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Probabilistic Type 2 Poly-Bernoulli Polynomials

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Abstract. The main purpose of this article is to introduce the probabilistic type 2 poly-Bernoulli polynomials under the condition that Y is a random variable. This means that we will consider the probabilistic extension of the type 2 poly-Bernoulli polynomials and study to obtain some new results. Furthermore, we also define the probabilistic unipoly-Bernoulli polynomials and numbers attached to p, and investigate their interesting basic properties. Based on these new definition, we derive some meaningful formulae of probabilistic type 2 poly-Bernoulli polynomials and probabilistic unipoly-Bernoulli polynomials and numbers attached to p.

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Key Words and Phrases: Bernoulli polynomials, Stirling numbers, Probabilistic type 2 poly-Bernoulli polynomials, Probabilistic unipoly-Bernoulli polynomials.

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1. Introduction

The Bernoulli polynomials are defined by

$$\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},\quad (\text{see}[1,\ 2,\ 7,\ 15,\ 27,\ 30],[12,\ 19,\ 20,\ 28]). \tag{1}$$

For $k\in\mathbb{Z}$, the polylogarithm function is defined by

$$Li_k(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^k},(|x|<1),\quad (\text{see}[4,\ 5,\ 24],[23]). \tag{2}$$

For $k\in\mathbb{Z}$, Kim defined the polyexponential function $e_k(x)$, which is given by

$$e_k(x)=\sum_{n=1}^{\infty}\frac{x^n}{(n-1)!n^k},\quad (\text{see}[6]). \tag{3}$$

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When $k = 1$, we note that

$$e_1(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!} = e^x - 1. \tag{4}$$

As we all know, the poly-Bernoulli polynomials are defined by Kaneko. It is given by

$$\frac{Li_k(1-e^{-t})}{1-e^{-t}} e^{xt} = \sum_{n=0}^{\infty} PB_n^{(k)}(x) \frac{t^n}{n!}, \quad \text{(see[5])}. \tag{5}$$

When $x = 0$, we note that $PB_n^{(k)} = PB_n^{(k)}(0)$ are called the poly-Bernoulli numbers.

In 2019, Kim considered the definition of type 2 poly-Bernoulli polynomials. It is given by

$$\frac{e_k(log(1+t))}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} \beta_n^{(k)}(x) \frac{t^n}{n!}, \quad \text{(see[6, 22])}. \tag{6}$$

When $x = 0$, we note that $\beta_n^{(k)} = \beta_n^{(k)}(0)$ are called the type 2 poly-Bernoulli numbers. Kim also studied the unipoly function attached to p. Its definition as follows.

$$u_k(x|p) = \sum_{n=1}^{\infty} \frac{p(n)}{n^k} x^n, (k \in \mathbb{Z}), \quad \text{(see[6])}. \tag{7}$$

Later, he defined the unipoly-Bernoulli polynomials attached to p by

$$\frac{1}{1-e^{-t}} u_k(1-e^{-t}|p) e^{xt} = \sum_{n=0}^{\infty} B_{n,p}^{(k)}(x) \frac{t^n}{n!}, \quad \text{(see[6])}. \tag{8}$$

Recently, Kim studied the probabilistic poly-Bernoulli polynomials associated with Y. Assume that Y is a random variable such that the moment generating function of Y given by

$$E[e^{Yt}] = \sum_{n=0}^{\infty} E[Y^n] \frac{t^n}{n!}, (|t| < r), \quad \text{([6, 14, 16])}. \tag{9}$$

exist for some $r \geq 0$. Then the definition of the probabilistic poly-Bernoulli polynomials are given by

$$\frac{Li_k(1-e^{-t})}{1-E[e^{-Yt}]} (E[e^{-Yt}])^x = \sum_{n=0}^{\infty} B_n^{(k,Y)}(x) \frac{t^n}{n!}, \quad \text{(see[3, 8, 9, 18, 31, 32])}. \tag{10}$$

When $k = 1$, it is obvious that $B_n^{(1,Y)} = (-1)^n B_n^Y(x)$. This type of polynomials is a new extension. Inspired by this, the aim of our paper is to explore the probabilistic type 2 poly-Bernoulli polynomials and obtain some new results. Meanwhile, the probabilistic unipoly-Bernoulli polynomials are also another research.

