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Polynomial Representations and Degree Sequences of Graphs Resulting From Some Graph Operations

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Abstract. Let $G = (V(G), E(G))$ be a graph with degree sequence $\langle d_1, d_2, \cdots, d_n\rangle$, where $d_1 \geq d_2 \geq \cdots \geq d_n$. The polynomial representation of G is given by $f_G(x) = \sum_{i=1}^{n} x^{d_i} = \sum_{k=1}^{\Delta(G)} a_k x^k$, where a_k is the number of vertices of G having degree k for each $i = 1, 2, \cdots n = \Delta(G)$. In this paper, we give the polynomial representation of the complement and line graph of a graph, the shadow graph, complementary prism, edge corona, strong product, symmetric product, and disjunction of two graphs.

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Key Words and Phrases: polynomial representation, line graph, edge corona, shadow graph, complementary prism, strong product, symmetric product, disjunction

1. Introduction

Let $G = (V(G), E(G))$ be a graph on n vertices and let $\Delta(G)$ be the maximum degree of G. If $s_1, s_2, \cdots, s_n$ are the degrees of the vertices of G, where $s_1 \geq s_2 \geq \cdots \geq s_n$, then the sequence $\langle s_1, s_2, \cdots, s_n\rangle$ is called the *degree sequence* of G. Here, $s_1 = \Delta(G)$. The polynomial $f_G(x) = \sum_{i=1}^{n} x^{s_i}$ is called the polynomial representation of G. A degree sequence $\langle s_1, s_2, \ldots, s_n\rangle$ of nonnegative integers is said to be *graphic* if a simple graph G with degree sequence $\langle s_1, s_2, \ldots, s_n\rangle$ can be found (see [1]). Using the polynomial representation of a graph, we can alternatively define a polynomial $P(x)$ to be graphic if there exists a graph G such that $P(x) = f_G(x)$. It is easy to verify not every polynomial is graphic. The degree sequence of a graph had been investigated in [2], [3], [4], [5], [6],

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[9], and [12]). Erdős and Gallai in [4] obtained a necessary and sufficient condition for a given polynomial to be graphic. The polynomial representations and degree sequences of the the join, corona, lexicographic product, Cartesian product, and Tensor product of two graphs had been obtained by Canoy et al. in [8]. These graphs were also investigated for other graph parameters in previous studies (see [7], [10], and [11]).

In this present study, the authors endeavored to determine expressions for the polynomial representations of the complement and line graph of a graph, shadow graph, complementary prism, edge corona, strong product, symmetric difference, and disjunction of two graphs.

2. Terminologies and Notations

Let $G = (V(G), E(G))$ be a simple undirected graph. The distance between two vertices u and v of G, denoted by $d_G(u, v)$, is equal to the length of a shortest path connecting u and v. Any path connecting u and v of length $d_G(u, v)$ is called a *u-v geodesic*. The *open neighborhood* of a vertex v of G is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and its *closed neighborhood* is the set $N_G[v] = N_G(v) \cup \{v\}$. The *open neighborhood* of a subset S of $V(G)$ is the set $N_G(S) = \cup_{v\in S} N_G(v)$ and its *closed neighborhood* is the set $N_G[S] = N_G(S) \cup S$. The *degree* of v, denoted by $deg_G(v)$, is equal to $|N_G(v)|$. The maximum degree of G, denoted by $\Delta(G)$, is equal to $\max\{deg_G(v) : v \in V(G)\}$. Suppose $\Delta(G) = n$. For each $i = 1, 2, \cdots n$, let a_i be the number of vertices of G with degree $i \geq 0$. Then the polynomial $f_G(x) = \displaystyle\sum_{i=1}^{n} a_i x^i$ is called the *polynomial representation* of G. Equivalently, $f_G(x) = \displaystyle\sum_{v\in V(G)} x^{|N_G(v)|}$.

