{2}\varrho(\mathrm{i};\xi)} B(\frac{\mathrm{i}s}{2}, \frac{1}{2}), \end{aligned} \tag{65}$$

where $\varrho(\mathrm{i};\xi)$ is the distance from the point i to the horocycle $h(\xi)$ and B is the Beta function.
Then, we find the function (62) with $(\xi_1, \xi_2) = (\xi_1, 1)$ and $f(w) = f_0(\mathrm{i})$:

$$\begin{aligned} \langle C\varphi_0, Cw_0 \rangle &= \frac{1}{f_0(\mathrm{i})} \int_{\mathbb{R}} \mathcal{W}^{\rho_k}_{\varphi_0} f_0((\xi_1, 1)) \left(\frac{\xi_1^2}{2\xi_1^2(\xi_1)^2} \right)^{\frac{1+is}{2}} d\xi_1 \\ &= 2^{\frac{\mathrm{i}s+1}{2}} \pi B(\frac{\mathrm{i}s}{2}, \frac{1}{2}) \int_{\mathbb{R}} e^{\frac{\mathrm{i}s-1}{2}\varrho(\mathrm{i};(\xi_1,1))} \left(2\xi_1^2\right)^{-\frac{1+is}{2}} d\xi_1 \\ &= \pi B(\frac{\mathrm{i}s}{2}, \frac{1}{2}) \int_{\mathbb{R}} (\xi_1^2+1)^{\frac{is-1}{2}} \left(\xi_1^2\right)^{-\frac{1+is}{2}} d\xi_1. \end{aligned} \tag{66}$$

Put $u = (\xi_1^2 + 1)^{-1}$, then (66) becomes

$$\begin{aligned} \langle C\varphi_0, Cw_0 \rangle &= \pi B(\frac{\mathrm{i}s}{2}, \frac{1}{2}) \int_0^1 u^{\frac{1}{2}-1}(1-u)^{-\frac{\mathrm{i}s}{2}-1} \, du \\ &= \pi B(\frac{\mathrm{i}s}{2}, \frac{1}{2}) B(-\frac{\mathrm{i}s}{2}, \frac{1}{2}) \\ &= \pi \frac{\Gamma(\frac{\mathrm{i}s}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{\mathrm{i}s+1}{2})} \frac{\Gamma(\frac{-\mathrm{i}s}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{-\mathrm{i}s+1}{2})} \\ &= \pi^2 \left| \Gamma\left(\frac{\mathrm{i}s}{2}\right) \right|^2 \left| \Gamma\left(\frac{\mathrm{i}s+1}{2}\right) \right|^{-2} \\ &= \pi^2 \frac{2}{s} \coth \frac{\pi s}{2}. \end{aligned} \tag{67}$$

Substituting this value in (60), we obtain

$$f(w) = \frac{1}{2\pi^2} s \tanh \frac{\pi s}{2} \int_{\mathbb{R}^2} \mathcal{W}^{\rho_k}_{\varphi_0} f(\xi) \left(\frac{v\xi_1^2}{2\xi_1^2(\xi_1 - \xi_2 u)^2 + 2u^2} \right)^{\frac{1+is}{2}} d\xi_1 d\xi_2. \tag{68}$$

And for $s \in \mathbb{R}$, we get the inversion formula

$$f(w) = \frac{1}{2\pi^2}\int_{\mathbb{R}} s\tanh\frac{\pi s}{2}\left(\int_{\mathbb{R}^2}\mathcal{W}^{\rho_k}_{\varphi_0}f(\xi)\rho_\tau(\mathbf{s}(\xi_1,\xi_2))\phi_0(w)\,d\xi_1 d\xi_2\right)ds, \quad (69)$$

where

$$\rho_\tau(\mathbf{s}(\xi_1,\xi_2))\phi_0(w) = \left(\frac{v\xi_1^2}{2\xi_1^2(\xi_1-\xi_2 u)^2+2u^2}\right)^{\frac{1+is}{2}}. \quad (70)$$