The Stirling number of the first kind are defined by

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k, \quad (\text{see}[10,\ 28,\ 29]). \tag{11}$$

Where $(x)_0 = 1, (x)_n = x(x-1)\cdots(x-n+1), (n \geq 1)$.

From (11), we can easily know

$$\frac{1}{k!}(log(1+t))^k = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!}, \quad (\text{see}[10,\ 11,\ 29]). \tag{12}$$

The Stirling number of the second kind are defined by

$$x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k, \quad (\text{see}[17,\ 21,\ 26]). \tag{13}$$

From (13), we also derive the generating function as follows.

$$\frac{1}{k!}(e^t-1)^k = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}, \quad (\text{see}[21,\ 26]). \tag{14}$$

In 2024, Kim defined the probabilistic Stirling number of the second kind associated with Y are given by

$$\frac{1}{k!}(E[e^{Yt}]-1)^k = \sum_{n=k}^{\infty} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_Y \frac{t^n}{n!}, \quad (\text{see}[3,\ 9,\ 18],[14]). \tag{15}$$

The Bell polynomials are defined by

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} Bel_n(x)\frac{t^n}{n!}, \quad (\text{see}[13,\ 16,\ 22,\ 23,\ 25]). \tag{16}$$

2. probabilistic type 2 poly-Bernoulli polynomials

Let $(Y_j)_{j\geq 1}$ be a sequence of mutually independent copies of the random variable Y, and let

$$S_0 = 0, S_k = Y_1 + Y_2 + \cdots + Y_k, (k \in \mathbb{N}). \tag{17}$$

In this section we consider probabilistic type 2 poly-Bernoulli polynomials.

$$\frac{e_k(log(1+t))}{E[e^{Yt}]-1}(E[e^{Yt}])^x = \sum_{n=0}^{\infty} \beta_n^{(k,Y)}(x)\frac{t^n}{n!}. \tag{18}$$

When $x = 0$, $\beta_n^{(k,Y)}(0) = \beta_n^{(k,Y)}$ are called probabilistic type 2 poly-Bernoulli numbers. From (18), we get

$$\begin{aligned}\sum_{n=0}^{\infty}\beta_n^{(k,Y)}(x)\frac{t^n}{n!} &= \frac{e_k(log(1+t))}{E[e^{Yt}]-1}(E[e^{Yt}])^x \qquad (19)\\ &= \sum_{j=0}^{\infty}\beta_j^{(k,Y)}\frac{t^j}{j!}\sum_{k=0}^{\infty}\binom{x}{k}k!\sum_{m=k}^{\infty}\left\{\begin{matrix} m \\ k \end{matrix}\right\}_Y\frac{t^m}{m!}\\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}\beta_{n-m}^{(k,Y)}(x)_k\left\{\begin{matrix} m \\ k \end{matrix}\right\}_Y\frac{t^n}{n!}.\end{aligned}$$

Therefore, by comparing the coefficients on both sides of (19), we have the following theorem.