Let G and H be graphs. The *complement* of G, denoted by $\overline{G}$ is the graph with $V(\overline{G}) = V(G)$ and $vw \in E(\overline{G})$ if and only if $vw \notin E(G)$. The *line graph* $L(G)$ of G is the graph with $V(L(G)) = E(G)$ and $e_1e_2 \in E((L(G))$ if and only if e_1 and e_2 have a common vertex in G. The *shadow graph* $D_2(G)$ of G is the graph obtained by taking two copies of G, say G_1 and G_2, and joining each vertex $u \in V(G_1)$ to the neighbors of the corresponding vertex $u' \in V(G_2)$. The *complementary prism* $G\overline{G}$ is the graph formed from the disjoint union of G and its complement $\overline{G}$ by adding a perfect matching between corresponding vertices of G and $\overline{G}$. For each $v \in V(G)$, let $\overline{v}$ denote the vertex in $\overline{G}$ corresponding to v. In simple terms, the graph $G\overline{G}$ is formed from $G \cup \overline{G}$ by adding the edge $v\overline{v}$ for every vertex $v \in V(G)$. The *edge corona* $G \diamond H$ of graphs G and H is the graph obtained by taking one copy of G and $|E(G)|$ copies of H and joining each of the end vertices u and v of every edge uv in G to every vertex of the copy H^{uv} of H (that is forming the join $\langle\{u, v\}\rangle + H^{uv}$ for each $uv \in E(G)$). The *strong product* $G \boxtimes H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and (u, v) is adjacent with (u', v') whenever $[uu' \in E(G)$ and $v = v']$ or $[vv' \in E(H)$ and $u = u']$ or $[uu' \in E(G)$ and $vv' \in E(H)]$. The *symmetric difference* $G \oplus H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and (u, v) is adjacent with (u', v') whenever $[uu' \in E(G)]$ or $[vv' \in E(H)]$ but not both.

The *disjunction* $G \vee H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and (u, v) is adjacent with (u', v') whenever $uu' \in E(G)$ or $vv' \in E(H)$.

3. Results

The first result gives the polynomial representation of the complement of a graph.

Theorem 1. *Let G be a non-trivial graph of order n. Then $f_{\overline{G}}(x) = x^{n-1} f_G(\frac{1}{x})$.*

Proof. Let $v \in V(\overline{G})$. Then $N_{\overline{G}}(v) = \{w \in V(\overline{G}) : w \in V(G) \setminus N_G[v]\}$. This implies that $|N_{\overline{G}}(v)| = n - |N_G(v)| - 1$. It follows that

$$\begin{aligned} f_{\overline{G}}(x) &= \sum_{v \in V(\overline{G})} x^{|N_{\overline{G}}(v)|} \\ &= \sum_{v \in V(G)} x^{n-|N_G(v)|-1} \\ &= x^{n-1} \sum_{v \in V(G)} x^{-|N_G(v)|} \\ &= x^{n-1} f_G(\frac{1}{x}). \quad \square \end{aligned}$$

Theorem 2. *Let G be a non-trivial graph of order n. If the degree sequence of G is $\langle d_1, d_2, \cdots d_n \rangle$, then the degree sequence of $\overline{G}$ is*

$$\langle n - d_n - 1, n - d_{n-1} - 1, \cdots, n - d_2 - 1, n - d_1 - 1 \rangle.$$

Proof. By Theorem 1,

$$\begin{aligned} f_{\overline{G}}(x) &= \sum_{j=1}^{n} x^{|N_{\overline{G}}(v)|} \\ &= \sum_{j=1}^{n} x^{n-d_j-1}. \end{aligned}$$

Hence, the degree sequence of $\overline{G}$ is $\langle n - d_n - 1, n - d_{n-1} - 1, \cdots, n - d_1 - 1 \rangle$. This proves the assertion. $\square$

Next, we give the polynomial representation of the line graph of a graph.

Theorem 3. *Let G be a non-trivial connected graph. Then*

$$f_{L(G)}(x) = \frac{1}{x^2} \sum_{uv \in E(G)} x^{|N_G(u)|+|N_G(v)|}.$$

Proof. Let $e = uv \in V(L(G))$. Then $|N_{L(G)}(e)| = |N_G(u)| + |N_G(v)| - 2$. It follows that

$$\begin{aligned} f_{L(G)}(x) &= \sum_{e \in V(L(G))} x^{|N_{L(G)}(e)|} \\ &= \sum_{uv \in V(L(G))} x^{|N_G(u)|+|N_G(v)|-2} \\ &= \frac{1}{x^2} \sum_{uv \in E(G)} x^{|N_G(u)|+|N_G(v)|}. \quad \square \end{aligned}$$

