EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 17, No. 4, 2024, 4050-4058
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

On L^∞ and L^2 Bounds for Weak Solutions of Time Flows for Certain Functionals of Linear Growth

Thomas Wunderli

Department of Mathematics and Statistics, The American University of Sharjah, Sharjah, United Arab Emirates

Abstract. We use recent approximation results in BV space to derive L^∞ bounds for u, L^∞ bounds in time for the BV seminorm $\int |Du|$ of u, and L^2 bounds for u_t for the weak solution $u \in C([0,\infty); L^2(\Omega) \cap BV(\Omega))$, $\Omega \subset \mathbb{R}^N$ open and bounded, of the time flow

$$\frac{\partial u}{\partial t} = div\nabla_p \varphi(x, Du) - \lambda(u - u_0),\ \lambda > 0,\ u(0, x) = u_0.$$

We assume Neumann boundary condition and $\varphi(x, p)$ is in a class of linear growth functions in p. Importantly, $\varphi(\cdot, p) \in L^1(\Omega)$ in contrast to the classical results stated in [1] where φ has a continuity assumption in the x variable. We also use the convergence of the solution above to derive an L^∞ bound for the solution u^* to the corresponding stationary problem, since $u(t) \to u^*$ in $L^1(\Omega)$.

2020 Mathematics Subject Classifications: 49Jxx, 35D30

Key Words and Phrases: Bounded variation, weak solution, variational problems, linear growth

1. Introduction

The theory of existence and qualitative properties on bounded, open $\Omega \subset \mathbb{R}^N$ of time flow problems of the form

$$\frac{\partial u}{\partial t} = div\nabla_p g(x, Du) \tag{1}$$

where $u \in L^2((0,T) : BV(\Omega) \cap L^2(\Omega))$ with initial data $u(0,\cdot) = u_0 \in L^2(\Omega)$, boundary data $u = h$ on $\partial\Omega$, and $g(x,p)$ convex in p with linear growth in p has been covered and summarized extensively in [1]. Since for each t, $u(t,\cdot) \in BV(\Omega)$ and that $W^1(\Omega) \subsetneq BV(\Omega)$ the divergence term on the right of the equation is not well defined. The solution has to be defined in the context of nonlinear semigroup theory as the authors do in the collection of results in [1]. In fact it is proved there that there is a solution to (1) in the sense of Definition 6.5 in [1] with initial and boundary conditions $u(0,x) = u_0(x)$ with

DOI: https://doi.org/10.29020/nybg.ejpam.v17i4.5328

Email address: twunderli@aus.edu (T. Wunderli)

$u_0 \in L^2(\Omega)$, $u(t,x) = h(x)$, and $h \in L^1(\partial\Omega)$. The solution $u \in C([0,T]; L^2(\Omega) \cap BV(\Omega))$, satisfies $u(0) = u_0$, $u'(t) \in L^2(\Omega)$, and $u'(t) = div \nabla_p g(x, Du)$ in $\mathcal{D}'(\Omega)$, that is, in the distributional sense. It is also assumed that g is continuous on $\overline{\Omega} \times \mathbb{R}^N$. Many of these results rely on the approximation of $\int_\Omega g(x, Du)$ for $u \in BV(\Omega)$ by $\int_\Omega g(x, \nabla u_k)\, dx$ for u_k smooth. However the approximation theorems assume continuity or lower semicontinuity of g in (x,p). We also note the more recent work of [17] for time flows in BV space using the Allen-Cahn equation.

In this work we use new approximation results (Proposition 1) to derive an L^∞ bound for u and an L^2 bound for u_t for the weak solution, as defined in [7] or [22], to the Neumann problem

$$\begin{cases} \frac{\partial u}{\partial t} = div \nabla_p \varphi(x, Du) - \lambda(u - u_0) & \text{in } (0,\infty) \times \Omega,\ \lambda > 0 \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } (0,\infty) \times \partial\Omega \\ u(0,x) = u_0(x) & \text{for } x \in \Omega,\ u_0 \in L^\infty(\Omega). \end{cases} \tag{2}$$

In fact, we show the weak solution to (2) satisfies $u \in L^\infty([0,\infty); BV(\Omega) \cap L^\infty(\Omega))$, $u_t \in L^2((0,\infty) \times \Omega)$. Importantly, while $\varphi(x,p)$ is also of linear growth, convex and C^2 in p, there is no continuity assumption in the x variable with only $\varphi(\cdot, p) \in L^1(\Omega)$. Using the L^∞ bound, as noted in Theorem 2 below, we easily prove an L^∞ bound for the solution to the corresponding time independent minimization problem of Theorem 1.