Theorem 1. *For $n, k \geq 0$, we have*

$$\beta_n^{(k,Y)} = \sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}\beta_{n-m}^{(k,Y)}(x)_k\left\{\begin{matrix} m \\ k \end{matrix}\right\}_Y.$$

From (18), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\beta_n^{(k,Y)}(x)\frac{t^n}{n!} &= \frac{e_k(log(1+t))}{t}\frac{t}{E[e^{Yt}]-1}(E[e^{Yt}])^x \qquad (20)\\ &= \sum_{l=0}^{\infty}B_l^Y(x)\frac{t^l}{l!}\sum_{i=1}^{\infty}\frac{(log(1+t))^i}{(i-1)!i^k}\\ &= \sum_{l=0}^{\infty}B_l^Y(x)\frac{t^l}{l!}\frac{1}{t}\sum_{i=1}^{\infty}\frac{1}{i^{k-1}}\sum_{j=i}^{\infty}S_1(j,i)\frac{t^j}{j!}\\ &= \sum_{l=0}^{\infty}B_l^Y(x)\frac{t^l}{l!}\sum_{j=0}^{\infty}\sum_{i=1}^{j+1}\frac{1}{i^{k-1}}\frac{S_1(j+1,i)}{j+1}\frac{t^j}{j!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\sum_{i=1}^{j+1}\binom{n}{j}\frac{S_1(j+1,i)}{i^{k-1}(j+1)}B_{n-j}^Y(x)\right)\frac{t^n}{n!}.\end{aligned}$$

Thus, by comparing the coefficients on both sides of (20), we have the following theorem.

Theorem 2. *For $n, j \geq 0$, we have*

$$\beta_n^{(k,Y)}(x) = \sum_{j=0}^{n}\sum_{i=1}^{j+1}\binom{n}{j}\frac{S_1(j+1,i)}{i^{k-1}(j+1)}B_{n-j}^Y(x).$$

Now, we observe that

$$\sum_{m=0}^{n} \left(E[e^{Yt}]\right)^m = \frac{E[e^{Yt}]^{n+1} - 1}{E[e^{Yt}] - 1}. \tag{21}$$

From (21), we have

$$\begin{aligned}
\sum_{m=0}^{n} E[e^{Yt}] &= \frac{1}{e_1(log(1+t))} \frac{e_1(log(1+t))}{E[e^{Yt}] - 1} (E[e^{Yt}]^{n+1} - 1) \qquad (22)\\
&= \frac{1}{t} \frac{t}{E[E^{Yt}] - 1} \left(E[e^{Yt}]^{n+1} - 1\right)\\
&= \frac{1}{t} \left(\sum_{l=0}^{\infty} \beta_l^{(1,Y)} - \sum_{l=0}^{\infty} \beta_l^{(1,Y)} \frac{t^l}{l!} \right)\\
&= \sum_{l=0}^{\infty} \frac{\beta_{l+1}^{(1,Y)}(n+1) - \beta_{l+1}^{(1,Y)}}{l+1} \frac{t^l}{l!}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\sum_{m=0}^{n} \left(E[e^{Yt}]\right)^m &= \sum_{m=0}^{n} E[e^{(Y_1+Y_2+\cdots+Y_m)t}] \qquad (23)\\
&= \sum_{m=0}^{n} \sum_{l=0}^{\infty} E[S_m^l] \frac{t^l}{l!}\\
&= \sum_{l=0}^{\infty} \sum_{m=0}^{n} E[S_m^l] \frac{t^l}{l!}.
\end{aligned}$$

Hence, comparing the coefficients on both sides of (22) and (23), we have the following theorem.

Theorem 3. *For $n \geq 0$, we have*

$$\sum_{m=0}^{n} E[S_m] = \frac{\beta_{l+1}^{(1,Y)}(n+1) - \beta_{l+1}^{(1,Y)}}{l+1}.$$

From (3), we have

$$\begin{aligned}
e_m(log(1+t)) &= \sum_{k=1}^{\infty} \frac{(log(1+t))^k}{(k-1)!k^m} \qquad (24)\\
&= \sum_{k=0}^{\infty} \frac{(log(1+t))^{k+1}}{k!(k+1)^m}
\end{aligned}$$

$$= \sum_{k=0}^{\infty} \frac{1}{(k+1)^{m-1}} \sum_{n=k+1}^{\infty} S_1(n,k+1)\frac{t^n}{n!}$$
$$= \sum_{n=k+1}^{\infty} \sum_{k=0}^{n-1} \frac{S_1(n,k+1)}{(k+1)^{m-1}} \frac{t^n}{n!}.$$

