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The Connections of Strongest Fuzzy Γ-Ideals on Ternary Γ-Semigroups

Warud Nakkhasen[1,*], Onnalin Yangnok[1], Kewarin Chaidet[1], Wichayaporn Jantanan[2]

[1] *Department of Mathematics, Faculty of Science, Mahasarakham University, Maha Sarakham 44150, Thailand*
[2] *Department of Mathematics, Faculty of Science, Buriram Rajabhat University, Buriram 31000, Thailand*

Abstract. The fuzzy relation R_μ on μ, where μ is a fuzzy set of a set X, is called a strongest fuzzy relation on X if $R_\mu(x, y) = \min\{\mu(x), \mu(y)\}$, for all $x, y \in X$. The notion of strongest fuzzy relations will be applied in our investigation on ternary Γ-semigroups. In order to achieve this, we will define the concepts of strongest fuzzy ternary Γ-subsemigroups, strongest fuzzy Γ-ideals (resp. left, right, and lateral), and strongest fuzzy bi-Γ-ideals on ternary Γ-semigroups. Then, we study the connections and characterizations of these concepts in ternary Γ-semigroups.

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1. Introduction

The notion of ternary Γ-semigroups was introduced by Madhusudhana Rao et al. [6] in 2015. The ternary Γ-semigroups were generalized the concepts of semigroups, Γ-semigroups and ternary semigroups. They characterized and examined about several some elements of ternary Γ-semigroups. Then Vasantha and Madhusudhana Rao [8] developed and characterized the terms completely semiprime ternary Γ-ideal and semiprime ternary Γ-ideal in ternary Γ-semigroups. After that, Vasantha et al. [11] introduced the concepts of trio L-trio TΓ-ideals, La-trio TΓ-ideals, R-trio TΓ-ideals, and trio TΓ-ideals in trio ternary Γ-semigroups. Afterwards, Ali et al. [1] introduced and discussed some properties of po-bi quasi-Γ-ideals, po-bi-Γ-ideals, and generalized po-bi quasi-Γ-ideals in po-bi-ternary Γ-semigroups. For other research related to ternary Γ-semigroups, additional studies can be done in general (e.g., [7, 9, 10]).

*Corresponding author.
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Email addresses: `warud.n@msu.ac.th` (W. Nakkhasen), `63010213011@msu.ac.th` (O. Yangnok), `63010213018@msu.ac.th` (K. Chaidet), `wichayaporn.jan@bru.ac.th` (W. Jantanan)

Fuzzy subsets or fuzzy sets are defined by Zadeh [13] as a function from a nonempty set X to the unit interval $[0,1]$. This idea is a mathematical extension of the classical sets in mathematics. Then, in 1985, Bhattacharya and Mukherjee [3] proved that a strongest fuzzy relation μ_σ on a group S is a fuzzy subgroup if and only if σ is a fuzzy subgroup. This concept of strongest fuzzy relations has been studied continuously. Mostafa et al. [5] presented some properties of KU-ideals in terms of strongest fuzzy relations in KU-algebras. Subsequently, the concept of strongest fuzzy relations in the Cartesian product of B-algebras was investigated by Yamini and Kailasavalli [12] in 2014. Following that, Bhargavi et al. [2] gave and analyzed the concept of the Cartesian product of fuzzy sets in ternary Γ-semigroups. In addition, they characterized different types of fuzzy Γ-ideals in terms of their Cartesian product of ternary Γ-semigroups. Recently, Derseh et al. [4] considered some properties of strongest intuitionistic fuzzy PMS-relations on PMS-algebras in 2023.

The purpose of this article is applying the fuzzy relation to define the concepts of strongest fuzzy Γ-subsemigroups, strongest fuzzy (resp. left, right, lateral) Γ-ideals, and strongest fuzzy bi-Γ-ideals of ternary Γ-semigroups. Later on, we consider the connections of strongest fuzzy Γ-subsemigroups, strongest fuzzy (resp. left, right, lateral) Γ-ideals, and strongest fuzzy bi-Γ-ideals on ternary Γ-semigroups.

2. Preliminaries

In this section, we will review important basic concepts for use in the next section. A *fuzzy set* [13] μ of a nonempty set X is a mapping form the set X into $[0,1]$. The *fuzzy relation* [3] R on a nonempty set X is a fuzzy set $R : X \times X \to [0,1]$. Let R be any fuzzy relation on a nonempty set X, and μ be a fuzzy set of X. Then R is said to be a *fuzzy relation on* μ [3] if $R(x,y) \leq \min\{\mu(x), \mu(y)\}$, for all $x, y \in X$.

Definition 1. *[3] Let μ be a fuzzy set of a nonempty X, and R_μ be a fuzzy relation on μ. Then R_μ is called a strongest fuzzy relation on X if $R_\mu(x,y) = \min\{\mu(x), \mu(y)\}$, for all $x, y \in X$.*

For any strongest fuzzy relation R_μ on a nonempty set X, and for each $t \in [0,1]$, we denote by $(R_\mu)_t$ the *level subset* of R_μ where $(R_\mu)_t := \{(x,y) \mid R_\mu(x,y) \geq t\}$ (see [3]).