The inversion formula is equivalent to the decomposition of the unitary representation ρ_k, $k=0$ (17) into irreducible components. We will describe the irreducible invariant subspaces H_s. Consider the eigenspace

$$\left\{f\in L_2(\mathbb{H}^+): \quad d\rho_k(C)f = (1+s^2)f\right\}. \quad (71)$$

This space is spanned by the functions (70). Thus, the elements of this eigenspace can be presented as a continuous linear combination over a set of such functions, that is

$$f_s(w) = \int_{\mathbb{R}^2}\mathcal{W}^{\rho_k}_{\varphi_0}f(\xi)\left(\frac{v\xi_1^2}{2\xi_1^2(\xi_1-\xi_2 u)^2+2u^2}\right)^{\frac{1+is}{2}} d\xi_1 d\xi_2, \quad (72)$$

where f_s belongs to the space $H_s\subset L_2(\mathbb{H}^+)$.
Introduce the projection operator $P_s: L_2(\mathbb{H}^+)\to H_s$ by

$$P_s f = f_s.$$

Thus, the problem of decomposing the space $L_2(\mathbb{H}^+)$ into irreducible subspaces consists in expanding the functions $f\in L_2(\mathbb{H}^+)$ in their projections f_s.
The solution of this problem is given by (69), since this formula can be written as follows:

$$f(w) = \frac{1}{2\pi^2}\int_{\mathbb{R}} s\tanh\frac{\pi s}{2} f_s ds, \qquad f_s = P_s f. \quad (73)$$

5. Conclusion

In this paper, we obtain the covariant and contravariant transforms using the representation itself like in Gelfand's approach [5], but the eigenvectors are selected by the derived representation as in Bargmann's works [2]. we use the relation between these transforms to find the inversion formula. Thus, the original contribution is using the covariant transform to find the inversion formula with eigenvectors selected by the derived representation. This new method will be easier to adopt for problems of decomposing a system into elementary bits in theoretical physics. Also, it is not restricted to $\mathrm{SL}_2(\mathbb{R})$, it can be successfully used for many other cases.

Acknowledgements

I would like to express my deepest gratitude to Dr. Vladimir Kisil, for his guidance, feedback and support.

References

[1] Syed Twareque Ali, Jean-Pierre Antoine, Jean-Pierre Gazeau, et al. *Coherent states, wavelets and their generalizations*, volume 3. Springer, 2000.

[2] Valentine Bargmann. Irreducible unitary representations of the lorentz group. *Annals of Mathematics*, pages 568–640, 1947.

[3] IM Gelfand, SG Gindikin, and MI Graev. Selected topics in integral geometry, translations of math. *Monographs, AMS, Providence, RI*, 2003.

[4] Izrail Moiseevich Gelfand, Mark I Graev, and Naum Iakovlevich Vilenkin. *Generalized Functions-Volume 5. Integral Geometry and Representation Theory.* Academic Press, 1966.

[5] Izrail Moiseevich Gel'fand and Mark Aronovich Naimark. Unitary representations of the lorentz group. *Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya*, 11(5):411–504, 1947.

[6] Vladimir V Kisil. The real and complex techniques in harmonic analysis from the point of view of covariant transform. *arXiv preprint arXiv:1209.5072*, 2012.

[7] Vladimir V. Kisil. Induced representations and hypercomplex numbers. *Advances in Applied Clifford Algebras*, 23(2):417–440, 2013.

[8] Vladimir V. Kisil. Symmetry, geometry, and quantization with hypercomplex numbers. *arXiv preprint arXiv:1611.05650*, 2016.

[9] V.V. Kisil. *Geometry Of Mobius Transformations: Elliptic, Parabolic And Hyperbolic Actions Of SL2(R) (With Dvd-rom).* World Scientific Publishing Company, 2012.