Corollary 1. *Let G be a non-trivial r-regular connected graph of size p. Then*

$$f_{L(G)}(x) = px^{2r-2}.$$

Proof. Since G is r-regular, $|N_G(v)| = r$ for all $v \in V(G)$. By Theorem 3, we have

$$\begin{aligned} f_{L(G)}(x) &= \frac{1}{x^2} \sum_{uv \in E(G)} x^{|N_G(u)|+|N_G(v)|} \\ &= \frac{1}{x^2} \sum_{uv \in E(G)} x^{2r} \\ &= px^{2r-2}. \quad \square \end{aligned}$$

Theorem 4. *Let G be a connected graph of size p and let H any graph of order n. Then*

$$f_{G \diamond H}(x) = f_G(x^{n+1}) + px^2 f_H(x).$$

Proof. Let $v \in V(G \diamond H)$. If $v \in V(G)$, then $N_{G\diamond H}(v) = N_G(v) \cup [\cup_{u \in N_G(v)} V(H^{uv})]$. If $v \in V(H^e)$ for $e = uw \in E(G)$, then $N_{G\diamond H}(v) = N_{H^e}(v) \cup \{u, w\}$. Thus,

$$\begin{aligned} f_{G\diamond H}(x) &= \sum_{v \in V(G\diamond H)} x^{|N_{G\diamond H}(v)|} \\ &= \sum_{v \in V(G)} x^{|N_{G\diamond H}(v)|} + \sum_{v \in V(G\diamond H)\setminus V(G)} x^{|N_{G\diamond H}(v)|} \\ &= \sum_{v \in V(G)} x^{|N_G(v)|+n|N_G(v)|} + \sum_{e \in E(G)} \sum_{v \in V(H^e)} x^{|N_{H^e}(v)|+2} \\ &= \sum_{v \in V(G)} x^{(n+1)|N_G(v)|} + px^2 \sum_{v \in V(H)} x^{|N_H(v)|} \\ &= f_G(x^{n+1}) + px^2 f_H(x). \quad \square \end{aligned}$$

The following result is immediate from the above result:

Corollary 2. *Let G be a connected graph of size p and let H be an r-regular graph of order n. Then*

$$f_{G\diamond H}(x) = f_G(x^{n+1}) + pnx^{r+2}.$$

In particular, $f_{G\diamond K_n}(x) = f_G(x^{n+1}) + pnx^{n+1}$.

Corollary 3. *Let m and n be positive integers. Then*

(i) $f_{P_m\diamond P_n}(x) = (m-2)x^{2n+2} + 2x^{n+1} + (m-1)[(n-2)x^4 + 2x^3]$ *for* $m, n \geq 2$;

(ii) $f_{P_m\diamond C_n}(x) = (m-2)x^{2n+2} + 2x^{n+1} + (m-1)nx^4$ *for* $m \geq 2$ *and* $n \geq 3$; *and*

(iii) $f_{C_m\diamond C_n}(x) = mx^{2n+2} + mnx^4$ *for* $m, n \geq 3$.

Proof. Clearly, $|E(P_r)| = r-1$, $|E(C_s)| = s$, $f_{P_r}(x) = 2x + (r-2)x^2$ and $f_{C_s}(x) = sx^2$ for positive integers $r \geq 2$ and $s \geq 3$. By Theorem 4, we find that (i), (ii), and (iii) hold. □

Theorem 5. *Let G be a connected graph of size p and let H be any graph with degree sequences $\langle d_1, d_2, \cdots d_m\rangle$ and $\langle r_1, r_2, \cdots r_n\rangle$, respectively. Then the terms of the degree sequence of $G\diamond H$ are the elements of the set $\{(n+1)d_i : 1 \leq i \leq m\} \cup \{r_j + 2 : 1 \leq j \leq n\}$, where p consecutive terms of the degree sequence are $r_j + 2$ for each j with $1 \leq i \leq n$.*