Let X be a nonempty set, and μ be a fuzzy set of X. For any subset A of X, the *characteristic function* χ_μ^A of A is a strongest fuzzy relation on X defined by for every $x, y \in X$,

$$\chi_\mu^A(x,y) = \begin{cases} 1 & \text{if } x, y \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2. *(cf. [6]) Let T and Γ be two nonempty sets. A ternary Γ-semigroup is an algebraic structure $(T, \Gamma, [\ ])$ if there exist a mapping $[\ ] : T \times \Gamma \times T \times \Gamma \times T \to T$, written as $(a, \alpha, b, \beta, c) \to [a\alpha b\beta c]$ satisfying the associative law:*

$$[[a\alpha b\beta c]\gamma d\delta e] = [a\alpha[b\beta c\gamma d]\delta e] = [a\alpha b\beta[c\gamma d\delta e]],$$

for all $a, b, c, d, e \in T$ *and* $\alpha, \beta, \gamma, \delta \in \Gamma$.

For the sake of simplicity, we will write $a\alpha b\beta c$ instead of $[a\alpha b\beta c]$, for each $a, b, c \in T$ and $\alpha, \beta \in \Gamma$. Let A, B and C be any nonempty subsets of a ternary Γ-semigroup T. We denote the set

$$A\Gamma B\Gamma C := \{a\alpha b\beta c \mid a \in A, b \in B, c \in C, \alpha, \beta \in \Gamma\}.$$

We now review the concepts of various kinds of Γ-ideals and fuzzy Γ-ideals in ternary Γ-semigroups that appeared in [2] in the following ways.

Definition 3. *[2] Let* A *be any nonempty subset of a ternary* Γ*-semigroup* T*. Then:*

(i) A *is called a ternary* Γ*-subsemigroup of* T *if* $A\Gamma A\Gamma A \subseteq A$*;*

(ii) A *is called a left (resp. right, lateral)* Γ*-ideal of* T *if* $T\Gamma T\Gamma A \subseteq A$ *(resp.* $A\Gamma T\Gamma T \subseteq A$, $T\Gamma A\Gamma T \subseteq A$*);*

(iii) A *is called a* Γ*-ideal of* T *if it is a left, a right, and a lateral* Γ*-ideal of* T*;*

(iv) a ternary Γ*-subsemigroup* A *of* T *is called a bi-*Γ*-ideal of* T *if* $T\Gamma A\Gamma T\Gamma A\Gamma T \subseteq A$.

Definition 4. *[2] Let* μ *be any fuzzy set of a ternary* Γ*-semigroup* T*. Then:*

(i) μ *is called a fuzzy ternary* Γ*-subsemigroup of* T *if* $\mu(a\alpha b\beta c) \geq \min\{\mu(a), \mu(b), \mu(c)\}$, *for all* $a, b, c \in T$ *and* $\alpha, \beta \in \Gamma$*;*

(ii) μ *is called a fuzzy left (resp. right, lateral)* Γ*-ideal of* T *if* $\mu(a\alpha b\beta c) \geq \mu(c)$ *(resp.* $\mu(a\alpha b\beta c) \geq \mu(a)$, $\mu(a\alpha b\beta c) \geq \mu(b)$*), for all* $a, b, c \in T$ *and* $\alpha, \beta \in \Gamma$*;*

(iii) μ *is called a fuzzy* Γ*-ideal of* T *if it is a fuzzy left* Γ*-ideal, a fuzzy right* Γ*-ideal, and a fuzzy lateral* Γ*-ideal of* T*;*

(iv) a fuzzy ternary Γ*-subsemigroup* μ *of* T *is called a fuzzy bi-*Γ*-ideal of* T *if* $\mu(a\alpha b\beta c\gamma d\delta e) \geq \min\{\mu(a), \mu(c), \mu(e)\}$, *for all* $a, b, c, d, e \in T$ *and* $\alpha, \beta, \gamma, \delta \in \Gamma$.

Let S and T be ternary Γ-semigroups with respect to the same set Γ. The mapping $\cdot : (S \times T) \times \Gamma \times (S \times T) \times \Gamma \times (S \times T) \to S \times T$ is defined by

$$(s_1, t_1)\alpha(s_2, t_2)\beta(s_3, t_3) = (s_1\alpha s_2\beta s_3, t_1\alpha t_2\beta t_3),$$

for all $(s_1, t_1), (s_2, t_2), (s_3, t_3) \in S \times T$ and $\alpha, \beta \in \Gamma$. Then $S \times T$ forms a ternary Γ-semigroup (see [2]).

