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From Anosov Closing Lemma to Global Data of Cohomological Nature

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Abstract. For diffeomorphisms with hyperbolic sets, the Anosov Closing Lemma ensures the existence of periodic orbits in the neighbourhood of orbits that return close enough to themselves. Moreover, it defines how the distance between the corresponding points of an initial orbit and the constructed periodic orbits is controlled. In the essential, this article presents proof of the estimate of this distance. The Anosov Closing Lemma is crucial in the statement of Livschitz Theorem that, based only on the periodic data, provides a necessary and sufficient condition so that cohomological equations have sufficiently regular solutions, Hölder solutions. It is one of the main tools to obtain global data of a cohomological nature based only on periodic data. As suggested by Katok and Hasselblat in [2], it is demonstrated, in detail and the cohomology context, the Livschitz Theorem for hyperbolic diffeomorphisms, where the mentioned distance control inequality is essential.

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1. Introduction

H. Poincaré found the modern theory of dynamical systems when he emphasised the qualitative approach instead of the traditional emphasis on explicit solutions of differential equations. In particular, when he considers the local theory of maps and vector fields near fixed and periodic orbits in the context of differentiable dynamics [16]. Other leading researchers on this broad subject were A. Lyapunov and J. Hadamard introduced several concepts of stability and developed analytic tools such as, for instance, the Hadamard-Perron Theorem [6]. Another essential advance in the study of differentiable dynamics was the concept of structural stability, particularly with the founding by S. Smale that systems with complicated orbit behaviour can be structurally stable. Afterwards, S. Smale, D. Anosov, Y. Sinai, R. Bowen and D. Ruelle developed the core of the hyperbolic dynamics theory [6]. Some results established by A. Livschitz in the 1970s [11, 12] address the

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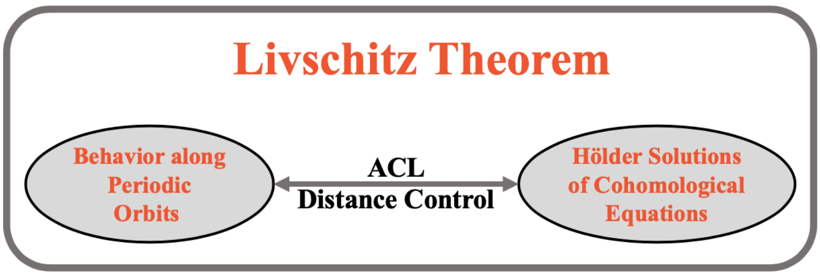
possibility of obtaining solutions of cohomological equations in the hyperbolic dynamics context. In turn, the Anosov Closing Lemma (ACL) is used to prove these results [6].

ACL formalizes how the combination of local hyperbolicity (from the linearized dynamical systems analysis) with nontrivial recurrence tends to produce an abundance of periodic orbits. Given a dynamical system with phase space X, $f : X \to X$, and fixed an initial condition $x_0 \in X$, it is crucial to identify those $x \in X$ which evolution under the iterates of f follows sufficiently close that of x_0, for a long time; also, to understand the asymptotic behavior of x orbit relative to x_0 orbit can be useful of this Lemma.

Besides presenting a detailed proof of the ACL for hyperbolic diffeomorphisms, an inequality that quantitatively estimates how the constructed periodic orbit differs from the initial orbit is here proved. This inequality is crucial in the Livschitz Theorem for hyperbolic diffeomorphisms proof. The Livschitz Theorem is an essential tool for obtaining global data of a cohomological nature from periodic data. Indeed, in the dynamical systems theory, several main problems can be reduced to solving the so-called cohomological equations

$$\varphi = \Phi \circ f - \Phi,$$

where $f : X \to X$ is a dynamical system and $\varphi : X \to \mathbb{R}$ is a function, both known, and $\Phi : X \to \mathbb{R}$ is an unknown function. The study of these equations is related, in particular, to the existence of absolutely continuous measures for expansive circle maps and the topological stability of hyperbolic torus automorphisms [7]. Such equations also arise naturally in celestial Mechanics and statistical mechanics (see, for instance, [2]). Given a hyperbolic dynamical system, the Livschitz Theorem provides a necessary and sufficient condition, based only on the data given by periodic orbits, for the existence of Hölder solutions of the cohomological equations. ACL is a key component in the proof of the Livschitz Theorem, as it provides the necessary link between the behavior along periodic orbits and the existence of solutions to cohomological equations with adequate regularity, as illustrated below.