For the integrand φ we first assume:

(1) $\varphi : \Omega \times \mathbb{R}^N \to \mathbb{R}$, where $\varphi(x,p)$ is convex in p, that is

$$\varphi(x, \lambda_1 p_1 + \lambda_2 p_2) \leq \lambda_1 \varphi(x, p_1) + \lambda_2 \varphi(x, p_2)$$

for each $z \in \mathbb{R}, p_1, p_2 \in \mathbb{R}^N$, $0 \leq \lambda_1, \lambda_2 \leq 1$, $\lambda_1 + \lambda_2 = 1$,

(2) $\varphi(x,p) = \varphi(x, |p|)$ is radially symmetric in p, and is of the form

$$\varphi(x,p) = \begin{cases} g(x,p) & \text{if } |p| \leq \beta \\ \psi(x)|p| + k(x) & \text{if } |p| > \beta \end{cases} \tag{3}$$

for $k \in L^1(\Omega)$ and $\psi \in C(\overline{\Omega})$.

(3) φ is a Carathéodory function, with $\varphi(\cdot, p) \in L^1(\Omega)$ for each p.

From (2), φ is of linear growth in the p variable as in [1], that is

$$\lim_{|p| \to \infty} \frac{\varphi(x,p)}{|p|} = \psi(x).$$

We note that time flows and functionals defined on BV include applications starting with the early examples of total variation flow in [16] and elastic plastic deformation in [11] and [12]. In fact, solving the time flow and letting $t \to \infty$ for the solution $u(t)$ gives the solution u to the stationary problem in these cases. For example, in [6], a functional with an integrand of the form φ as in (3) has applications to anisotropic noise removal in image processing with the assumption $\varphi(\cdot, p) \in L^\infty(\Omega)$. The model used there is

$$\min_{u \in BV(\Omega)} \int_\Omega \varphi(x, Du) + \lambda/2 \left\| u - u_0 \right\|^2_{L^2(\Omega)}$$

$\lambda > 0$, where the solution u is taken to be the restored image and u_0 is the noisy or corrupted image. The integrand φ in [6] is

$$\varphi(x,p)=\begin{cases}\frac{1}{r(x)}|p|^{r(x)} & \text{if } |p|\leq 1\\ |p|-\frac{r(x)-1}{r(x)} & \text{if } |p|>1\end{cases}$$

with $1<\alpha\leq r(x)\leq 2$, $r\in L^{\infty}(\Omega)$, which corresponds to $k(x)=-\frac{r(x)-1}{r(x)}$ and $\psi(x)\equiv 1$ for (3). In addition, the authors provide numerical examples and prove existence results for the corresponding time flow, including the convergence of the time flow solution $u(t)$ to u in $L^1(\Omega)$ as $t\to\infty$, where u is the solution to the above minimization problem. We note that many of the results proved there are based on the specific form of φ used in [6]. In our case, we include the more general condition that $\varphi(\cdot,p)\in L^1(\Omega)$ and that φ takes a more general form than in [6].