3. Strongest fuzzy Γ-ideals on ternary Γ-semigroups

In this section, we introduce the concepts of strongest fuzzy ternary Γ-subsemigroups, strongest fuzzy (resp. left, right, and lateral) Γ-ideals, and strongest fuzzy bi-Γ-ideals on ternary Γ-semigroups. Then we study the relationships and characterizations of these concepts in ternary Γ-semigroups.

Definition 5. *Let T be a ternary Γ-semigroup, μ be a fuzzy set of T, and R_μ be a strongest fuzzy relation on T. Then R_μ is called a strongest fuzzy ternary Γ-subsemigroup on T if*

$$R_\mu(a_1\alpha b_1\beta c_1, a_2\alpha b_2\beta c_2) \geq \min\{R_\mu(a_1,a_2), R_\mu(b_1,b_2), R_\mu(c_1,c_2)\},$$

for all $a_1,a_2,b_1,b_2,c_1,c_2 \in T$ and $\alpha,\beta \in \Gamma$.

Definition 6. *Let T be a ternary Γ-semigroup, μ be a fuzzy set of T, and R_μ be a strongest fuzzy relation on T. Then R_μ is called:*

(i) a strongest fuzzy left Γ-ideal on T if $R_\mu(a_1\alpha b_1\beta c_1, a_2\alpha b_2\beta c_2) \geq R_\mu(c_1,c_2)$, for all $a_1,a_2,b_1,b_2,c_1,c_2 \in T$ and $\alpha,\beta \in \Gamma$;

(ii) a strongest fuzzy right Γ-ideal on T if $R_\mu(a_1\alpha b_1\beta c_1, a_2\alpha b_2\beta c_2) \geq R_\mu(a_1,a_2)$, for all $a_1,a_2,b_1,b_2,c_1,c_2 \in T$ and $\alpha,\beta \in \Gamma$;

(iii) a strongest fuzzy lateral Γ-ideal on T if $R_\mu(a_1\alpha b_1\beta c_1, a_2\alpha b_2\beta c_2) \geq R_\mu(b_1,b_2)$, for all $a_1,a_2,b_1,b_2,c_1,c_2 \in T$ and $\alpha,\beta \in \Gamma$;

(iv) a strongest fuzzy Γ-ideal on T if it is a strongest fuzzy left Γ-ideal, a strongest fuzzy right Γ-ideal, and a strongest fuzzy lateral Γ-ideal on T.

Definition 7. *Let T be a ternary Γ-semigroup, μ be a fuzzy set of T, and R_μ be a strongest fuzzy ternary Γ-subsemigroup on T. Then R_μ is said to be a strongest fuzzy bi-Γ-ideal on T if*

$$R_\mu(a_1\alpha b_1\beta c_1\gamma d_1\delta e_1, a_2\alpha b_2\beta c_2\gamma d_2\delta e_2) \geq \min\{R_\mu(a_1,a_2), R_\mu(c_1,c_2), R_\mu(e_1,e_2)\},$$

for all $a_1,a_2,b_1,b_2,c_1,c_2,d_1,d_2,e_1,e_2 \in T$ and $\alpha,\beta,\gamma,\delta \in \Gamma$.

By Definition 7, it is clear that every strongest fuzzy bi-Γ-ideal on a ternary Γ-semigroup is also a strongest fuzzy ternary Γ-subsemigroup, but the converse is not always true, as the following example.

Example 1. *Let $T = \{a,b,c\}$ and $\Gamma = T$. Define the operation $\cdot$ on T by $x\alpha y\beta z = (x*y)*z$, for all $x,y,z \in T$ and $\alpha,\beta \in \Gamma$ where the binary operation $*$ on T is defined by the following table:*

$*$	a	b	c
a	a	a	a
b	a	b	b
c	a	c	c

Then, T is a ternary Γ-semigroup [6]. Next, we define a fuzzy set μ of T by

$$\mu(a) = 0.2, \mu(b) = 0.5, \text{ and } \mu(c) = 0.9.$$

Following a careful analysis, we have R_μ is a strongest fuzzy ternary Γ-subsemigroup on T, but it is not a strongest fuzzy bi-Γ-ideal on T, since $R_\mu(c\alpha a\beta c\gamma a\delta c, c\alpha a\beta c\gamma a\delta c) = 0.2 < 0.9 = \min\{R_\mu(c,c)\, R_\mu(c,c), R_\mu(c,c)\}$, for all $\alpha,\beta,\gamma,\delta \in \Gamma$.

Proposition 1. *Let T be a ternary Γ-semigroup. Then:*

(i) every strongest fuzzy left Γ-ideal on T is also a strongest fuzzy bi-Γ-ideal;

(ii) every strongest fuzzy right Γ-ideal on T is also a strongest fuzzy bi-Γ-ideal;

(iii) every strongest fuzzy lateral Γ-ideal on T is also a strongest fuzzy bi-Γ-ideal;

(iv) every strongest fuzzy Γ-ideal on T is also a strongest fuzzy bi-Γ-ideal.