On the other hand,

$$e_m(log(1+t)) = \sum_{l=0}^{\infty} \beta_l^{(m,Y)} \frac{t^l}{l!} \left(E[e^{Yt}-1]\right) \tag{25}$$
$$= \sum_{l=0}^{\infty} \beta_l^{(m,Y)} \frac{t^l}{l!} \left(\sum_{j=0}^{\infty} E[Y^j]\frac{t^j}{j!} - 1 \right)$$
$$= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} \beta_l^{(m,Y)} E[Y^{n-l}] - \beta_n^{(m,Y)} \right) \frac{t^n}{n!}.$$

Therefore, by comparing the coefficients on both sides of (24) and (25), we have the following theorem.

Theorem 4. *For $n, k \geq 0$, we have*

$$\sum_{k=0}^{n-1} \frac{S_1(n,k+1)}{(k+1)^{m-1}} = \begin{cases} \sum_{l=0}^{n} \left(\binom{n}{l} \beta_l^{(m,Y)} E[Y^{n-l}] - \beta_n^{(m,Y)} \right), & \text{if } n \geq k+1, \\ 0, & \text{if } n < k+1. \end{cases}$$

Let Y be the Poisson random variable with parameter $\alpha > 0$, then we have

$$\frac{e_k(log(1+t))}{E[e^{Yt}]-1} \left(E[e^{Yt}]\right)^x = \frac{e_k(log(1+t))}{e^{\alpha(e^t-1)}-1} e^{\alpha x(e^t-1)} \tag{26}$$
$$= \frac{\alpha(e^t-1)}{\alpha(e^t-1)} \frac{e_k(log(1+t))}{e^{\alpha(e^t-1)-1}} e^{\alpha x(e^t-1)}$$
$$= \frac{1}{\alpha} \sum_{j=0}^{\infty} \beta_j^{(k)} \frac{t^j}{j!} \frac{\alpha(e^t-1)}{e^{\alpha(e^t-1)}-1} e^{\alpha x(e^x-1)}$$
$$= \frac{1}{\alpha} \sum_{j=0}^{\infty} \beta_j^{(k)} \frac{t^j}{j!} \sum_{l=0}^{\infty} \alpha^l B_l(x) \frac{(e^x-1)^l}{l!}$$
$$= \sum_{j=0}^{\infty} \beta_j^{(k)} \frac{t^j}{j!} \sum_{m=0}^{\infty} \sum_{l=0}^{m} \alpha^{l-1} B_l(x) S_2(m,l) \frac{t^m}{m!}$$
$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{l=0}^{m} \binom{n}{m} \beta_{n-m}^{(k)} \alpha^{l-1} B_l(x) S_2(m,l) \frac{t^n}{m}.$$

From (18) and (26), we have the following theorem.

Theorem 5. *Let Y be the Poisson random variable with parameter α, we have*

$$\beta_n^{(k,Y)}(x) = \sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}\beta_{n-m}^{(k)}\alpha^{l-1}B_l(x)S_2(m,l).$$

From (18), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_n^{(k,Y)}(\alpha+1) &= \frac{e_k(log(1+t))}{E[e^{Yt}]-1}\left(E[e^{Yt}]\right)^{\alpha} E[e^{Yt}] \\ &= \sum_{l=0}^{\infty} B_l^{(k,Y)}(\alpha)\frac{t^l}{l!}\sum_{m=0}^{\infty} E[Y^m]\frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}B_l^{(k,Y)}(\alpha)E[Y^{n-l}]\frac{t^n}{n!}.\end{aligned} \tag{27}$$

From (18), we also have

$$\begin{aligned}\sum_{n=0}^{\infty} B_n^{(k,Y)}(\alpha)\frac{t^n}{n!} &= \sum_{n=0}^{\infty} B_n^{Y}\frac{t^n}{n!}E[e^{(Y_1+Y_2+\cdots+Y_\alpha)t}] \\ &= \sum_{l=0}^{\infty} B_l^{(k,Y)}\frac{t^n}{n!}\sum_{m=0}^{\infty} E[S_\alpha^m]\frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}B_l^{(k,Y)}E[S_\alpha^m]\frac{t^n}{n!}.\end{aligned} \tag{28}$$

Therefore, by (27) and (28), we have the following theorem.