2. Preliminary Results

We first recall Lemma 1 in [20]

Lemma 1. *Assume φ satisfies the conditions (1)-(3) above:*

$$\varphi(x,p)=\begin{cases} g(x,p) & \text{if } |p|\leq\beta\\ \psi(x)|p|+k(x) & \text{if } |p|>\beta,\end{cases}$$

with $\psi\in C\left(\overline{\Omega}\right)$, $\psi\geq 0$, $k(x)\in L^1(\Omega)$ for each $u\in L^1(\Omega)$. Also assume for some G

$$\varphi(x,p)=G(r_1(x),...,r_K(x),p) \text{ for all } p$$

where

$$G(z_1,...,z_K,p)=\begin{cases} g_1(z_1,...,z_K,p) \text{ if } |p|\leq\beta\\ z_K|p|+g_2(z_1,...,z_K) \text{ if } |p|>\beta\end{cases}$$

and where for each $|p|\leq\beta$, g_1 is C^1 in the variable $\mathbf{z}=(z_1...,z_K)\in U\subset\mathbb{R}^K$, U open, $r_i\in L^1(\Omega)$ each i, $(r_1(x),...,r_K(x))\in U$ a.e. x, and $|(\nabla_{\mathbf{z}}g_1)(\mathbf{z},p)|\leq C$, C independent of $(\mathbf{z},p)$. Note that $r_K(x)=\psi(x)$ and hence $z_k\geq 0$.

Then for all $u\in BV(\Omega)$ we have

$$\begin{aligned}\mathcal{G}(u) &= \int_{\Omega}\varphi(x,\nabla u)\,dx+\int_{\Omega}\psi(x)|D^s u|\\ &= \sup_{\left\{\phi\in C_0^1(\Omega,\mathbb{R}^N):|\phi(x)|\leq\psi(x) \text{ for all } x\in\Omega\right\}}\left\{-\int_{\Omega}u div\phi+\varphi^*(x,\phi(x))\,dx\right\},\end{aligned}$$

and hence $\mathcal{G}$ is lower semicontinuous in $L^1(\Omega)$.

In order to prove the bounds for the weak solution u, we need the following proposition to extend the approximation Lemma from [21] to include time dependence, which covers the case where we only have $\varphi(\cdot,p)\in L^1(\Omega)$.

Proposition 1. *If φ satisfies conditions in Lemma 1 and $\varphi(x,p) \geq 0$ for a.e. x, each p, then for each $u \in L^2([0,T];BV(\Omega) \cap L^2(\Omega))$, there exists a sequence $u_k \in L^2([0,T];W^{1,1}(\Omega) \cap C^\infty(\Omega) \cap L^2(\Omega))$ with*

$$\begin{aligned} \int_0^T \int_\Omega \varphi(x,Du_k)\,dxdt &\to \int_0^T \int_\Omega \varphi(x,Du)\,dt \text{ and} \\ u_k &\to u \text{ in } L^2([0,T]\times\Omega). \end{aligned}$$

If $\partial\Omega$ is Lipschitz, we can choose $u_k \in L^2([0,T];C^\infty(\overline{\Omega}))$.

Proof. We follow the proof in [21] (also see [9], [10] for the pure total variation case) with the same partition of unity Ω_i for Ω resulting in the partition $\{[0,T]\times\Omega_i\}$, with the standard smoothing

$$(\eta_\varepsilon * u)(t,x) = \int_{B_\varepsilon(x)} \eta_\varepsilon(x-y)u(t,y)\,dy$$

in the x variable only. Noting that each $[0,T]\times$support(ϕ_i) is compact, we choose

1. each $0<\varepsilon_i<\varepsilon$, $i \geq 1$
2. $\int_0^T \int_\Omega |\eta_{\varepsilon_i} * (u\phi_i) - u\phi_i|^2\,dx \leq \varepsilon 2^{-i}$
3. $\int_0^T \int_\Omega |\eta_{\varepsilon_i} * (u\nabla\phi_i) - u\nabla\phi_i|\,dx \leq \varepsilon 2^{-i}$
4. support $\eta_{\varepsilon_i} * (u\phi_i) \subset [0,T]\times\Omega_{i+2} - [0,T]\times\overline{\Omega}_{i-2}$.