Proof. (i) Let R_μ be a strongest fuzzy left Γ-ideal on T. It is not difficult to verify that R_μ is a strongest fuzzy ternary Γ-subsemigroup on T. For any $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, e_1, e_2 \in T$, and any $\alpha, \beta, \gamma, \delta \in \Gamma$, we have

$$(a_1\alpha b_1)\beta(c_1\gamma d_1)\delta e_1 = x_1\beta y_1\delta e_1 \text{ and } (a_2\alpha b_2)\beta(c_2\gamma d_2)\delta e_2 = x_2\beta y_2\delta e_2,$$

for some $x_1 = a_1\alpha b_1$, $y_1 = c_1\gamma d_1$, $x_2 = a_2\alpha b_2$, and $y_2 = c_2\gamma d_2$. It follows that

$$\begin{aligned} R_\mu(a_1\alpha b_1\beta c_1\gamma d_1\delta e_1, a_2\alpha b_2\beta c_2\gamma d_2\delta e_2) &= R_\mu(x_1\beta y_1\delta e_1, x_2\beta y_2\delta e_2) \\ &\geq R_\mu(e_1, e_2) \\ &\geq \min\{R_\mu(a_1, a_2), R_\mu(c_1, c_2), R_\mu(e_1, e_2)\}. \end{aligned}$$

Hence, R_μ is a strongest fuzzy bi-Γ-ideal on T.

The proofs of (ii) and (iii) are similar to the proof of (i).

(iv) It is obvious.

The converses of statements in Proposition 1 don't have to be true as shown in the following example.

Example 2. *Let $T = \{a, b, c\}$ and $\Gamma = T$. Define the mapping $\cdot$ on T in Example 1. Now, we define a fuzzy set μ of T by*

$$\mu(a) = 0.7, \mu(b) = 0.7 \text{ and } \mu(c) = 0.2.$$

After a thorough examination, we obtain R_μ is a strongest fuzzy bi-Γ-ideal on T. Nevertheless, R_μ is not a strongest fuzzy left Γ-ideal on T, since

$$R_\mu(c\alpha c\beta b, c\alpha c\beta b) = 0.2 < 0.7 = R_\mu(b, b), \text{ for all } \alpha, \beta \in \Gamma.$$

Furthermore, it is not a strongest fuzzy lateral Γ-ideal on T either, because

$$R_\mu(c\alpha b\beta c, c\alpha b\beta c) = 0.2 < 0.7 = R_\mu(b, b), \text{ for all } \alpha, \beta \in \Gamma.$$

Next, we present the characterizations of strongest fuzzy ternary Γ-subsemigroups, strongest fuzzy (resp. left, right, lateral) Γ-ideals, and strongest fuzzy bi-Γ-ideals on ternary Γ-semigroups.

Theorem 1. *Let T be a ternary Γ-semigroup, μ be a fuzzy set of T, and R_μ be a strongest fuzzy relation on T. Then, μ is a fuzzy ternary Γ-subsemigroup of T if and only if R_μ is a strongest fuzzy ternary Γ-subsemigroup on T.*

Proof. Assume that μ is a fuzzy ternary Γ-subsemigroup of T. Let $a_1, a_2, b_1, b_2, c_1, c_2 \in T$ and $\alpha, \beta \in \Gamma$. Then, we have

$$\begin{aligned} R_\mu(a_1\alpha b_1\beta c_1, a_2\alpha b_2\beta c_2) &= \min\{\mu(a_1\alpha b_1\beta c_1), \mu(a_2\alpha b_2\beta c_2)\} \\ &\geq \min\{\min\{\mu(a_1), \mu(b_1), \mu(c_1)\}, \min\{\mu(a_2), \mu(b_2), \mu(c_2)\}\} \\ &= \min\{\min\{\mu(a_1), \mu(a_2)\}, \min\{\mu(b_1), \mu(b_2)\}, \min\{\mu(c_1), \mu(c_2)\}\} \\ &= \min\{R_\mu(a_1, a_2), R_\mu(b_1, b_2), R_\mu(c_1, c_2)\}. \end{aligned}$$

Thus, R_μ is a strongest fuzzy ternary Γ-subsemigroup on T.

Conversely, assume that R_μ is a strongest fuzzy ternary Γ-subsemigroup on T. Let $a, b, c \in T$ and $\alpha, \beta \in \Gamma$. Then, we have

$$\begin{aligned} \mu(a\alpha b\beta c) &= \min\{\mu(a\alpha b\beta c), \mu(a\alpha b\beta c)\} \\ &= R_\mu(a\alpha b\beta c, a\alpha b\beta c) \\ &\geq \min\{R_\mu(a, a), R_\mu(b, b), R_\mu(c, c)\} \\ &= \min\{\min\{\mu(a), \mu(a)\}, \min\{\mu(b), \mu(b)\}, \min\{\mu(c), \mu(c)\}\} \\ &= \min\{\mu(a), \mu(b), \mu(c)\}. \end{aligned}$$

Hence, μ is a fuzzy ternary Γ-subsemigroup of T.