This article aims to present a proof of ACL oriented towards the study of cohomology in discrete dynamical systems; in this way, it provides a self-contained approach to the Livschitz Theorem, which proof is presented for pedagogical purposes also. As far as we know, there is no detailed treatment of ACL in the literature, pragmatically oriented to the cohomological context, except for hyperbolic flows and the generalization to a class of suspension flows in the article [10]. The novelty of this research lies in the comprehensive treatment of the Livschitz Theorem in discrete time, building upon the fundamental relationship between the existence of solutions to cohomological equations and the behavior of cocycles along periodic orbits. It extends the Livschitz Theorem to a broader class of dynamical systems, providing a unified framework for analyzing the regularity of solutions to cohomological equations. For a broader knowledge of the ACL and the Livschitz Theorem, article [10] can be consulted for development of the cohomology in continuous time and proofs of the Livschitz Theorem for hyperbolic flows and also for suspension flows are presented, concerning [1, 3–5, 8, 9, 15, 17, 18].

The Livschitz Theorem and its connection to the ACL have implications in various fields, such as the study of transport properties in dynamical systems, the analysis of Markov chains, and the study of spectral properties of operators associated with dynamical systems. Furthermore, the techniques developed in this research can be leveraged to study the behavior of numerical schemes for simulating wave phenomena, such as the Kuramoto-Sivashinsky equation and fourth-order reaction-diffusion equations [13, 14].

We introduce fundamental notions for the cohomology study in discrete dynamical systems (Section 2). In particular, we introduce the concepts of cocycle, coboundary and cohomology between cocycles in discrete time. We present cohomological equations and emphasize the fundamental relationship between the existence of solutions of these equations and the behavior of the cocycles along periodic orbits. Intending to present a detailed proof of the Livschitz Theorem in a version for hyperbolic diffeomorphisms, we begin Section 3 by outlining the proof of ACL for diffeomorphisms (Subsection 3.1), with emphasis on the statement of the inequality that quantitatively estimates distances between constructed periodic orbit and the initial orbit. The statement of that distance control estimate is crucial for the Livschitz Theorem's proof (Subsection 3.2). Except for the statement of that inequality as a distance control estimate, the proofs presented here closely follow the suggestions of Katok and Hasselblatt in [6]. All the proofs are given in great detail and connected to the cohomology theory, with an additional pedagogical nature. We finalize presenting some conclusions and comments (Section 4).

2. Cocycles and cohomology in discrete time

Let $f : \mathbb{Z} \times X \to X$ be a dynamical system with phase space X and discrete time. So, they are valid the group properties

$$\begin{aligned} f(m+n,x) &= f(m,f(n,x)) \\ f(0,x) &= x, \end{aligned}$$

for each $x \in X$ and $n, m \in \mathbb{Z}$.

Given $n \in \mathbb{Z}$, we define the map $f(n) : X \to X$ by $f(n)x = f(n,x)$ through the dynamical system f. We designate by cocycle over f each function $\alpha : \mathbb{Z} \times X \to \mathbb{R}$ verifying the property

$$\alpha(m+n,x) = \alpha(m,f(n)x) + \alpha(n,x) \tag{1}$$

whenever $x \in X$ and $n, m \in \mathbb{Z}$. The cocycles over f constitute a linear space. Defining, for each $n \in \mathbb{Z}$, the map $\widetilde{f}(n) : X \times \mathbb{R} \to X \times \mathbb{R}$ by $\widetilde{f}(n)(x, y) = (f(n)x, y + \alpha(n, x))$, the property (1) is equivalent to $\widetilde{f}(m+n) = \widetilde{f}(m) \circ \widetilde{f}(n)$.

Each function $\Phi : X \to \mathbb{R}$ induces a cocycle by defining

$$\alpha(n, x) = \Phi(f(n)x) - \Phi(x). \tag{2}$$

In fact, the function α defined this way satisfies property (1) since

$$\begin{aligned} \Phi(f(m+n)x) - \Phi(x) &= \Phi(f(m)(f(n)x)) - \Phi(x) \\ &= \Phi(f(m)f(n)x) - \Phi(f(n)x) + \Phi(f(n)x) - \Phi(x) \\ &= \alpha(m, f(n)x) + \alpha(n, x). \end{aligned}$$

The cocycles defined by (2) are designated by coboundaries. A natural equivalence relationship between cocycles is the cohomology. Two cocycles α and β over a dynamical system f are cohomologous if they differ by a coboundary, that is, if there is a function $\Phi : X \to \mathbb{R}$ such that

$$\alpha(n, x) - \beta(n, x) = \Phi(f(n)x) - \Phi(x).$$

We note that a cocycle α is a coboundary if and only if α is cohomologous to the trivial cocycle $\beta(n, x) = 0$. In this case it is said that α is cohomologically trivial and a function Φ satisfying (2) is a trivialization of α. Also, for a cocycle α to be a coboundary it is necessary that $\alpha(n, x) = 0$ for all $n \in \mathbb{Z}$ and $x \in X$ such that $f(n)x = x$. Equation (2) is said to be a cohomological equation.