Proof. The polynomial representations of G and H are, respectively, $f_G(x) = \sum_{i=1}^{m} x^{d_i}$ and $f_H(x) = \sum_{j=1}^{n} x^{r_j}$. By Theorem 4,

$$\begin{aligned} f_{G\diamond H}(x) &= \sum_{i=1}^{m} x^{(n+1)d_i} + px^2 \sum_{j=1}^{n} x^{r_j} \\ &= \sum_{i=1}^{m} x^{(n+1)d_i} + p \sum_{j=1}^{n} x^{r_j+2}. \end{aligned}$$

It follows that the terms of the degree sequence of $G \diamond H$ are the elements of the set $\{(n+1)d_i : 1 \leq i \leq m\} \cup \{r_j + 2 : 1 \leq j \leq n\}$. Moreover, p consecutive terms of the degree sequence are $r_j + 2$ for each j with $1 \leq j \leq n$. □

Theorem 6. *Let G be a non-trivial connected graph and let G_1 and G_2 be copies of G in the shadow graph $D_2(G)$. Then*

$$f_{D_2(G)}(x) = 2f_G(x^2).$$

Proof. Let $v \in V(G_1)$ and let v' be the vertex of G_2 corresponding to v. Then

$$N_{D_2(G)}(v) = N_{G_1}(v) \cup N_{G_2}(v') = N_{D_2(G)}(v').$$

This implies that $|N_{D_2(G)}(v)| = |N_{D_2(G)}(v')| = 2|N_G(v)|$. Thus,

$$\begin{aligned} f_{D_2(G)}(x) &= \sum_{v \in V(D_2(G))} x^{|N_{D_2(G)}(v)|} \\ &= \sum_{v \in V(G_1)} x^{|N_{D_2(G)}(v)|} + \sum_{v' \in V(G_2)} x^{|N_{D_2(G)}(v')|} \\ &= 2 \sum_{v \in V(G)} x^{2|N_G(v)|} \\ &= 2f_G(x^2). \end{aligned}$$

Corollary 4. *Let m and n be positive integers such that $m \geq 2$ and $n \geq 3$. Then*

(i) $f_{D_2(P_m)}(x) = 2(m-2)x^4 + 4x^2$;

(ii) $f_{D_2(C_n)}(x) = 2nx^4$; *and*

(iii) $f_{D_2(S_m)}(x) = 2x^{2m} + 2mx^2$, *where S_m is a star of order $m+1$.*

Proof. The polynomial representations of P_m, C_n, and S_m are, respectively, $f_{P_m}(x) = (m-2)x^2 + 2x$, $f_{C_n}(x) = nx^2$, and $f_{S_m}(x) = x^m + mx$. It follows from Theorem 6 that

$$f_{D_2(P_m)}(x) = 2f_{P_m}(x^2) = 2(m-2)x^4 + 4x^2,$$

$$f_{D_2(C_n)}(x) = 2f_{C_n}(x^2) = 2nx^4,$$

and

$$f_{D_2(S_m)}(x) = 2f_{S_m}(x^2) = 2x^{2m} + 2mx^2.$$

This proves the assertion. □

Theorem 7. *Let G be a non-trivial connected graph. Then $a \in \mathbb{R}$ is a zero of $f_{D_2(G)}(x)$ if and only if a^2 is a zero of $f_G(x)$.*

Proof. By Theorem 6, $f_{D_2(G)}(x) = 2f_G(x^2)$. Hence, if a is a zero of $f_{D_2(G)}(x)$, then $2f_G(a^2) = 0$. This implies that a^2 is a zero of $f_G(x)$.

Conversely, if a^2 is a zero of $f_G(x)$, then $f_{D_2(G)}(a) = 2f_G(a^2) = 0$. Thus, a is a zero of $f_{D_2(G)}(x)$. □

Theorem 8. *Let G be a non-trivial connected graph with degree sequence $\langle d_1, d_2, \cdots d_n \rangle$. Then the degree sequence of $D_2(G)$ is $\langle 2d_1, 2d_1, 2d_2, 2d_2, \cdots, 2d_n, 2d_n \rangle$.*

Proof. Given the degree sequence $\langle d_1, d_2, \cdots d_n\rangle$ of G, it follows that $f_G(x) = \sum_{i=1}^{n} x^{d_i}$. Hence, by Theorem 6,

$$f_{D_2(G)}(x) = 2f_G(x^2) = 2\sum_{i=1}^{n} x^{2d_i}.$$

It follows that the degree sequence of $D_2(G)$ is $\langle 2d_1, 2d_1, 2d_2, 2d_2, \cdots, 2d_n, 2d_n\rangle$. □