Theorem 2. *Let T be a ternary Γ-semigroup, μ be a fuzzy set of T, and R_μ be a strongest fuzzy relation on T. Then the following statements hold:*

(i) *μ is a fuzzy left Γ-ideal of T if and only if R_μ is a strongest fuzzy left Γ-ideal on T;*

(ii) *μ is a fuzzy right Γ-ideal of T if and only if R_μ is a strongest fuzzy right Γ-ideal on T;*

(iii) *μ is a fuzzy lateral Γ-ideal of T if and only if R_μ is a strongest fuzzy lateral Γ-ideal on T;*

(iv) *μ is a fuzzy Γ-ideal of T if and only if R_μ is a strongest fuzzy Γ-ideal on T.*

Proof. (i) Assume that μ is a fuzzy left Γ-ideal of T. Let $a_1, a_2, b_1, b_2, c_1, c_2 \in T$ and $\alpha, \beta \in \Gamma$. Thus, we have

$$\begin{aligned} R_\mu(a_1\alpha b_1\beta c_1, a_2\alpha b_2\beta c_2) &= \min\{\mu(a_1\alpha b_1\beta c_1), \mu(a_2\alpha b_2\beta c_2)\} \\ &\geq \min\{\mu(c_1), \mu(c_2)\} \\ &= R_\mu(c_1, c_2). \end{aligned}$$

This implies that R_μ is a strongest fuzzy left Γ-ideal on T. Conversely, assume that R_μ is a strongest fuzzy left Γ-ideal on T. Let $a, b, c \in T$ and $\alpha, \beta \in \Gamma$. Then, we have

$$\mu(a\alpha b\beta c) = \min\{\mu(a\alpha b\beta c), \mu(a\alpha b\beta c)\}$$

$$
\begin{aligned}
&= R_\mu(a\alpha b\beta c, a\alpha b\beta c)\\
&\geq R_\mu(c,c)\\
&= \min\{\mu(c),\mu(c)\}\\
&= \mu(c).
\end{aligned}
$$

We obtain that μ is a fuzzy left Γ-ideal of T.

For the proofs of (ii) and (iii), we can prove similarly.

(iv) It follows by the conditions of (i), (ii), and (iii).

Theorem 3. *Let T be a ternary Γ-semigroup, μ be a fuzzy set of T, and R_μ be a strongest fuzzy relation on T. Then, μ is a fuzzy bi-Γ-ideal of T if and only if R_μ is a strongest fuzzy bi-Γ-ideal on T.*

Proof. Assume that μ is a fuzzy bi-Γ-ideal of T. Then μ is a fuzzy ternary Γ-subsemigroup of T. By Theorem 1, we get R_μ is a strongest fuzzy ternary Γ-subsemigroup on T. Let $a_1,a_2,b_1,b_2,c_1,c_2,d_1,d_2,e_1,e_2 \in T$ and $\alpha,\beta,\gamma,\delta \in \Gamma$. Thus, we have

$$
\begin{aligned}
&R_\mu(a_1\alpha b_1\beta c_1\gamma d_1\delta e_1, a_2\alpha b_2\beta c_2\gamma d_2\delta e_2)\\
&\quad= \min\{\mu(a_1\alpha b_1\beta c_1\gamma d_1\delta e_1), \mu(a_2\alpha b_2\beta c_2\gamma d_2\delta e_2)\}\\
&\quad\geq \min\{\min\{\mu(a_1),\mu(c_1),\mu(e_1)\}, \min\{\mu(a_2),\mu(c_2),\mu(e_2)\}\}\}\\
&\quad= \min\{\min\{\mu(a_1),\mu(a_2)\}, \min\{\mu(c_1),\mu(c_2)\}, \min\{\mu(e_1),\mu(e_2)\}\}\\
&\quad= \min\{R_\mu(a_1,a_2), R_\mu(c_1,c_2), R_\mu(e_1,e_2)\}.
\end{aligned}
$$

Hence, R_μ is a strongest fuzzy bi-Γ-ideal on T.

Conversely, assume that R_μ is a strongest fuzzy bi-Γ-ideal on T. Again, by Theorem 1, we have μ is a fuzzy ternary Γ-subsemigroup of T. Now, let $a,b,c,d,e \in T$ and $\alpha,\beta,\gamma,\delta \in \Gamma$. It follows that

$$
\begin{aligned}
\mu(a\alpha b\beta c\gamma d\delta e) &= \min\{\mu(a\alpha b\beta c\gamma d\delta e), \mu(a\alpha b\beta c\gamma d\delta e)\}\\
&= R_\mu(a\alpha b\beta c\gamma d\delta e, a\alpha b\beta c\gamma d\delta e)\\
&\geq \min\{R_\mu(a,a), R_\mu(c,c), R_\mu(e,e)\}\\
&= \min\{\mu(a),\mu(c),\mu(e)\}.
\end{aligned}
$$

Therefore, μ is a fuzzy bi-Γ-ideal of T.