Each cocycle α over a dynamical system $f : \mathbb{Z} \times X \to X$ is uniquely determined by the function $\varphi : X \to \mathbb{R}$ defined by $\varphi(x) = \alpha(1, x)$. In fact, it is immediate to verify that

$$\alpha(n, x) = \begin{cases} \sum_{i=0}^{n-1} \varphi(f^i x) & \text{if } n > 0 \\ -\sum_{i=n}^{-1} \varphi(f^i x) & \text{if } n \leq 0 \end{cases}, \tag{3}$$

where $f^0 x = x$ and $fx = f(1, x)$. So, we can identify the dynamical system with the invertible map $f : X \to X$ (without danger of notation confusion) being the inverse given by $f^{-1}x = f(-1, fx)$. There is then a one-to-one correspondence between cocycles and real functions defined on X. Two function $\varphi, \psi : X \to \mathbb{R}$ are called cohomologous respecting to f if $\varphi - \psi = \Phi \circ f - \Phi$ for some function $\Phi : X \to \mathbb{R}$. We can easily verify that two functions are cohomologous if and only if the respective cocycles are cohomologous. Furthermore, a function is called a coboundary if it is cohomologous to the zero function.

Given a function $\varphi : X \to \mathbb{R}$, let α be the cocycle over f defined by (3). To show that the equation

$$\varphi = \Phi \circ f - \Phi, \tag{4}$$

also called cohomological equation, has a solution is equivalent to show that the cocycle α is a coboundary. In fact, if the cohomological equation (4) is satisfied then, for each $n > 0$, we have

$$\alpha(n, x) = \sum_{i=0}^{n-1} \varphi(f^i x) = \sum_{i=0}^{n-1} [\Phi(f^{i+1}x) - \Phi(f^i x)] = \Phi(f^n x) - \Phi(x)$$

(with similar identities for $n \leq 0$) and α is a coboundary. On the other hand, if α is a coboundary then there is a function $\Phi : X \to \mathbb{R}$ such that $\alpha(n, x) = \Phi(f^n x) - \Phi(x)$. Making $n = 1$ we conclude that the cohomological equation (4) is satisfied by Φ.

Suppose now that the cohomological equation (4) has a solution. If x is a m-periodic point of the dynamical system f, $f^m x = x$, then

$$\sum_{i=0}^{m-1} \varphi(f^i x) = \alpha(m, x) = \Phi(f^m x) - \Phi(x) = 0.$$

Therefore, it is necessary that $\sum_{i=0}^{m-1} \varphi(f^i x) = 0$ for all m-periodic point x so that there is a solution Φ to the cohomological equation (4).

If it is not required any additional property to a solution of the cohomological equation then there is no difficulty in showing their existence, provided that $\sum_{i=0}^{m-1} \varphi(f^i x) = 0$ for each m-periodic point x. Indeed, we can pick up one point x from each orbit of f, arbitrarily choose $\Phi(x) \in \mathbb{R}$ and then define Φ over the points of each orbit by

$$\Phi(f^n x) = \Phi(x) + \sum_{i=0}^{n-1} \varphi(f^i x).$$

However, if there is some additional structure in the phase space X which we intend to maintain, this procedure may be unsatisfactory. For example, in the case of irrational rotation of the circle, this construction necessarily start from a collection of non-measurable points and so, in general, we obtain a non-measurable solution of the cohomological equation.

3. Livschitz Theorem for hyperbolic diffeomorphisms

Let M be a Riemannian manifold, with norm $\|\cdot\|_x$ and inner product $\langle \cdot, \cdot \rangle_x$ in the tangent space $T_x M$ of each point $x \in M$. In what follows we will write only $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ without reference to the point x. A distance d is defined in M by

$$d(x, y) = \inf_{\gamma} \int_0^1 \|\gamma'(t)\| \, dt,$$

where the infimum is taken over all differentiable curves $\gamma : [0, 1] \to M$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Let $f : M \to M$ be a diffeomorphism and $\Lambda \subset M$ an f-invariant set, $f\Lambda = \Lambda$. The map $f|_\Lambda$ is topologically transitive if there is $x_0 \in M$ whose orbit $\{f^n x_0 : n \in \mathbb{Z}\}$ is dense in Λ. If there is an open neighborhood V of Λ such that $\Lambda = \bigcap_{n \in \mathbb{Z}} f^n V$, then Λ is locally maximal for f. An f-invariant set $\Lambda \subset M$ is hyperbolic for f if:

- The tangent space restricted to Λ can be written as a continuous direct sum of df-invariant bundles, that is, for each $x \in \Lambda$ there is a decomposition of the tangent space in the stable and unstable subspaces,

$$T_x M = E^s(x) \oplus E^u(x),$$

that varies continuously with x and verifies the equalities $d_x f E^s(x) = E^s(fx)$ and $d_x f E^u(x) = E^u(fx)$;

- There are constants $C > 0$ and $\tau \in (0,1)$ such that, for all $x \in \Lambda$ and $n \in \mathbb{N}$, we have $\|d_x f^n v\| \leq C\tau^n \|v\|$ for $v \in E^s(x)$, and $\|d_x f^{-n} v\| \leq C\tau^n \|v\|$ for $v \in E^u(x)$.

Given $f : M \to M$, consider the function $\varphi : M \to \mathbb{R}$ and the cohomological equation (4). As presented in Section 2, if equation (4) has a solution then $\sum_{i=0}^{m-1} \varphi(f^i x) = 0$ whenever x is a m-periodic point of f, $f^m x = x$. In addition, we have shown that this is a necessary and sufficient for the existence of solutions of the cohomological equation (4). However, the solutions can be discontinuous or even not measurable. Naturally arises the question of how to ensure the existence of continuous solutions or even with some additional regularity. The Livschitz Theorem formulated below answers this question in the context of hyperbolic dynamics.

Theorem 1. ***(Livschitz Theorem for hyperbolic diffeomorphisms)*** *Let $f : M \to M$ be a C^1 diffeomorphism defined on a Riemannian manifold M. Let $\Lambda \subset M$ be a compact invariant hyperbolic set locally maximal with $f|_\Lambda$ topologically transitive. Suppose that $\varphi : \Lambda \to \mathbb{R}$ is a Hölder function such that*

$$\sum_{i=0}^{m-1} \varphi(f^i x) = 0$$

whenever $f^m(x) = x$. Then there is a Hölder function $\Phi : \Lambda \to \mathbb{R}$, with at least the same Hölder exponent as φ, and unique up to an additive constant, such that $\varphi = \Phi \circ f - \Phi$.

Given the relationship exposed in Section 2 between solving cohomological equations and obtaining coboundaries, we can interpret the Theorem 1 as follows: for hyperbolic dynamics and Hölder functions, the periodic data are necessary and sufficient to identify Hölder coboundaries. Note that a function $\varphi : M \to \mathbb{R}$ is called Hölder with exponent θ, $0 < \theta \leq 1$, if there is $K > 0$ such that

$$|\varphi(x) - \varphi(y)| \leq K d(x,y)^\theta$$

for all $x, y \in M$.

While closely following the suggestions of Katok and Hasselblatt [6], the proof of the Theorem 1 here presented is pragmatically oriented to the study of cohomology in dynamical systems, and consists of the following steps:

1) the function Φ is determined along a dense orbit, guaranteed by the existence of topological transitivity of f in Λ, by choosing an arbitrary value in one of the orbit points;

2) the Hölder regularity of φ is then used to ensure the Hölder regularity of Φ while Φ is extended to the whole set Λ.

The ACL for diffeomorphism is crucial for the first part of the proof. For diffeomorphisms with hyperbolic sets, it ensures that there are always periodic orbits in the neighborhood of orbits that turn close enough of themselves. As a consequence, we obtain an control estimate to the distance between the corresponding points in the initial orbit and the periodic orbit (regarding this see for instance [5]).

3.1. Proof of the Anosov Closing Lemma for diffeomorphisms: distance control inequality

Since it is involved in many details of the ACL proof, we first present the Hadamard-Perron Theorem in the context of hyperbolic sets.

Theorem 2. ***(Hadamard–Perron Theorem for hyperbolic diffeomorphisms)*** *Let M be a Riemannian manifold, $f : M \to M$ a C^1 diffeomorphism and $\Lambda \subset M$ a compact hyperbolic set. Then, for each $x \in \Lambda$, there are stable and unstable local embedded C^1 manifolds, respectively $W^s(x)$ and $W^u(x)$, such that:*

i. $T_xW^s(x) = E^s(x)$ *and* $T_xW^u(x) = E^u(x)$*;*

ii. $f(W^s(x)) \subset W^s(fx)$ *and* $f^{-1}(W^u(x)) \subset W^u(f^{-1}x)$*;*

iii. *For each $\delta > 0$ there is $D = D(\delta) > 0$ such that, for each $n \in \mathbb{N}$, we have $d(f^n x, f^n y) \leq D(\tau+\delta)^n d(x, y)$ for $y \in W^s(x)$, and $d(f^{-n}x, f^{-n}y) \leq D(\tau+\delta)^n d(x, y)$ for $y \in W^u(x)$;*

iv. *There are $\beta > 0$ and an unique family U_x of neighborhoods containing the ball around $x \in \Lambda$ of radius β such that*

$$\begin{aligned} W^s(x) &= \{y \in M : f^n y \in U_{f^n x} \text{ for } n \in \mathbb{N}\} \\ W^u(x) &= \left\{y \in M : f^{-n} y \in U_{f^{-n} x} \text{ for } n \in \mathbb{N}\right\} \end{aligned}$$

The proof of the Theorem 2 presents a methodology that plays a central role in hyperbolic dynamical systems theory (see [6]). It involves the use of the Contraction Mapping Principle in appropriately constructed functional spaces.