Then for u_ε defined by $u_\varepsilon = \sum_{i=1}^\infty \eta_{\varepsilon_i} * (u\phi_i)$ we have $u_\varepsilon \to u$ in $L^2([0,T]\times\Omega)$. Passing to a subsequence of ε we have $u_\varepsilon \to u$ in $L^2(\Omega)$ for a.e. t. Thus for a.e. t,

$$\int_\Omega \varphi(x,Du) \leq \liminf_{\varepsilon\to0} \int_\Omega \varphi(x,Du_\varepsilon)\,dx.$$

Since $\varphi(x,p)\geq 0$, by Fatou's Lemma we have

$$\int_0^T \int_\Omega \varphi(x,Du) \leq \liminf_{\varepsilon\to0} \int_0^T \int_\Omega \varphi(x,Du_\varepsilon)\,dx. \tag{4}$$

For the above subsequence in ε, proceed as in the proof there to get for a.e. t, after taking the supremum over relevant $\phi \in C_0^1(\Omega,\mathbb{R}^N)$ with $|\phi(x)| \leq \psi(x)$ for each x

$$\begin{aligned} \int_\Omega \varphi(x,Du_\varepsilon) \leq\; & \int_\Omega \varphi(x,Du) + \int_\Omega \omega(\varepsilon_1)|\nabla u|\,dx \\ & +\omega(\varepsilon_1)\int_\Omega d|D^s u| + 2\beta|\psi|_\infty\varepsilon \\ & +(\sup_\phi II + \sup_\phi |III| + \sup_\phi |IV| + \omega(\varepsilon_1)|\psi|_\infty|\Omega|), \end{aligned}$$

where ω is a modulus of continuity for ψ with $\omega(t)\to 0$ as $t\to 0^+$ and II, III, IV are the same terms as in [21]. From the proof of the approximation Lemma in [21] and [10] we have $\int_0^T \sup_\phi II\,dt \to 0$ as $\varepsilon\to0$ since $u \in L^2([0,T];BV(\Omega)\cap L^2(\Omega))$, $\int_0^T \sup_\phi |III|\,dt \leq$

$|\psi|_\infty \varepsilon T$ from item 3, and $\int_0^T \sup_\phi |IV| \leq (\beta\varepsilon + 2\beta|\psi|_\infty \varepsilon)T$. Now integrate with respect to t to get

$$\begin{aligned}\int_0^T \int_\Omega \varphi(x, Du_\varepsilon)\, dt \;\; \leq \;\; & \int_0^T \int_\Omega \varphi(x, Du)\, dt + \int_0^T \int_\Omega \omega(\varepsilon_1)|\nabla u|\, dxdt \\ & + \omega(\varepsilon_1) \int_0^T \int_\Omega d|D^s u|\, dt + 2\beta|\psi|_\infty \varepsilon T + \int_0^T \sup_\phi II\, dt \\ & + [\varepsilon|\psi|_\infty + \beta\varepsilon + 2\beta|\psi|_\infty \varepsilon + \omega(\varepsilon_1)|\psi|_\infty\, |\Omega|]T.\end{aligned}$$

Send $\varepsilon \to 0$ to obtain

$$\limsup_{\varepsilon\to 0} \int_0^T \int_\Omega \varphi(x, Du_\varepsilon)\, dxdt \leq \int_0^T \int_\Omega \varphi(x, Du).$$

This combined with (4) proves the first part.

If $\partial\Omega$ is Lipschitz, using the fact that $L^2([0,T]; C^\infty(\overline{\Omega}))$ is dense in $L^2([0,T]; W^{1,1}(\Omega) \cap L^2(\Omega))$ (from a simple modification of Theorem 3, section 4.2 in [9]), a modification of Remark 2.2.8 in [7] and by noting from Lemma 1 in [18] that

$$\int_0^T \int_\Omega |\varphi(x, \nabla v) - \varphi(x, \nabla u)|\, dxdt \leq \|\psi\|_\infty \int_0^T \int_\Omega |\nabla v - \nabla u|\, dxdt$$

for each $u, v \in L^2([0,T]; W^{1,1}(\Omega))$, we can choose $u_k \in L^2([0,T]; C^\infty(\overline{\Omega}))$.

Remark 1. *We note that the assumption $\psi \in C(\overline{\Omega})$ is used here so that ψ is uniformly continuous, as the original assumption of $\psi \in C(\Omega) \cap L^\infty(\Omega)$ in Lemma from [21] was incorrect.*

3. Bounds for the Weak Solution

We recall the definition of a weak solution as used [22] or [7] for the time flow problem.