In the following, we will write $T \times T$ instead of a ternary Γ-semigroup $T \times T$, where T is a ternary Γ-semigroup.

Theorem 4. *Let T be a ternary Γ-semigroup, μ be a fuzzy set of T, and R_μ be a strongest fuzzy relation on T. Then, R_μ is a strongest fuzzy ternary Γ-subsemigroup on T if and only if for every $t \in [0,1]$, $(R_\mu)_t$ is a ternary Γ-subsemigroup of $T \times T$ if it is nonempty.*

Proof. Assume that R_μ is a strongest fuzzy ternary Γ-subsemigroup on T. Let $t \in [0,1]$ be such that $(R_\mu)_t \neq \emptyset$, and let $(a_1,a_2),(b_1,b_2),(c_1,c_2) \in (R_\mu)_t$ and $\alpha,\beta \in \Gamma$. Then $R_\mu(a_1,a_2) \geq t, R_\mu(b_1,b_2) \geq t$, and $R_\mu(c_1,c_2) \geq t$. It turns out that

$$R_\mu(a_1\alpha b_1\beta c_1, a_2\alpha b_2\beta c_2) \geq \min\{R_\mu(a_1,a_2), R_\mu(b_1,b_2), R_\mu(c_1,c_2)\} \geq t.$$

This means that

$$(a_1,a_2)\alpha(b_1,b_2)\beta(c_1,c_2) = (a_1\alpha b_1\beta c_1, a_2\alpha b_2\beta c_2) \in (R_\mu)_t.$$

So, $(R_\mu)_t\Gamma(R_\mu)_t\Gamma(R_\mu)_t \subseteq (R_\mu)_t$. Hence, $(R_\mu)_t$ is a ternary Γ-subsemigroup of $T \times T$.

Conversely, for any $t \in [0,1]$, $(R_\mu)_t \neq \emptyset$ is a ternary Γ-subsemigroup of $T \times T$. Let $a_1, a_2, b_1, b_2, c_1, c_2 \in T$ and $\alpha, \beta \in \Gamma$. Choose $R_\mu(a_1,a_2) = t_1, R_\mu(b_1,b_2) = t_2$, and $R_\mu(c_1,c_2) = t_3$, for some $t_1, t_2, t_3 \in [0,1]$. Let $t = \min\{t_1,t_2,t_3\}$. Then, we have $(a_1,a_2), (b_1,b_2), (c_1,c_2) \in (R_\mu)_t$. By the hypothesis, we get

$$(a_1,a_2)\alpha(b_1,b_2)\beta(c_1,c_2) \in (R_\mu)_t\Gamma(R_\mu)_t\Gamma(R_\mu)_t \subseteq (R_\mu)_t.$$

Thus, $(a_1\alpha b_1\beta c_1, a_2\alpha b_2\beta c_2) = (a_1,a_2)\alpha(b_1,b_2)\beta(c_1,c_2) \in (R_\mu)_t$. This implies that

$$\begin{aligned} R_\mu(a_1\alpha b_1\beta c_1, a_2\alpha b_2\beta c_2) &\geq t = \min\{t_1,t_2,t_3\} \\ &= \min\{R_\mu(a_1,a_2), R_\mu(b_1,b_2), R_\mu(c_1,c_2)\}. \end{aligned}$$

Therefore, R_μ is a strongest fuzzy ternary Γ-subsemigroup on T.

Theorem 5. *Let T be a ternary Γ-semigroup, μ be a fuzzy set of T, and R_μ be a strongest fuzzy relation on T. Then the following statements hold:*

(i) R_μ is a strongest fuzzy left Γ-ideal on T if and only if for any $t \in [0,1]$, $(R_\mu)_t$ is a left Γ-ideal of $T \times T$ if it is nonempty;

(ii) R_μ is a strongest fuzzy right Γ-ideal on T if and only if for any $t \in [0,1]$, $(R_\mu)_t$ is a right Γ-ideal of $T \times T$ if it is nonempty;

(iii) R_μ is a strongest fuzzy lateral Γ-ideal on T if and only if for any $t \in [0,1]$, $(R_\mu)_t$ is a lateral Γ-ideal of $T \times T$ if it is nonempty;

(iv) R_μ is a strongest fuzzy Γ-ideal on T if and only if for any $t \in [0,1]$, $(R_\mu)_t$ is a Γ-ideal of $T \times T$ if it is nonempty.