It follows from Properties ***iii.*** and ***iv.*** that given any two stable local manifolds $W_1^s(x)$ and $W_2^s(x)$ of x satisfying the properties of Hadamard-Perron Theorem, their intersection contains an open neighborhood of x in each of them. Thus it can be concluded that on a certain $n \geq 0$ we have $f^n(W_1^s(f^{-n}x)) \subset W_2^s(x)$ and $f^n(W_2^s(f^{-n}x)) \subset W_1^s(x)$. Such a number n can be chosen uniformly for all $x \in \Lambda$. The same holds for unstable local manifolds with n replaced by $-n$. This implies that the stable and unstable global manifolds given by

$$\widetilde{W}^s(x) = \bigcup_{n=0}^{\infty} f^{-n}(W^s(f^n x)) \quad \text{and} \quad \widetilde{W}^u(x) = \bigcup_{n=0}^{\infty} f^{n}(W^u(f^{-n} x))$$

are independent of a particular choice of stable and unstable local manifolds and can be topologically characterized by the sets

$$\begin{aligned} \widetilde{W}^s(x) &= \{y \in M : d(f^n x, f^n y) \to 0 \text{ when } n \to +\infty\}, \\ \widetilde{W}^u(x) &= \{y \in M : d(f^n x, f^n y) \to 0 \text{ when } n \to -\infty\}. \end{aligned}$$

The balls with radius ε and center x belonging to $\widetilde{W}^s(x)$ and $\widetilde{W}^u(x)$ are denoted by $W^s_\varepsilon(x)$ and $W^u_\varepsilon(x)$, respectively. Now, we have all the elements and notation to write the ACL for hyperbolic diffeomorphisms and proceed to its detailed proof.

Lemma 1. ***(ACL for hyperbolic diffeomorphisms)*** *Given a Riemannian manifold M, let $\Lambda \subset M$ be a compact hyperbolic set locally maximal for the C^1 diffeomorphism $f : M \to M$. Then, for all $\lambda \in (0,1)$ sufficiently large, there are an open neighborhood V of Λ and constants $C, \delta > 0$ such that for $x \in \Lambda$ satisfying $d(f^n x, x) < \delta$ there is a point $y \in \Lambda$ such that $f^n y = y$ and is valid the inequality*

$$d(f^k x, f^k y) \leq C\lambda^{\min\{k, n-k\}} d\left(f^n x, x\right) \tag{5}$$

for $k = 0, 1, \ldots, n$.

In what follows, we present an outline of the proof for $M = \mathbb{R}^n$ where, excepting the proof of inequality (5), the suggestions of Katok and Hasselblatt in [6] are followed. The inequality (5) provides an important quantitative data since it establishes how the constructed periodic orbit differs from the initial orbit: it states how the distance between corresponding points of the initial orbit and the constructed periodic orbit is controlled.

For each $x \in \Lambda$ we fix a local coordinate system in $T_x M$ such that the decomposition $E^u(x) \oplus E^s(x)$ is identified with the decomposition $\mathbb{R}^n = \mathbb{R}^l \oplus \mathbb{R}^{n-l}$ and the metric in $T_x M$ is the usual metric in $\mathbb{R}^n$. For each $x \in \Lambda$ there is an open neighborhood $V_{f^k x}$ of $f^k x$ for each $k \in \mathbb{Z}$ such that $f \left| V_{f^k x} \right.$ can be written as

$$f_k(u, v) = (A_k u + A_k(u, v), B_k v + \beta_k(u, v))$$

where $A_k : \mathbb{R}^l \to \mathbb{R}^l$ and $B_k : \mathbb{R}^{n-l} \to \mathbb{R}^{n-l}$ are linear maps defined by

$$A_k = d_{f^k x} f|_{E^u(f^k x)} \quad \text{and} \quad B_k = d_{f^k x} f|_{E^s(f^k x)}.$$