Definition 1. *We define the weak solution $u \in L^2([0,\infty); BV(\Omega) \cap L^2(\Omega))$ of the initial value Neumann problem*

$$\begin{cases} \frac{\partial u}{\partial t} = div \nabla_p \varphi(x, Du) - \lambda(u - u_0) & in\ (0,\infty) \times \Omega,\ \lambda > 0 \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & on\ (0,\infty) \times \partial\Omega \\ u(0,x) = u_0(x) & for\ x \in \Omega,\ u_0 \in L^\infty(\Omega) \end{cases} \tag{5}$$

to be the following: $u \in L^2([0,\infty) : BV(\Omega) \cap L^2(\Omega))$ with $u_t := \frac{\partial u}{\partial t} \in L^2(\Omega \times [0,\infty))$ is a weak solution of (5) if

$$\begin{aligned}\int_0^s \int_\Omega u_t(v-u)\, dxdt + \int_0^s \int_\Omega \varphi(x, Dv)\, dt + \int_0^s \int_\Omega (v-u_0)^2\, dxdt \geq \\ \int_0^s \int_\Omega \varphi(x, Du)\, dt + \int_0^s \int_\Omega (u-u_0)^2\, dxdt\end{aligned} \tag{6}$$

for all $v \in L^2([0,\infty) : BV(\Omega) \cap L^2(\Omega))$ for a.e. $s \in [0,\infty)$.

We now assume φ satisfies the coercivity condition (4) $\varphi(x,p) \geq c|p|$, $c > 0$, for a.e. x, each p.

We first note that the existence of a semigroup solution $u \in C([0,\infty); L^2(\Omega))$ with $u_t \in L^\infty((0,\infty); L^2(\Omega))$ is guaranteed by the standard theory of nonlinear semigroups for maximal monotone operators since the functional

$$\Phi(u) := \begin{cases} \int_\Omega \varphi(x, Du) + \frac{\lambda}{2}\int_\Omega (u-u_0)^2\,dx, \ \lambda > 0, & \text{for } u \in BV(\Omega) \cap L^2(\Omega) \\ \infty & \text{for } \quad u \in L^2(\Omega) \setminus BV(\Omega) \end{cases} \tag{7}$$

is convex and lower semicontinuous on $L^2(\Omega)$ due to Lemma 1, and is hence a maximal monotone operator. We also have

Theorem 1. *If φ satisfies the condition of Lemma 1 and the coercivity condition (4) , then problem*

$$\min_{u \in BV(\Omega) \cap L^2(\Omega)} \Phi(u)$$

for Φ defined by (7) has a unique solution.

Proof. This follows from standard results due to coercivity, lower semicontinuity of $\Phi(u)$ in $L^2(\Omega)$ from Lemma 1, compactness of BV, and strict convexity of Φ.

The semigroup solution $u(t)$ for (7) satisfies

$$\begin{aligned} u(0) &= u_0 \\ u(t) &\in D(\partial\Phi) \text{ for each } t > 0 \\ -u'(t) &\in \partial\Phi[u(t)] \text{ for a.e. } t \geq 0, \end{aligned}$$

where $\partial\Phi$ is the subdifferential of Φ (see for example [5], [8]). From the definition of $\partial\Phi$ it follows

$$\int_\Omega u_t(v - u(t))\,dx \ + \Phi(v) \geq \Phi(u(t)) \text{ for a.e. } t \geq 0$$

for each $v \in L^2(\Omega)$. We again note that necessity of lower semicontinuity of the $\Phi(u)$ term. For other recent cases where lower semicontinuity holds for functions defined on BV, see for example [2], [3], [13], [14], and [15].

In Theorem 2 we additionally prove $u \in L^\infty([0,\infty); BV(\Omega) \cap L^\infty(\Omega))$, $u_t \in L^2((0,\infty) \times \Omega)$ for the weak solution given in Definition 1.