Proof. (i) Assume that R_μ is a strongest fuzzy left Γ-ideal on T. Let $(a_1,a_2), (b_1,b_2) \in T \times T$ and $(c_1,c_2) \in (R_\mu)_t$, and $\alpha, \beta \in \Gamma$. Then

$$R_\mu(a_1\alpha b_1\beta c_1, a_2\alpha b_2\beta c_2) \geq R_\mu(c_1,c_2) \geq t.$$

We obtain that $(a_1,a_2)\alpha(b_1,b_2)\beta(c_1,c_2) = (a_1\alpha b_1\beta c_1, a_2\alpha b_2\beta c_2) \in (R_\mu)_t$; that is,

$$(T \times T)\Gamma(T \times T)\Gamma(R_\mu)_t \subseteq (R_\mu)_t.$$

This shows that $(R_\mu)_t$ is a left Γ-ideal of $T \times T$.

Conversely, assume that for any $t \in [0,1]$, $(R_\mu)_t \neq \emptyset$ is a left Γ-ideal of $T \times T$. Let $a_1, a_2, b_1, b_2, c_1, c_2 \in T$ and $\alpha, \beta \in \Gamma$. Take $R_\mu(c_1, c_2) = t$, for some $t \in [0,1]$. It follows that $(c_1, c_2) \in (R_\mu)_t$, and then $(R_\mu)_t \neq \emptyset$. By the given assumption, we have

$$(a_1\alpha b_1\beta c_1, a_2\alpha b_2\beta c_2) = (a_1, a_2)\alpha(b_1, b_2)\beta(c_1, c_2) \in (T \times T)\Gamma(T \times T)\Gamma(R_\mu)_t \subseteq (R_\mu)_t.$$

This implies that $R_\mu(a_1\alpha b_1\beta c_1, a_2\alpha b_2\beta c_2) \geq t = R_\mu(c_1, c_2)$. Therefore, R_μ is a strongest fuzzy left Γ-ideal on T.

The proofs of (ii) and (iii) can proved in a similar way.

(iv) It obtains from (i), (ii), and (iii).

Theorem 6. *Let T be a ternary Γ-semigroup, μ be a fuzzy set of T, and R_μ be a strongest fuzzy relation on T. Then, R_μ is a strongest fuzzy bi-Γ-ideal on T if and only if for each $t \in [0,1]$, $(R_\mu)_t$ is a bi-Γ-ideal of $T \times T$ when it is nonempty.*

Proof. Assume that R_μ is a strongest fuzzy bi-Γ-ideal on T. Let $t \in [0,1]$ be such that $(R_\mu)_t \neq \emptyset$. Let $(a_1, a_2), (c_1, c_2), (e_1, e_2) \in (R_\mu)_t$ and $(b_1, b_2), (d_1, d_2) \in T \times T$, and let $\alpha, \beta, \gamma, \delta \in \Gamma$. Thus, we have

$$R_\mu(a_1\alpha c_1\beta e_1, a_2\alpha c_2\beta e_2) \geq \min\{R_\mu(a_1, a_2), R_\mu(c_1, c_2), R_\mu(e_1, e_2)\} \geq t$$

and

$$R_\mu(a_1\alpha b_1\beta c_1\gamma d_1\delta e_1, a_2\alpha b_2\beta c_2\gamma d_2\delta e_2) \geq \min\{R_\mu(a_1, a_2), R_\mu(b_1, b_2), R_\mu(e_1, e_2)\} \geq t.$$

Also,

$$(a_1, a_2)\alpha(c_1, c_2)\beta(e_1, e_2) = (a_1\alpha c_1\beta e_1, a_2\alpha c_2\beta e_2) \in (R_\mu)_t$$

and

$$(a_1, a_2)\alpha(b_1, b_2)\beta(c_1, c_2)\gamma(d_1, d_2)\delta(e_1, e_2) = (a_1\alpha b_1\beta c_1\gamma d_1\delta e_1, a_2\alpha b_2\beta c_2\gamma d_2\delta e_2) \in (R_\mu)_t,$$

respectively. This shows that

$$(R_\mu)_t\Gamma(R_\mu)_t\Gamma(R_\mu)_t \subseteq (R_\mu)_t \text{ and } (R_\mu)_t\Gamma(T \times T)\Gamma(R_\mu)_t\Gamma(T \times T)\Gamma(R_\mu)_t \subseteq (R_\mu)_t.$$

Therefore, $(R_\mu)_t$ is a bi-Γ-ideal of $T \times T$.

Conversely, let $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, e_1, e_2 \in T$ and $\alpha, \beta, \gamma, \delta \in \Gamma$. Choose $R_\mu(a_1, a_2) = t_1$, $R_\mu(c_1, c_2) = t_2$, and $R_\mu(e_1, e_2) = t_3$, for some $t_1, t_2, t_3 \in [0,1]$. Let $t = \min\{t_1, t_2, t_3\}$. It turns out that $(a_1, a_2), (c_1, c_2), (e_1, e_2) \in (R_\mu)_t$. By assumption, we have $(R_\mu)_t$ is a bi-Γ-ideal of $T \times T$. So, we obtain

$$(a_1\alpha c_1\beta e_1, a_2\alpha c_2\beta e_2) = (a_1, a_2)\alpha(c_1, c_2)\beta(e_1, e_2) \in (R_\mu)_t$$

and

$$(a_1\alpha b_1\beta c_1\gamma d_1\delta e_1, a_2\alpha b_2\beta c_2\gamma d_2\delta e_2) = (a_1, a_2)\alpha(b_1, b_2)\beta(c_1, c_2)\gamma(d_1, d_2)\delta(e_1, e_2) \in (R_\mu)_t.$$

