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Prime Labeling of Union of Some Graphs

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Abstract. A prime labeling of a graph G is a map from the vertex set of G, $V(G)$, to the set $\{1, 2, ..., |V(G)|\}$ such that any two adjacent vertices in the graph G have labels that are relatively prime. In this paper, we discuss when the disjoint union of some graphs is a prime graph.

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1. Introduction

A path P_m in a graph is an alternative sequence of vertices and edges with no repeated vertices, a cycle C_m in a graph is a path that begins and ends at the same vertex and a wheel graph W_m is formed by joining a single vertex, known as the apex vertex, to all vertices of a cycle C_m, these vertices are known as the rim vertices.

A bijective map f from the vertex set of a graph G to $\{1, 2, ..., |V(G)|\}$ such that $f(u)$ and $f(v)$ are relatively prime whenever u and v are adjacent in G is called a prime labeling (PL) of G and a graph G is called a prime graph (PG) if G has a PL. Entringer defined the PL that was introduced by Tout et. al. in [1]. Entringer conjectured that all trees could be prime labeled, a hypothesis supported by Haxell et. al. in [8] proving that all sufficiently large trees have this property. Seoud et. al. in [7] further contributed by providing necessary and sufficient conditions for a graph to admit a prime labeling. For more details about prime graphs see for example [2], [5], [6], [10].

In this paper, we discuss when the disjoint union of some graphs is a PG. We prove that $W_m \cup P_n$ is a PG if and only if m is even or n is odd. Also, we show that $C_{2n} \cup C_{2n} \cup W_{2m}$ and $C_{2n} \cup C_{2n} \cup C_{2n} \cup W_{2m}$ are PGs. Finally, we study some properties of the disjoint union between a complete graph and any graph such that this union is a PG. Readers are advised to refer to the appropriate references or sources for clarification on terms and concepts that have not been defined in the text in [3] and [4].

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2. Prime labeling of union of some graphs

In this section, we generalize a result in [11], we prove that $W_m \cup P_n$ is a PG if and only if m is even or n is odd. Also, we show that $C_{2n} \cup C_{2n} \cup W_{2m}$ and $C_{2n} \cup C_{2n} \cup C_{2n} \cup W_{2m}$ are PGs.

The following lemma imposes certain restrictions on the independence number of PGs.

Lemma 1. *[14] "For any PG G, we have* $\alpha(G) \geq \left[\frac{|V(G)|}{2}\right]$*."*

The authors in [11] proved that "the disjoint union of a PG of even order and a graph of order 3 is a PG." In the following theorem, we generalize this result.

Theorem 1. *Let* G_1 *and* G_2 *be PGs of orders* n *and* m *respectively. If for any prime* $p \leq m-1$*, we get* p *divides* n*, then* $G_1 \cup G_2$ *is a PG.*

Proof. Let $u_1, u_2, ..., u_n$ be the vertices of G_1, $v_1, v_2, ...v_m$ be the vertices of G_2,
$f : V(G_1) \longrightarrow \{1, 2, ..., n\}$ be a PL of G_1 and
$g : V(G_2) \longrightarrow \{1, 2, ..., m\}$ be a PL of G_2.
Define $h : V(G_1 \cup G_2) \longrightarrow \{1, 2, ..., n+m\}$ by

$$\begin{aligned} h(u_i) &= f(u_i) \text{ for all } 1 \leq i \leq n, \text{ and} \\ h(v_j) &= n + g(v_j) \text{ for all } 1 \leq j \leq m. \end{aligned}$$

If u_i and u_j are adjacent in G_1. Then $(h(u_i), h(u_j)) = (f(u_i), f(u_j)) = 1$ because f is a PL.
Suppose v_i and v_j are adjacent in G_2 and

$$d = (h(v_i), h(v_j)) = (n + g(u_i), n + g(u_j)).$$

Thus d divides $g(u_i) - g(u_j)$ and $|g(u_i) - g(u_j)| \leq m-1$. If $d > 1$, then d has a prime divisor say p. Therefore, $p \leq d \leq m-1$ and by assumption p divides n. But p divides $n + g(u_i)$ and p divides $n + g(u_j)$. Thus p divides $g(u_i)$ and p divides $g(u_j)$ and hence $(g(u_i), g(u_j)) \geq p$ which is a contradiction, because g is a PL. Therefore, $(h(v_i), h(v_j)) = 1$ and so h is a PL of $G_1 \cup G_2$.