Theorem 9. *Let G be a non-trivial connected graph of order n. Then*

$$f_{G\overline{G}}(x) = xf_G(x) + x^n f_G(\frac{1}{x}).$$

Proof. Let $v \in V(G)$. Then

$$N_{G\overline{G}}(v) = N_G(v) \cup \{\overline{v}\}$$

and

$$N_{G\overline{G}}(\overline{v}) = N_{\overline{G}}(\overline{v}) \cup \{v\} = \{\overline{z} \in V(\overline{G}) : z \in V(G) \setminus N_G[v]\} \cup \{v\}.$$

Thus, $|N_{G\overline{G}}(v)| = |N_G(v)| + 1$ and $|N_{G\overline{G}}(\overline{v})| = (n - |N_G[v]|) + 1 = n - |N_G(v)|$. Therefore,

$$\begin{aligned}
f_{G\overline{G}}(x) &= \sum_{p \in V(G\overline{G})} x^{|N_{G\overline{G}}(p)|} \\
&= \sum_{v \in V(G)} x^{|N_{G\overline{G}}(v)|} + \sum_{\overline{v} \in V(\overline{G})} x^{|N_{G\overline{G}}(\overline{v})|} \\
&= \sum_{v \in V(G)} x^{|N_G(v)|+1} + \sum_{\overline{v} \in V(\overline{G})} x^{n-|N_G(v)|} \\
&= x \sum_{v \in V(G)} x^{|N_G(v)|} + x^n \sum_{\overline{v} \in V(\overline{G})} x^{-|N_G(v)|} \\
&= xf_G(x) + x^n f_G(\frac{1}{x}). \quad \square
\end{aligned}$$

Corollary 5. *Let n be a positive integer. Then*

(*i*) $f_{P_n\overline{P}_n}(x) = 2x^{n-1} + (n-2)x^{n-2} + (n-2)x^3 + 2x^2$ *for* $n \geq 2$*;*

(*ii*) $f_{C_n\overline{C}_n}(x) = nx^{n-2} + nx^3$ *for* $n \geq 3$*; and*

(*iii*) $f_{S_n\overline{S}_n}(x) = x^{n+1} + nx^n + nx^2 + x$ *for* $n \geq 2$.

Proof. From Theorem 9 and the polynomial representations $f_{P_n}(x) = (n-2)x^2 + 2x$, $f_{C_n}(x) = nx^2$, and $f_{S_n}(x) = x^n + nx$ of P_n, C_n, and S_n, respectively, we have

$$f_{P_n\overline{P}_n}(x) = x[(n-2)x^2 + 2x] + x^n[(n-2)\frac{1}{x^2} + 2\frac{1}{x}] = 2x^{n-1} + (n-2)x^{n-2} + (n-2)x^3 + 2x^2,$$

$$f_{C_n\overline{C}_n}(x) = x(nx^2) + x^n(n\frac{1}{x^2}) = nx^{n-2} + nx^3,$$

and

$$f_{S_n\overline{S}_n}(x) = x(x^n + nx) + x^{n+1}(\frac{1}{x^n} + n\frac{1}{x}) = x^{n+1} + nx^n + nx^2 + x.$$

This proves the assertion. □

Theorem 10. *Let G be a non-trivial connected graph with degree sequence $\langle d_1, d_2, \cdots d_n\rangle$. Then the terms of the degree sequence of $G\overline{G}$ are the elements of the set $\{d_i + 1 : 1 \leq i \leq n\} \cup \{n - d_i : 1 \leq i \leq n\}$.*

Proof. From the polynomial representation $f_G(x) = \sum_{i=1}^{p} x^{d_i}$ of G and from Theorem 9, we find that

$$\begin{aligned} f_{G\overline{G}}(x) &= xf_G(x) + x^n f_G\frac{1}{x}) \\ &= x\sum_{i=1}^{n} x^{d_i} + x^n \sum_{i=1}^{n} x^{-d_i} \\ &= \sum_{i=1}^{n} x^{d_i+1} + \sum_{i=1}^{n} x^{n-d_i}. \end{aligned}$$