It means that

$$\begin{aligned} R_\mu(a_1\alpha c_1\beta e_1, a_2\alpha c_2\beta e_2) &\geq t = \min\{t_1, t_2, t_3\} \\ &= \min\{R_\mu(a_1, a_2), R_\mu(c_1, c_2), R_\mu(e_1, e_2)\} \end{aligned}$$

and

$$\begin{aligned} R_\mu(a_1\alpha b_1\beta c_1\gamma d_1\delta e_1, a_2\alpha b_2\beta c_2\gamma d_2\delta e_2) &\geq t = \min\{t_1, t_2, t_3\} \\ &= \min\{R_\mu(a_1, a_2), R_\mu(c_1, c_2), R_\mu(e_1, e_2)\}. \end{aligned}$$

Consequently, R_μ is a strongest fuzzy bi-Γ-ideal on T.

Example 3. *By Example 1, we obtain R_μ is a strongest fuzzy bi-Γ-ideal on a ternary Γ-semigroup T. It turns out that the set of all level subsets of R_μ are $(R_\mu)_{0.7} = \{(a,a),(a,b),(b,a),(b,b)\}$ and $(R_\mu)_{0.2} = T \times T$. By Theorem 6, we have $(R_\mu)_{0.7}$ and $(R_\mu)_{0.2}$ are bi-Γ-ideals of a ternary Γ-semigroup $T \times T$. This is the process of finding some bi-Γ-ideals of a ternary Γ-semigroup $T \times T$ using Theorem 6 such as the sets $\{(a,a),(a,b),(b,a),(b,b)\}$ and $T \times T$.*

Let X be a nonempty set, and μ be a fuzzy set of X. We observe that all level subsets of the strongest fuzzy relation χ_μ^A on X only include that the sets A and X, for each subset A of X. Therefore, we obtain the following results by Theorem 4, Theorem 5, and Theorem 6, respectively.

Corollary 1. *Let T be a ternary Γ-semigroup, μ be a fuzzy set of T, and A be a nonempty subset of T. Then, χ_μ^A is a strongest fuzzy ternary Γ-subsemigroup on T if and only if A is a ternary Γ-subsemigroup of T.*

Corollary 2. *Let T be a ternary Γ-semigroup, μ be a fuzzy set of T, and A be a nonempty subset of T. Then the following conditions hold:*

(i) χ_μ^A is a strongest fuzzy left Γ-ideal on T if and only if A is a left Γ-ideal of T;

(ii) χ_μ^A is a strongest fuzzy right Γ-ideal on T if and only if A is a right Γ-ideal of T;

(iii) χ_μ^A is a strongest fuzzy lateral Γ-ideal on T if and only if A is a lateral Γ-ideal of T;

(iv) χ_μ^A is a strongest fuzzy Γ-ideal on T if and only if A is a Γ-ideal of T.

Corollary 3. *Let T be a ternary Γ-semigroup, μ be a fuzzy set of T, and A be a nonempty subset of T. Then, χ_μ^A is a strongest fuzzy bi-Γ-ideal on T if and only if A is a bi-Γ-ideal of T.*

4. Conclusions

The concept of fuzzy relation was applied to define the notions of strongest fuzzy ternary Γ-subsemigroups, strongest fuzzy (resp. left, right, lateral) Γ-ideals, and strongest fuzzy bi-Γ-ideals on ternary Γ-semigroups. Following this, we investigated the connections of these concepts that every strongest fuzzy (resp. left, right, lateral) Γ-ideal is also a strongest fuzzy bi-Γ-ideal, while every strongest fuzzy bi-Γ-ideal is also a strongest fuzzy ternary Γ-subsemigroup on a ternary Γ-semigroup. In addition, as Example 1 and Example 2 indicate, the converses of the above mentioned relationships are not true. After that, we studied the links between different types of fuzzy Γ-ideals of ternary Γ-semigroups and their respective types of strongest fuzzy Γ-ideals on ternary Γ-semigroups, which occurred in Theorem 1, Theorem 2, and Theorem 3. Finally, the characterizations of strongest fuzzy ternary Γ-subsemigroups, strongest fuzzy (resp. left, right, lateral) Γ-ideals, and strongest fuzzy bi-Γ-ideals on ternary Γ-semigroups by the various types if their level subsets in ternary Γ-semigroups are presented in Theorem 4, Theorem 5, and Theorem 6. Future studies will be possible to investigate some decompositions of many types of strongest fuzzy Γ-ideals on ordered ternary Γ-semigroups or other algebraic structures.

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