Vaidya et. al. in [13] proved the following theorem

Theorem 2. *[13] "*$W_{2k} \cup P_m$ *is a PG."*

Next, we show when, in general, $W_m \cup P_n$ is a PG.

Theorem 3. $W_m \cup P_n$ *is a PG if and only if* m *is even or* n *is odd.*

Proof. We separate the proof in the following cases,

(i) Suppose m is odd and n is even. Let $m = 2k+1$ and $n = 2h$. Then

$$\alpha(W_m \cup P_n) = \alpha(W_m) + \alpha(P_n) = k + h < \left[\frac{|W_m \cup P_n|}{2}\right] = \left[\frac{2k+2+2h}{2}\right] = k+h+1.$$

By Lemma 1, we get $W_m \cup P_n$ is not a PG.

(ii) Suppose m is even. By Theorem 2, $W_m \cup P_n$ is a PG.

(iii) Suppose m and n are odd. Let u_0 be the apex vertex of W_m, $u_1, u_2, ..., u_m$ be the consecutive rim vertices of W_m and $v_1v_2...v_n$ be the path P_n and define $f : V(W_m \cup P_n) \longrightarrow \{1, 2, ..., m+n+1\}$ as follows:
$f(u_i) = \begin{cases} i+1 \ , \ 0 \le i \le 2 \\ i+2 \ , \ 3 \le i \le m \end{cases}$ and
$f(v_j) = \begin{cases} m+j+2 \ , \ 1 \le j \le n-1 \\ 4 \quad , \quad j = n \end{cases}$.
Since $f(u_0) = 1$, $f(u_0)$ is relatively prime to $f(u_i)$ for all $1 \le i \le m$. Also,

$$(f(u_2), f(u_3)) = (3, 5) = 1, (f(u_1), f(u_m)) = (2, n+2) = 1, \text{ because } m \text{ is odd.}$$

$$\text{Now, } (f(v_{n-1}), f(v_n)) = (m+n+1, 4) = 1, \text{ because } m+n+1 \text{ is odd.}$$

The labels assigned to adjacent vertices within the graph $W_m \cup P_n$ exhibit a property of being mutually prime because these labels are two consecutive integers. So f is a PL.

Theorem 4. *The disjoint union of two wheels is not a PG.*

Proof. Let W_n and W_m be any two wheels. Then

$$\begin{aligned} \alpha(W_n \cup W_m) &= \alpha(W_n) + \alpha(W_m) = \left[\frac{n}{2}\right] + \left[\frac{m}{2}\right] \\ &\le \left[\frac{n+m}{2}\right] < \left[\frac{|W_n \cup W_m|}{2}\right] = \left[\frac{n+m+2}{2}\right] = \left[\frac{n+m}{2}\right] + 1. \end{aligned}$$

By Lemma 1, we get $W_n \cup W_m$ is not a PG.

Patel et. al. in [9] proved that "the disjoint union of an even wheel and an even cycle is a PG." In Theorem 5, we prove that $C_{2n} \cup C_{2n} \cup W_{2m}$ and $C_{2n} \cup C_{2n} \cup C_{2n} \cup W_{2m}$ are PGs.

Theorem 5. *$C_{2n} \cup C_{2n} \cup W_{2m}$ and $C_{2n} \cup C_{2n} \cup C_{2n} \cup W_{2m}$ are PGs for all n, m.*

Proof. Let $u_1, u_2, ..., u_{2n}$ be the vertices of the first cycle, $u_{2n+1}, u_{2n+2}, ..., u_{4n}$ be the vertices of the second cycle, $u_{4n+1}, u_{4n+2}, ..., u_{6n}$ be the vertices of the third cycle, v_0 be the apex vertex of W_n and $v_1, v_2, ..., v_{2m}$ be the consecutive rim vertices of W_{2m}.