Therefore, the terms of the degree sequence of $G\overline{G}$ are exactly the elements of the set $\{d_i + 1 : 1 \leq i \leq n\} \cup \{n - d_i : 1 \leq i \leq n\}$. □

Theorem 11. *Let G and H be non-trivial connected graphs. Then*

$$f_{G\boxtimes H}(x) = \sum_{v\in V(G)} f_H(x^{|N_G(v)|+1})x^{|N_G(v)|} = \sum_{p\in V(H)} f_G(x^{|N_H(p)|+1})x^{|N_H(p)|}.$$

Proof. Let $(v, p) \in V(G \boxtimes H)$. Let $D_1 = N_G(v) \times \{p\}$, $D_2 = \{v\} \times N_H(p)$, and $D_3 = N_G(v) \times N_H(p)$. By definition of strong product of two graphs, it follows that $N_{G\boxtimes H}((v, p)) = D_1 \cup D_2 \cup D_3$. Hence,

$$|N_{G\boxtimes H}((v, p))| = |N_G(v)| + |N_H(p)| + |N_G(v)||N_H(p)|.$$

Thus,

$$\begin{aligned} f_{G\boxtimes H}(x) &= \sum_{(v,p)\in V(G\boxtimes H)} x^{|N_{G\boxtimes H}(v,p)|} \\ &= \sum_{(v,p)\in V(G\boxtimes H)} x^{|N_G(v)|+|N_H(p)|+|N_G(v)||N_H(p)|} \\ &= \sum_{v\in V(G)} x^{|N_G(v)|} \sum_{p\in V(H)} x^{(|N_G(v)|+1)|N_G(p)|} \end{aligned}$$

$$= \sum_{v\in V(G)} x^{|N_G(v)|} f_H(x^{|N_G(v)|+1}).$$

Since

$$\begin{aligned}\sum_{(v,p)\in V(G\boxtimes H)} x^{|N_G(v)|+|N_H(p)|+|N_G(v)||N_H(p)|} &= \sum_{p\in V(H)} x^{|N_H(p)|} \sum_{v\in V(G)} x^{(|N_H(p)|+1)|N_G(v)|} \\ &= \sum_{p\in V(H)} x^{|N_H(p)|} f_G(x^{|N_H(p)|+1}),\end{aligned}$$

it follows that $f_{G\boxtimes H}(x) = \sum_{v\in V(G)} f_H(x^{|N_G(v)|+1})x^{|N_G(v)|} = \sum_{p\in V(H)} f_G(x^{|N_H(p)|+1})x^{|N_H(p)|}$. □

Corollary 6. *Let G and H be non-trivial r_1-regular and r_2-regular connected graphs of orders m and n, respectively. Then*

$$f_{G\boxtimes H}(x) = mnx^{r_1+r_2+r_1r_2}.$$

Proof. Since G and H are, respectively, r_1-regular and r_2-regular graphs, $f_G(x) = mx^{r_1}$ and $f_H(x) = nx^{r_2}$. It follows that $f_H(x^{r_1+1}) = nx^{r_2(r_1+1)}$. Thus, by Theorem 6,

$$\begin{aligned}f_{G\boxtimes H}(x) &= \sum_{v\in V(G)} x^{|N_G(v)|} f_H(x^{|N_G(v)|+1}) \\ &= \sum_{v\in V(G)} x^{r_1} nx^{r_1r_2+r_2} \\ &= n\sum_{v\in V(G)} x^{r_1+r_2+r_1r_2} \\ &= mnx^{r_1+r_2+r_1r_2}. \quad \square\end{aligned}$$

Theorem 12. *Let G and H be non-trivial connected graphs of orders m and n, respectively. Then*

$$f_{G\oplus H}(x) = \sum_{v\in V(G)} x^{n|N_G(v)|} f_H(x^{m-2|N_G(v)|}).$$

Proof. Let $(v,p) \in V(G\oplus H)$. From the definition of $G\oplus H$, it follows that

$$N_{G\oplus H}((v,p)) = [N_G(v) \times (V(H) \setminus N_H(p)] \cup [(V(G) \setminus N_G(v)) \times N_H(p)].$$