Redefining the norms on the stable and unstable bundles (see [17]), we can suppose that $C = 1$ in the definition of the hyperbolic set, there is $\tau' \in (\tau, 1)$ such that

$$\left\| A_k^{-1} \right\| \leq \tau' \quad \text{and} \quad \|B_k\| \leq \tau' \tag{6}$$

for all $k \in \mathbb{Z}$ (when $C > 1$ we can remake the proof with minor changes). In addition, eventually by further choice of $V_{f^k x}$, we can guarantee that exists $\varepsilon < d(f^n x, x)$ such that

$$\|A_k\|_{C^1} < \varepsilon \quad \text{and} \quad \|\beta_k\|_{C^1} < \varepsilon \tag{7}$$

for all $k \in \mathbb{Z}$. Note that the points $(u_k, v_k) \in V_{f^k x}$ for $k = 0, 1, \ldots, n-1$ constitute a n-periodic orbit of f if and only if $(u, v) = ((u_0, v_0), (u_1, v_1), \ldots, (u_{n-1}, v_{n-1}))$ is a fixed point of the map $F : \mathbb{R}^N \to \mathbb{R}^N$, where $N = n \dim M$, given by

$$F(u, v) = (f_{n-1}(u_{n-1}, v_{n-1}), f_0(u_0, v_0), \ldots, f_{n-2}(u_{n-2}, v_{n-2})).$$

We write F as $F(u, v) = L(u, v) + G(u, v)$ where $L(u, v)$ is given by

$$((A_{n-1}u_{n-1}, B_{n-1}v_{n-1}), (A_0 u_0, B_0 v_0), \ldots, (A_{n-2}u_{n-2}, B_{n-2}v_{n-2})).$$

It follows from (7) that

$$\left\|G(u, v) - G(u', v')\right\| \leq \varepsilon \left\|(u, v) - (u', v')\right\| \tag{8}$$

with the norm $\|(u, v)\| = \max\{|u|, |v|\}$.
On the other hand, it follows from (6) that the matrix $L - \mathrm{Id}$ is invertible. By using the decomposition

$$(L - \mathrm{Id})^{-1} = ((L - \mathrm{Id})^{-1}|_{E_s}, L^{-1}(\mathrm{Id} - L^{-1})^{-1}|_{E_u})$$

we obtain

$$\|(L - \mathrm{Id})^{-1}\| = \|(L - \mathrm{Id})^{-1}|_{E_s}\| + \|L^{-1}(\mathrm{Id} - L^{-1})^{-1}|_{E_u}\| \leq \frac{1}{1 - \|L|_{E_s}\|} + \frac{\|L^{-1}|_{E_u}\|}{1 - \|L^{-1}|_{E_u}\|}$$

and then

$$\|(L - \mathrm{Id})^{-1}\| \leq C_1 \tag{9}$$

for some constant $C_1 > 0$ that only depend on τ. So, the solutions of $F(z) = z$ are precisely the solutions of $\mathcal{F}(z) = z$ where $\mathcal{F}(z) = -(L - \mathrm{Id})^{-1} G(z)$. It follows from (8) and (9) that

$$\|\mathcal{F}(z) - \mathcal{F}(z')\| \leq C_1 \varepsilon \|z - z'\|. \tag{10}$$

Taking ε small enough we obtain $C_1 \varepsilon < 1$ that allows to conclude that $\mathcal{F} : \mathbb{R}^N \to \mathbb{R}^N$ is a contraction. By the Contraction Mapping Principle there is a unique fixed point $y_0 \in \mathbb{R}^N$ of $\mathcal{F}$. In addition, $y_0 = \lim_{k \to +\infty} \mathcal{F}^k(s)$ where $s = (x, fx, \ldots, f^{n-1}x)$. So we have

$$\|y_0 - s\| \leq \sum_{k=1}^{\infty} \|\mathcal{F}^k(s) - \mathcal{F}^{k-1}(s)\|.$$

From (10) we obtain $\|\mathcal{F}^k(s) - \mathcal{F}^{k-1}(s)\| \leq (C_1\varepsilon)^{k-1} \|\mathcal{F}(s) - s\|$, and hence

$$\|y_0 - s\| \leq \sum_{k=1}^{\infty} \|\mathcal{F}^k(s) - \mathcal{F}^{k-1}(s)\| \leq \|\mathcal{F}(s) - s\| \sum_{k=1}^{\infty} (C_1\varepsilon)^{k-1}.$$

Since $L(s)+G(s) = F(s) = s+v$ for some v with $\|v\| < \varepsilon$, we have $G(s) = -(L - \mathrm{Id})\, s+v$, that is $\mathcal{F}(s) = s - (L - \mathrm{Id})^{-1} v$. Using (9) we have $\|\mathcal{F}(s) - s\| \leq C_1\varepsilon$ and hence

$$\|y_0 - s\| \leq C_1\varepsilon \sum_{k=1}^{\infty} (C_1\varepsilon)^{k-1} = \frac{C_1\varepsilon}{1 - C_1\varepsilon}.$$