(i) To show that $C_{2n} \cup C_{2n} \cup W_{2m}$ is a PG. We have the following two cases:

(a) i. If 3 does not divide $n+1$, define
$f : V(C_{2n} \cup C_{2n} \cup W_{2m}) \longrightarrow \{1, 2, ..., 4n+2m+1\}$ as follows:

$$f(u_i) = i+2, \text{ for all } 1 \le i \le 4n,$$

$$\begin{aligned} f(v_j) &= j+1 \text{ for } j=0 \text{ and } 1, \\ f(v_j) &= 4n+j+1, \text{ for all } 2 \leq j \leq 2m. \end{aligned}$$

We get $(f(u_1), f(u_{2n})) = (3, 2n+2) = 1$, because 3 does not divide $n+1$. Also, $(f(u_{2n+1}), f(u_{4n})) = (2n+3, 4n+2) = 1$ because if $d = (f(u_{2n+1}), f(u_{4n}))$, then d divids $2n+3$ and hence d is odd and d divids $2(2n+3)-(4n+2) = 4$. Thus $d = 1$. Clearly, any other adjacent vertices have relatively prime labels. So, f is a PL.

ii. If 3 divides $n+1$, define $f : V(C_{2n} \cup C_{2n} \cup W_{2m}) \longrightarrow \{1, 2, ..., 4n+2m+1\}$ as follows:

$$\begin{aligned} f(u_i) &= i+3, \text{ for all } 1 \leq i \leq 4n-1, \\ f(u_{4n}) &= 3, \\ f(v_j) &= j+1 \text{ for } j=0 \text{ and } 1, \\ f(v_j) &= 4n+j+1, \text{ for all } 2 \leq j \leq 2m. \end{aligned}$$

Since 3 divides $n+1$, 3 does not divide $2n+4$ and $4n+2$. So, $(f(u_{2n+1}), f(u_{4n})) = (2n+4, 3) = 1$ and $(f(u_{4n-1}), f(u_{4n})) = (4n+2, 3) = 1$. It is clear that all other adjacent vertices have relatively prime labels. Therefore, f is a PL.

(ii) To show that $C_{2n} \cup C_{2n} \cup C_{2n} \cup W_{2m}$ is a PG. We have the following two cases:

(a) If 3 does not divide $4n+1$, define $f : V(C_{2n} \cup C_{2n} \cup C_{2n} \cup W_{2m}) \longrightarrow \{1, 2, ..., 6n+2m+1\}$ as follows

$$\begin{aligned} f(u_i) &= 6n+i \text{ for } i=1,2, \\ f(u_i) &= i \text{ for all } 3 \leq i \leq 6n, \\ f(v_j) &= j+1 \text{ for } j=0 \text{ and } 1, \\ f(v_j) &= 6n+j+1 \text{ for all } 2 \leq j \leq 2m. \end{aligned}$$

We have

$$(f(u_2), f(u_3)) = (6n+2, 3) = 1, \text{ because 3 does not divide } 6n+2,$$

$$(f(u_1), f(u_{2n})) = (6n+1, 2n) = 1, \text{ because } 1 = (6n+1) - 3(2n)$$

and

$$\begin{aligned} (f(u_{2n+1}), f(u_{4n})) &= (2n+1, 4n) = 1, \\ \text{because } 2 &= 2(2n+1) - 4n \text{ and 2 does not divide } 2n+1. \end{aligned}$$

Also,

$$\begin{aligned} (f(u_{4n+1}), f(u_{6n})) &= (4n+1, 6n) = 1, \\ \text{because } 3 &= 3(4n+1) - 2(6n) \text{ and 3 does not divide } 4n+1. \end{aligned}$$

Thus f is a PL.