Hence,

$$|N_{G\oplus H}((v,p))| = |N_G(v)|(n - |N_H(p)|) + |N_H(p)|(m - |N_G(v)|).$$

Therefore,

$$\begin{aligned} f_{G\oplus H}(x) &= \sum_{(v,p)\in V(G\oplus H)} x^{|N_{G\oplus H}(v,p)|} \\ &= \sum_{(v,p)\in V(G\oplus H)} x^{|N_G(v)|(n-|N_H(p)|)+|N_H(p)|(m-|N_G(v)|)} \\ &= \sum_{v\in V(G)}\sum_{p\in V(H)} x^{|N_G(v)|(n-|N_H(p)|)+|N_H(p)|(m-|N_G(v)|)} \\ &= \sum_{v\in V(G)} x^{n|N_G(v)|} \sum_{p\in V(H)} x^{(m-2|N_G(v)|)|N_H(p)|} \\ &= \sum_{v\in V(G)} x^{n|N_G(v)|} f_H(x^{m-2|N_G(v)|}). \quad \square \end{aligned}$$

Corollary 7. *Let G and H be non-trivial of orders m and n, respectively. If G is r-regular, then*

$$f_{G\oplus H}(x) = mx^{nr} f_H(x^{m-2r}).$$

Proof. Since G is r-regular, $|N_G(v)| = r$ for all $v \in V(G)$. Thus, from Theorem 12, we have

$$\begin{aligned} f_{G\oplus H}(x) &= \sum_{v\in V(G)} x^{n|N_G(v)|} f_H(x^{m-2|N_G(v)|}) \\ &= \sum_{v\in V(G)} x^{nr} f_H(x^{m-2r}) \\ &= mx^{nr} f_H(x^{m-2r}). \quad \square \end{aligned}$$

Theorem 13. *Let G and H be non-trivial connected graphs of orders m and n, respectively. If $\langle d_1, d_2, \cdots d_m\rangle$ and $\langle q_1, q_2, \cdots q_n\rangle$ are the degree sequences of G and H, respectively, then the terms of the degree sequence of $G \oplus H$ are the elements of the set $\{nd_i + (m-2d_i)q_j : 1 \le i \le m$ and $1 \le j \le n\}$.*

Proof. From Theorem 12 and the polynomial representation $f_H(x) = \sum_{j=2}^{n} x^{q_j}$, we have

$$\begin{aligned} f_{G\oplus H}(x) &= \sum_{v\in V(G)} x^{n|N_G(v)|} f_H(x^{m-2|N_G(v)|}) \\ &= \sum_{i=1}^{m} x^{nd_i} \sum_{j=1}^{n} x^{(m-2d_i)q_j} \end{aligned}$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} x^{nd_i} x^{(m-2d_i)q_j}$$
$$= \sum_{i=1}^{m}\sum_{j=1}^{n} x^{nd_i+(m-2d_i)q_j}.$$

It follows that the terms of the degree sequence of $G \oplus H$ are the elements of the set $\{nd_i + (m-2d_i)q_j : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$. □

Theorem 14. *Let G and H be non-trivial connected graphs of orders m and n, respectively. Then*

$$f_{G\vee H}(x) = f_G(x^n) f_H(x^m).$$

Proof. Let $(v,p) \in V(G \vee H)$. From the definition of $G \vee H$, it follows that

$$N_{G\vee H}((v,p)) = (N_G(v) \times V(H)) \cup (V(G) \times N_H(p)).$$

Hence,

$$|N_{G\vee H}((v,p))| = n|N_G(v)| + m|N_H(p)|.$$

Therefore,

$$\begin{aligned} f_{G\vee H}(x) &= \sum_{(v,p)\in V(G\vee H)} x^{|N_{G\vee H}(v,p)|} \\ &= \sum_{(v,p)\in V(G\vee H)} x^{n|N_G(v)|+m|N_H(p)|} \\ &= \sum_{v\in V(G)} x^{n|N_G(v)|} \sum_{p\in V(H)} x^{m|N_H(p)|} \\ &= \sum_{v\in V(G)} x^{n|N_G(v)|} f_H(x^m) \\ &= f_G(x^n) f_H(x^m). \quad \square \end{aligned}$$