Theorem 6. *For any $\alpha \in \mathbb{Z}$ and $n, \alpha \geq 0$, we have*

$$B_n^{(k,Y)}(\alpha+1) - B_n^{(k,Y)}(\alpha) = \sum_{l=0}^{n}\binom{n}{l}\left(B_l^{(k,Y)}(\alpha)E[Y^{n-l}] - B_l^{(k,Y)}E[S_\alpha^m]\right).$$

3. The probabilistic unipoly-Bernoulli polynomials

In this section, we give the definition of the probabilistic unipoly-Bernoulli polynomials attached to p as follows.

$$\frac{1}{1-E[e^{-Yt}]}u_k(1-e^{-t}|p)(E[e^{-Yt}])^x = \sum_{n=0}^{\infty} B_{n,p}^{(k,Y)}(x)\frac{t^n}{n!}. \tag{29}$$

If $x = 0$, $B_{n,p}^{(k,Y)} = B_{n,p}^{(k,Y)}(0)$ are called the probabilistic unipoly-Bernoulli numbers.

Particularly, if $p(n) = 1$, then $B_{n,1}^{(k,Y)} = B_n^{(k,Y)}(x)$.

From (29)

$$\begin{aligned}\frac{1}{1-E[e^{-Yt}]}u_k(1-e^{-t}|p) &= \frac{1}{1-E[e^{-Yt}]}\sum_{m=1}^{\infty}\frac{P(m)(1-e^{-t})^m}{m^k} \\ &= \frac{t}{1-E[e^{-Yt}]}\frac{1}{t}\sum_{m=1}^{\infty}\frac{p(m)}{m^k}\frac{(1-e^{-t})^m}{m!}m! \\ &= \sum_{j=0}^{\infty}B_j^Y(-1)^j\frac{t^j}{j!}\frac{1}{t}\sum_{m=1}^{\infty}\frac{p(m)m!}{m^k}\sum_{l=m}^{\infty}S_2(l,m)(-1)^{l-m}\frac{t^l}{l!} \\ &= \sum_{j=0}^{\infty}B_j^Y(-1)^j\frac{t^j}{j!}\sum_{l=0}^{\infty}\sum_{m=1}^{l+1}\frac{p(m)m!}{m^k}\frac{S_2(l+1,m)(-1)^{l+1-m}}{l+1}\frac{t^l}{l!} \\ &= \sum_{n=0}^{\infty}\sum_{l=0}^{n}\sum_{m=1}^{l+1}\binom{n}{l}\frac{p(m)(m-1)!}{m^{k-1}}(-1)^{n-m+1}\frac{S_2(l+1,m)}{l+1}B_{n-l}^Y\frac{t^n}{n!}.\end{aligned} \tag{30}$$

Therefore, by compring the coefficients on both sides of (29) and (30), we have the following theorem.