Let y be such that $y_0 = (y, fy, \ldots, f^{n-1}y)$, which is a n-periodic point. By the choice of ε we have

$$d(f^k x, f^k y) \leq C_2 d(f^n x, x), \tag{11}$$

for $k = 0, 1, \ldots, n-1$, for some constant $C_2 > 0$. It follows from (11) that

$$d(f^n x, f^n y) = d(f^n x, y) \leq d(f^n x, x) + d(x, y) \leq (1 + C_2) d(f^n x, x).$$

We can thus claim that

$$d(f^k x, f^k y) \leq (1 + C_2)\varepsilon \tag{12}$$

for $k = 0, 1, \ldots, n$. Being Λ a locally maximal set, there is an open neighborhood V of Λ such that $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n V$. As y is a periodic point, we have $y \in \bigcap_{n\in\mathbb{Z}} f^n V$ (eventually choosing again ε and the neighborhoods $V_{f^k x}$), and then $y \in \Lambda$.

It remains now to establish the inequality (5). Since Λ is a compact locally maximal hyperbolic set, it has local product structure. Thus, for each $\gamma > 0$ small enough, there is $\varepsilon > 0$ such that if the points $x, y \in \Lambda$ verify $d(x, y) < \varepsilon$ then the intersection of $W^s_\gamma(x)$ with $W^u_\gamma(y)$ is not empty, but constituted by a single point which we denote by $[x, y] = w$. We have $f^k w = \left[f^k x, f^k y\right]$ and the estimate

$$d(f^k x, f^k y) \leq d(f^k x, f^k w) + d(f^k w, f^k y).$$

Since $w \in W^s_\gamma(x)$, the Theorem 2 guarantees that, for each $\delta > 0$, there is a constant $C_3 = C_3(\delta)$ such that

$$d(f^k x, f^k w) \leq C_3 (\tau + \delta)^k d(x, w). \tag{13}$$

Again by the Theorem 2, since $w \in W^u_\gamma(y)$, we can claim that

$$d(f^k w, f^k y) = d(f^{k-n}(f^n w), f^{k-n}(f^n y)) \leq C_3 (\tau + \delta)^{n-k} d(f^n w, f^n y).$$

Using (13), it follows that

$$\begin{aligned} d(f^k x, f^k y) &\leq C_3 (\tau + \delta)^k d(x, w) + C_3 (\tau + \delta)^{n-k} d(f^n w, y) \\ &\leq C_3 (\tau + \delta)^{\min\{k, n-k\}} \left[d(x, w) + d(f^n w, f^n x) + d(f^n x, y)\right] \\ &\leq C_3 (\tau + \delta)^{\min\{k, n-k\}} \left[(1 + C_3) d(x, w) + d(f^n x, y)\right]. \end{aligned}$$

By (12) with $k = n$, we obtain $d(f^n x, y) \leq (1+C_2) d(f^n x, x)$. As a consequence of uniform transversally, there is $C_4 > 0$ such that if ε is small enough then $d(x, w) \leq C_4 d(x, y)$. Again by (12), now with $k = 0$, we obtain

$$d(x, w) \leq C_4 (1 + C_2) d(f^n x, x).$$

Choosing $\lambda \in (0, 1)$ such that $\lambda \geq \tau + \delta$, we finally obtain the intended inequality (5) with $C = C_3(1 + C_3)C_4(1 + C_2) + C_3(1 + C_2)$.

3.2. Proof of the Livschitz Theorem for hyperbolic diffeomorphisms

Note that the assumption that the hyperbolic set Λ is locally maximal is essential to ensure that the periodic point constructed in the development is still in Λ. This apparent detail is crucial in the following proof of Theorem 1.

Since $f\,|_\Lambda$ is topologically transitive there is a point $x_0 \in \Lambda$ with orbit dense in Λ. By choosing an arbitrary real value $\Phi(x_0)$ we define

$$\Phi\left(f^n x_0\right) = \Phi(x_0) + \alpha(n, x_0),$$

where $\alpha(n,x)$ is defined as (3). Consider $n, m \in \mathbb{N}$ such that the distance $d(f^n x_0, f^m x_0)$ is small enough in order to apply the ACL for hyperbolic diffeomorphisms. Assuming that $m > n$, this Lemma provides constants $C > 0$, $\lambda \in (0,1)$ and a point $y \in \Lambda$ such that $y = f^{m-n} y$ and

$$d(f^{n+i} x_0, f^i y) \leq C \lambda^{\min\{i, m-n-i\}} d(f^n x_0, f^m x_0) \tag{14}$$