Corollary 8. *Let n and m be positive integers. Then*

(*i*) $f_{P_m\vee P_n}(x) = (m-2)(n-2)x^{2m+2n} + 2(m-2)x^{m+2n} + 2(n-2)x^{2m+n} + 4x^{m+n}$ *for $m, n \geq 2$;*

(*ii*) $f_{P_m\vee C_n}(x) = n(m-2)x^{2m+2n} + 2nx^{2m+n}$ *for $m \geq 2$ and $n \geq 3$; and*

(*iii*) $f_{C_m\vee C_n}(x) = mnx^{2m+2n}$ *for $m, n \geq 3$.*

Proof. For any $k \geq 2$ and $r \geq 3$, $f_{P_k}(x) = (k-2)x^2 + 2x$ and $f_{C_r}(x) = rx^2$. By Theorem 14, we have

$$f_{P_m\vee P_n}(x) = f_{P_m}(x^n) f_{P_n}(x^m)$$

$$\begin{aligned} &= [(m-2)(x^n)^2 + 2(x^n)][(n-2)(x^m)^2 + 2(x^m)] \\ &= (m-2)(n-2)x^{2m+2n} + 2(m-2)x^{m+2n} + 2(n-2)x^{2m+n} + 4x^{m+n}, \end{aligned}$$

$$\begin{aligned} f_{P_m \vee C_n}(x) &= f_{P_m}(x^n) f_{C_n}(x^m) \\ &= [(m-2)(x^n)^2 + 2(x^n)](n(x^m)^2) \\ &= n(m-2)x^{2m+2n} + 2nx^{2m+n}, \end{aligned}$$

and

$$\begin{aligned} f_{C_m \vee C_n}(x) &= f_{C_m}(x^n) f_{C_n}(x^m) \\ &= (m(x^n)^2)(n(x^m)^2) \\ &= mnx^{2m+2n}. \quad \square \end{aligned}$$

Theorem 15. *Let G and H be non-trivial connected graphs of orders m and n, respectively. Then $a \in \mathbb{R}$ is a zero of $f_{G \vee H}(x)$ if and only if a^n is a zero of $f_G(x)$ or a^m is a zero of $f_H(x)$.*

Proof. By Theorem 14, $f_{G \vee H)}(x) = f_G(x^n) f_H(x^m)$. Suppose a is a zero of $f_{G \vee H)}(x)$. Then $f_{G \vee H)}(a) = f_G(a^n) f_H(a^m) = 0$. This implies that $f_G(a^n) = 0$ or $f_H(a^m) = 0$. Hence, a^n is a zero of $f_G(x)$ or a^m is a zero of $f_H(x)$.

Conversely, suppose a^n is a zero of $f_G(x)$ or a^m is a zero of $f_H(x)$. Then, clearly, $f_{G \vee H)}(a) = f_G(a^n) f_H(a^m) = 0$, showing that a is a zero of $f_{G \vee H)}(x)$. □

Theorem 16. *Let G and H be non-trivial connected graphs of orders m and n, respectively. If $\langle d_1, d_2, \cdots d_m \rangle$ and $\langle q_1, q_2, \cdots q_n \rangle$ are the degree sequences of G and H, respectively, then the terms of the degree sequence of $G \vee H$ are the elements of the set $\{nd_i + mq_j : 1 \leq i \leq m$ and $1 \leq j \leq n\}$.*

Proof. From Theorem 14,

$$\begin{aligned} f_{G \vee H}(x) &= f_G(x^n) f_H(x^m) \\ &= \sum_{i=1}^{m} x^{nd_i} \sum_{j=1}^{n} x^{mq_j} \\ &= \sum_{i=1}^{m} \sum_{j=1}^{n} x^{nd_i} x^{mq_j} \\ &= \sum_{i=1}^{m} \sum_{j=1}^{n} x^{nd_i + mq_j}. \end{aligned}$$

It follows that the terms of the degree sequence of $G \vee H$ are the elements of the set $\{nd_i + mq_j : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$. □

4. Conclusion and Recommendation

The polynomial representations of some graphs have been obtained in this study. The authors were not able to describe the degree sequence of some graphs resulting from some operations. However, for particular graphs, the degree sequence of the graphs may be obtained. Determining real roots or zeros of the polynomial representation of a graph, if they exist, can be an aspect for further investigation.

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