Theorem 7. *For $n, k \geq 0$, we have*

$$B_{n,p}^{(k,Y)} = \sum_{l=0}^{n}\sum_{m=1}^{l+1}\binom{n}{l}\frac{p(m)(m-1)!}{m^{k-1}}(-1)^{n-m+1}\frac{S_2(l+1,m)}{l+1}B_{n-l}^Y.$$

Let Y be the Poisson random variable with parameter $\alpha > 0$. Then we have

$$\begin{aligned}u_k(1-e^{-t}|p)e^{x\alpha(e^t-1)} &= \sum_{n=0}^{\infty}B_{n,p}^{(k,Y)}(x)\frac{t^n}{n!}(1-e^{\alpha(e^{-t}-1)}) \\ &= \sum_{n=0}^{\infty}B_{n,p}^{(k,Y)}(x)\frac{t^n}{n!} - \sum_{m=0}^{\infty}B_m^{(k,Y)}(x)\frac{t^m}{m!}\sum_{l=0}^{\infty}\frac{\alpha^l(e^{-t}-1)^l}{l!} \\ &= \sum_{n=0}^{\infty}B_{n,p}^{(k,Y)}(x)\frac{t^n}{n!} - \sum_{m=0}^{\infty}B_m^{(k,Y)}(x)\frac{t^m}{m!}\sum_{l=0}^{\infty}\alpha^l\sum_{i=l}^{\infty}S_2(i,l)(-1)^i\frac{t^i}{i!} \\ &= \sum_{n=0}^{\infty}B_{n,p}^{(k,Y)}(x)\frac{t^n}{n!} - \sum_{n=0}^{\infty}\sum_{i=0}^{n}\sum_{l=0}^{i}\binom{n}{i}(-1)^i\alpha^lS_2(i,l)B_{n-i}^{(k,Y)}(x)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(B_{n,p}^{(k,Y)}(x) - \sum_{i=0}^{n}\sum_{l=0}^{i}\binom{n}{i}(-1)^i\alpha^lS_2(i,l)B_{n-i}^{(k,Y)}(x)\right)\frac{t^n}{n!}.\end{aligned} \tag{31}$$

On the other hand

$$u_k(1-e^{-t}|p)e^{x\alpha(e^t-1)} = \sum_{m=1}^{\infty}\frac{p(m)}{m^k}(1-e^{-t})^m\sum_{i=0}^{\infty}Bel_i(x)\frac{\alpha^i(e^{-t}-1)^i}{i!} \tag{32}$$
$$= \sum_{j=1}^{\infty}\sum_{i=0}^{j}\frac{p(j-i)}{(j-i)^k}Bel_i(x)\frac{\alpha^i}{i!}(-1)^{j-i}\frac{(e^{-t}-1)^j}{j!}j!$$
$$= \sum_{j=1}^{\infty}\sum_{i=0}^{j}\frac{p(j-i)}{(j-i)^k}Bel_i(x)\frac{\alpha^i j!}{i!}(-1)^{j-i}\sum_{n=j}^{\infty}S_2(n,j)(-1)^n\frac{t^n}{n!}$$
$$= \sum_{n=1}^{\infty}\sum_{j=1}^{n}\sum_{i=0}^{j}\frac{p(j-i)}{(j-i)^k}Bel_i(x)\frac{\alpha^i j!}{i!}(-1)^{j-i+n}S_2(n,j)\frac{t^n}{n!}.$$

Therefore, by comparing the coefficients on both sides of (31) and (32), we have the following theorem.

Theorem 8. *Let Y be the Poisson random variable with parameter $\alpha(>0)$. Then we have*

$$B_{n,p}^{(k,Y)}(x) = \sum_{j=1}^{n}\sum_{i=0}^{j}\frac{p(j-i)}{(j-i)^k}Bel_i(x)\frac{\alpha^i j!}{i!}(-1)^{j-i+n}S_2(n,j)+\sum_{i=0}^{n}\sum_{l=0}^{i}\binom{n}{i}(-1)^i\alpha^l S_2(i,l)B_{n-i}^{(k,Y)}(x).$$

From (29), we have

$$\sum_{n=0}^{\infty}B_{n,p}^{(k,Y)}(\alpha)\frac{t^n}{n!} = \sum_{n=0}^{\infty}B_{l,p}^{(k,Y)}\frac{t^l}{l!}\left(E[e^{-Yt}]\right)^{\alpha} \tag{33}$$
$$= \sum_{l=0}^{\infty}B_{l,p}^{(k,Y)}\frac{t^l}{l!}\sum_{m=0}^{\infty}(-1)^m E[Y_1+Y_2+\cdots+Y_\alpha]\frac{t^m}{m!}$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}(-1)^m\binom{n}{m}B_{n-m,p}^{(k,Y)}E[S_\alpha^m]\frac{t^n}{n!}.$$

Therefore, by compring the coefficients on bosides (33), we have the following theorem.