Theorem 2. *If φ satisfies the assumptions of Lemma 1, the coercivity condition (4) , is C^2 in p, $\varphi(x,p) \geq \varphi(x,0)$ a.e. x for all p and $\partial\Omega$ Lipschitz, then there exists a weak solution u to (5) where $u \in L^\infty([0,\infty); BV(\Omega) \cap L^\infty(\Omega))$, $u_t \in L^2((0,\infty) \times \Omega)$ and*

$$\begin{aligned} \int_0^\infty \int_\Omega (u_t)^2\,dxdt + \int_\Omega \varphi(x, Du) &\leq \int_\Omega \varphi(x, Du_0) \textit{ for a.e. } t \in [0,\infty) \\ \|u\|_{L^\infty([0,\infty)\times\Omega)} &\leq C(\Omega)\,\|u_0\|_\infty . \end{aligned}$$

Proof. The proof essentially follows the earlier works of [6], [7], or [22], by considering the solution $u_\delta^\varepsilon \in L^2\left([0,\infty); H^1(\Omega)\right)$ to the approximation problem

$$\begin{cases} \frac{\partial u}{\partial t} = \varepsilon\Delta u + div\nabla_p\varphi(x,\nabla u) - \lambda(u-u_0^\delta) & \text{in } [0,T]\times\Omega \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } [0,T]\times\partial\Omega \\ u(0,x) = u_0^\delta(x) & \text{for } x\in\Omega,\ u_0^\delta \in BV(\Omega)\cap C^\infty\left(\overline{\Omega}\right). \end{cases}$$

We use the fact that u_δ^ε satisfies the form of the weak solution given in (6) with φ replaced by $\varphi(x,p)+\frac{\varepsilon}{2}|p|^2$ and $v\in L^2([0,\infty);H^1(\Omega))$, passing to limits $\varepsilon\to 0, \delta\to 0$ after obtaining the appropriate L^∞ and L^2 bounds, and finally using the Lipschitz assumption of $\partial\Omega$ and Proposition 1 to get (6) for $v\in L^2([0,\infty): BV(\Omega)\cap L^2(\Omega))$.

From the works cited in the proof of Theorem 2 above, the weak solution also satisfies

$$\int_0^s\int_\Omega u_t(v-u)\,dxdt + \int_0^s\int_\Omega \varphi(x,Dv)\,dt \geq \tag{8}$$
$$\int_0^s\int_\Omega \varphi(x,Du)\,dt - \lambda\int_0^s\int_\Omega (u-u_0)(v-u)\,dxdt.$$

It is then straightforward to show if u is a weak solution of (5) then for each $t>0$

$$\int_\Omega u_t(v-u)\,dxdt + \int_\Omega \varphi(x,Dv)\,dt \geq \tag{9}$$
$$\int_\Omega \varphi(x,Du)\,dt - \lambda\int_\Omega (u-u_0)(v-u)\,dxdt$$

and hence using Young's inequality for the last term on the right

$$\int_\Omega u_t(v-u)\,dxdt + \int_\Omega \varphi(x,Dv)\,dt + \int_\Omega (v-u_0)^2\,dxdt \geq \tag{10}$$
$$\int_\Omega \varphi(x,Du)\,dt + \int_\Omega (u-u_0)^2\,dxdt \quad \text{for each } v\in BV(\Omega)\cap L^2(\Omega).$$

Thus u also a semigroup solution. Letting $v=u+\lambda\phi$ for $\phi\in C_c^\infty(\Omega)$ in (10) and letting $\lambda\to 0^+$ and $\lambda\to 0^-$ we have

Corollary 1. *If u is a solution to (6) then we have*

$$\frac{\partial u}{\partial t} = div\nabla_p\varphi(x,\nabla u) - \lambda(u-u_0) \quad \text{in } [0,T]\times\Omega$$

in $\mathcal{D}'(\Omega)$.

4. Conclusion

In this work we proved L^∞ and L^2 bounds for weak solutions in BV for time flows (2) of the minimization problem from Theorem 1 for a class of integrands $\varphi(\cdot,p)\in L^1(\Omega)$; whereas most of the previous results include a continuity assumption in x. For future consideration, we may consider integrands φ that are not C^2 in the variable p as well as more general integrands g that are not specifically of the form φ as stated in this work, but with $g(\cdot,p)\in L^1(\Omega)$ and g convex p.