for $i = 0, 1, \ldots, m-n$. Taking into account the definition of Φ in the dense orbit of x_0, we observe that

$$\begin{aligned} \left|\Phi\left(f^n x_0\right) - \Phi\left(f^m x_0\right)\right| &= \left|\Phi(x_0) + \alpha(n, x_0) - \Phi(x_0) - \alpha(m, x_0)\right| \\ &= \left|\sum_{i=0}^{n-1} \varphi(f^i x_0) - \sum_{j=0}^{m-1} \varphi(f^j x_0)\right| = \left|\sum_{i=0}^{m-n-1} \varphi(f^{n+i} x_0)\right|. \end{aligned}$$

Given the hypothesis concerning the periodic points, we have

$$\left|\Phi\left(f^n x_0\right) - \Phi\left(f^m x_0\right)\right| = \left|\sum_{i=0}^{m-n-1} \left[\varphi(f^{n+i} x_0) - \varphi(f^i y)\right]\right| \leq \sum_{i=0}^{m-n-1} \left|\varphi(f^{n+i} x_0) - \varphi(f^i y)\right|.$$

Since φ is Hölder with exponent $\theta \in (0,1]$, there is a constant $K > 0$ such that $|\varphi(x_1) - \varphi(x_2)| \leq K d(x_1, x_2)^\theta$. Then we obtain

$$\left|\Phi\left(f^n x_0\right) - \Phi\left(f^m x_0\right)\right| \leq \sum_{i=0}^{m-n-1} K\, d(f^{n+i} x_0, f^i y)^\theta.$$

It follows from (14) that

$$\left|\Phi\left(f^n x_0\right) - \Phi\left(f^m x_0\right)\right| \leq \sum_{i=0}^{m-n-1} K (C \lambda^{\min\{i, m-n-i\}}\, d(f^n x_0, f^m x_0))^\theta,$$

leading to

$$\left|\Phi\left(f^n x_0\right) - \Phi\left(f^m x_0\right)\right| \leq 2 K C^\theta\, d(f^n x_0, f^m x_0)^\theta \sum_{i=0}^{m-n-1} \lambda^{\theta i} < \frac{2 K C^\theta}{1 - \lambda^\theta}\, d(f^n x_0, f^m x_0)^\theta.$$

So, we obtain the inequality

$$|\Phi(f^n x_0) - \Phi(f^m x_0)| \ < \ \frac{2KC^\theta}{1-\lambda^\theta}\, d(f^n x_0, f^m x_0)^\theta. \tag{15}$$

In a very similar way we can prove that the inequality (15) is also valid for any $n, m \in \mathbb{Z}$. Since Φ is Hölder in the orbit of x_0 and this orbit is dense in Λ, the function Φ is uniquely extendable to a continuous function in Λ, which we still denote by Φ. Immediately follows from (15) that the extension of Φ into Λ has at least the same Hölder exponent as φ. Since f is continuous and φ is Hölder continuous, then φ and $\Phi \circ f - \Phi$ are continuous functions in Λ which coincide on the dense orbit of x_0. Therefore, they coincide in the whole set Λ and Φ is a continuous solution of the cohomological equation. The claim of uniqueness is a consequence of the choice of $\Phi(x_0)$ determine Φ in a unique way.

4. Conclusions

ACL plays a pivotal role in establishing the Livschitz Theorem, which provides a necessary and sufficient condition for the existence of cohomological equations with sufficiently regular solutions. This Lemma ensures that for hyperbolic diffeomorphisms, any orbit that comes close to a periodic orbit can be perturbed to an actual periodic orbit. This property is crucial for the Livschitz Theorem, as it allows relating the behavior of cocycles along periodic orbits to the existence of solutions to the cohomological equation. Throughout the article, the necessary elements to demonstrate the Livschitz Theorem in discrete time are detailed, emphasizing the development that leads to a distance control inequality provided by the ACL for hyperbolic diffeomorphisms. In this way, it is worth mentioning that the ACL is crucial in the Livschitz Theorem proof and consequently ensures the existence of cohomological equations with sufficiently regular solutions. The inequality (5) is crucial quantitative data that states the control of the distance between corresponding points of an initial orbit and the constructed periodic orbit. Given the exposed relationship between solving cohomological equations and obtaining coboundaries, the Livschitz Theorem can be understood as: in hyperbolic dynamics and Hölder functions, the periodic data are necessary and sufficient to identify Hölder coboundaries. The class of hyperbolic dynamical systems contains several examples of invertible smooth dynamical systems with complicated orbit structure, namely hyperbolic toral automorphisms, their C^1-perturbations, as well as expanding maps of the circle.

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