Theorem 9. *For $\alpha, n \geq 0$ and $\alpha \in \mathbb{Z}$, we have*

$$B_{n,p}^{(k,Y)}(\alpha) = \sum_{m=0}^{n}(-1)^m\binom{n}{m}B_{n-m,p}^{(k,Y)}E[S_\alpha^m].$$

Let Y be the Bernoulli random variable with probability of success A. Then we have

$$\sum_{n=0}^{\infty}B_{n,p}^{(k,Y)}(x)\frac{t^n}{n!} = \frac{1}{A(e^{-t}-1)}u_k(1-e^{-t}|p)\left(A(e^{-t}-1)+1\right)^x \tag{34}$$

$$= \frac{1}{A(e^{-t}-1)}\sum_{l=1}^{\infty}\frac{p(l)}{l^k}(1-e^{-t})^l\sum_{m=0}^{\infty}\binom{x}{m}A^m(e^{-t}-1)^m$$
$$= \frac{1}{A(e^{-t}-1)}\sum_{i=1}^{\infty}\sum_{l=1}^{i}\frac{p(l)}{l^k}\binom{x}{i-l}A^{i-l}(e^{-t}-1)^i$$
$$= \sum_{i=1}^{\infty}\sum_{l=1}^{i}(-1)^l\frac{p(l)}{l^k}\binom{x}{i-l}A^{i-l-1}(e^{-t}-1)^{i-1}$$
$$= \sum_{i=0}^{\infty}\sum_{l=1}^{i+1}(-1)^l\frac{p(l)}{l^k}\binom{x}{i-l+1}A^{i-l}(e^{-t}-1)^i$$
$$= \sum_{i=0}^{\infty}\sum_{l=1}^{i+1}(-1)^l\frac{p(l)}{l^k}i!\binom{x}{i-l+1}A^{i-l}\sum_{n=i}^{\infty}(-1)^nS_2(n,i)\frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty}\sum_{i=0}^{n}\sum_{l=1}^{i+1}(-1)^{l+n}\frac{p(l)}{l^k}(i)_{l-1}(x)_{i-l}A^{i-l}S_2(n,i)\frac{t^n}{n!}.$$

Therefore, by compring the coefficients on both sides of (34), we have the following theorem.

Theorem 10. *Let Y be the Bernoulli random variable with probability of success A, then we have*

$$B_{n,p}^{(k,Y)}(x) = \sum_{i=0}^{n}\sum_{l=1}^{i+1}(-1)^{l+n}\frac{p(l)}{l^k}(i)_{l-1}(x)_{i-l}A^{i-l}S_2(n,i).$$

4. Conclusion

In this paper, we present a probabilistic version of the type 2 poly-Bernoulli polynomials associated with a random variable Y satisfying suitable moment conditions. We call it probabilistic type 2 poly-Bernoulli polynomials. We study some properties of such polynomials and obtain relevant results. More specifically, we derived an exact expression for $\beta_n^{k,Y}(x)$, and establish a relation between the type 2 poly-Bernoulli numbers and the Stirling number of the first kind, and obtain a explicit formula of $\beta_n^{(k,Y)}(x)$, In the case where Y is the Poisson variable with parameter α. Similarly, we define the unipoly-Bernoulli polynomials attached to p. Then we show the explicit expression of $B_{n,p}^{k,Y}(x)$ and other results by skilful calculations. As a next step in our research, we will study this probability type of polynomials more deeply so that give better and generalizable results.

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