References

[1] F. Andreu-Vaillo, V. Caselles, José M. Mazón, *Parabolic quasilinear equations minimizing linear growth functionals,* Progress in Mathematics (Boston, Mass.) 223. Basel: Birkhuser (ISNB 3-7643-6691-2/HBK). xiv, 340 p. (2004).

[2] M. Baía, M. Chermisi, J. Matias, and P. M. Santos, *Lower semicontinuity and relaxation of signed functionals with linear growth in the context of $\mathcal{A}$-quasiconvexity.* Calc. Var. 47 (2013), 465-498.

[3] L. Beck, Thomas Schmidt, *Convex duality and uniqueness for BV minimizers,* J. Funct. Anal., Vol. 268 (2015) pp. 3061-3107.

[4] J. M. Borwein, A. S. Lewis, *Convex Analysis and Nonlinear Optimization: Theory and Examples (2 ed.)*, Springer, 2006, pp. 76-79.

[5] H. Brézis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert,* North-Holland Mathematics Studies, No.5, North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973.

[6] Y. Chen, S. Levine, M. Rao, *Variable exponent, linear growth functionals in image restoration,* SIAM J. Appl. Math, Vol. 66, No. 4, (2006), pp. 1383-1406.

[7] Chen, Y. and Wunderli, T., *Adaptive total variation for image restoration in BV space.* J. of Math. Anal. and Appl., 272, (2002), pp.117-137.

[8] I. Ekeland and R. Temam, *Convex analysis and variational problems*, Society for Industrial and Applied Mathematics, Philadelphia, 1999.

[9] L. Evans, R. Gariepy, *Measure theory and fine properties of functions,* CRC Press, Boca Raton, (1992).

[10] E. Giusti, *Minimal surfaces and functions of bounded variation,* Monogr. Math. 80, Birkhauser, Basel-Boston-Stuttgart (1984).

[11] R. Hardt and D. Kinderlehrer, *Elastic plastic deformation,* Appl. Math. Optim. 10 (1983), pp. 203–246.

[12] R.Hardt, X. Zhou, *An evolution problem for linear growth functionals*, Commun. Partial Differential Equations, 19 (1994), pp. 1879–1907.

[13] J. Kristensen and F. Rindler, *Relaxation of signed integral functionals in BV.* Calc. Var. 37 (2010), pp. 29-62.

[14] J. Kristensen, F. Rindler, *Characterization of generalised gradient young measures generated by sequences in $W^{1,1}$ and BV.* Archive for Rational Mechanics and Analysis 197 (2010), pp. 539-598.

[15] F. Rindler, G. Shaw, *Liftings, Young Measures, and Lower Semicontinuity,* Arch. Rational Mech. Anal. 232 (2019), pp. 1227–1328.

[16] L. Rudin, S. Osher, and E. Fatemi, *Nonlinear total variation based noise removal algorithms*, Phys. D 60, (1992), pp. 259–268.

[17] K. Tashiro, *Time-global existence of generalized BV flow via the Allen–Cahn equation.* Interfaces Free Bound. (2024),

[18] T. Wunderli, *On Functionals with Convex Carathéodory Integrands with a Linear Growth Condition,* Journal of Mathematical Analysis and Applications, 463 (2018), pp. 611-622.

[19] T. Wunderli, *Lower Semicontinuity and* Γ*-convergence of a Class of Linear Growth Functionals,* Nonlinear Analysis, 188 (2019), pp. 80-90

[20] T. Wunderli, *Lower Semicontinuity in L1 of a Class of Functionals Defined on BV with Carathéodory Integrands,* Abstract and Applied Analysis, 2021, Article ID 6709303

[21] T. Wunderli, *Approximation of BV space-defined functionals containing piecewise integrands with* L^1 *condition*, European Journal of Pure and Applied Mathematics, Vol. 16, No. 4, (2023), pp. 2025-2034.

[22] X. Zhou, *An evolution problem for plastic antiplanar shear*, Appl. Math. Optim., 25 (1992), pp. 263–285.