(b) If 3 divides $4n+1$, define
$f : V(C_{2n} \cup C_{2n} \cup C_{2n} \cup W_{2m}) \longrightarrow \{1, 2, ..., 6n+2m+1\}$ as follows

$$\begin{aligned}
f(u_i) &= 6n+i \text{ for } i=1 \text{ and } 2,\\
f(u_{2n}) &= 4n,\\
f(u_{4n}) &= 6n,\\
f(u_{6n}) &= 2n,\\
f(u_i) &= i \text{ for all } i \neq 1, 2, 2n, 4n \text{ and } 6n,\\
f(v_j) &= j+1 \text{ for } j=0 \text{ and } 1,\\
f(v_j) &= 6n+j+1 \text{ for all } 2 \leq j \leq 2m.
\end{aligned}$$

Then,

$$\begin{aligned}
(f(u_1), f(u_{2n})) &= (6n+1, 4n) = 1,\\
\text{because } 2 &= 2(6n+1) - 3(4n) \text{ and } 6n+1 \text{ is odd},
\end{aligned}$$

$$(f(u_{2n-1}), f(u_{2n})) = (2n-1, 4n) = (2n-1, 2n) = 1,$$

$$\begin{aligned}
(f(u_2), f(u_3)) &= (6n+2, 3) = 1,\\
&\quad \text{because 3 does not divide } 6n+2.
\end{aligned}$$

$$\begin{aligned}
(f(u_{4n+1}), f(u_{6n})) &= (4n+1, 2n) = 1,\\
\text{because } 1 &= (4n+1) - 2(2n).
\end{aligned}$$

$$\begin{aligned}
(f(u_{6n-1}), f(u_{6n})) &= (6n-1, 2n) = 1,\\
\text{because } 1 &= 3(2n) - (6n-1).
\end{aligned}$$

Now, since $1 = 2(2n+1) - (4n+1)$ and 3 divides $4n+1$, 3 does not divide $2n+1$. Therefore,

$$(f(u_{2n+1}), f(u_{4n})) = (2n+1, 6n) = (2n+1, 2n) = 1.$$

Also, 3 does not divide $4n-1$ because 3 divides $4n+1$. Thus

$$(f(u_{4n-1}), f(u_{4n})) = (4n-1, 6n) = (4n-1, 2n) = 1.$$

Therefore f is a PL.

3. prime labeling of union of complete graphs and graphs with maximal size

In this section, we will study some properties of the disjoint union between a complete graph and any graph such that this union is a PG.
Seoud et. al. in [12] define a maximal PG as follows:

Definition 1. *[12] "A maximal PG is a PG of n vertices such that adding any new edge yields a non-PG. Usually this graph is denoted by $R(n)$."*

Theorem 6. *[14] "The largest complete subgraph in the maximal PG of n vertices is of order $\pi(n)+1$, where $\pi(n)$ is the number of primes less than or equal to n."*

Remark 1. *Let H be the largest complete subgraph in the maximal PG of n vertices. Then we can label the vertices of H by the primes less than or equal to n together with 1 namely, $1, p_1, p_2, ..., p_{\pi(n)}$. Also, we can replace the label p_i by p_i^k for some $k \geq 2$ and $p_i^k \leq n$ because for any $a \in \mathbb{Z}^+$, $(a, p_i) = 1$ if and only if $\left(a, p_i^k\right) = 1$.*

Theorem 7. *Suppose K_n is the complete graph of order n and G_m is any graph of order m such that $K_n \cup G_m$ is a PG. Then*

(i) $\pi(n+m) \geq n-1$.

(ii) $\alpha(G_m) \geq \left[\frac{n+m}{2}\right] - 1$.

Proof.

(i) By Theorem 6, $n = |V(K_n)| \leq \pi(n+m) + 1$. So, $\pi(n+m) \geq n-1$.

(ii) Since at most one of the vertices of K_n has even label, the set $S = \{u \in V(G_m) : \text{the label of } u \text{ is even}\}$ is an independent set of G_m with cardinality at least $\left[\frac{n+m}{2}\right] - 1$. So, $\alpha(G_m) \geq \left[\frac{n+m}{2}\right] - 1$.

Let G_m be a graph with maximum size such that $K_n \cup G_m$ be a PG. We will examine when G_m is connected. Firstly, we need the following lemma and corollary.

Lemma 2. *[4]"(Bonse's inequality) Let $k \geq 5$ and $p_1, p_2, ..., p_k$ be the first k primes. Then $p_{k+1}^2 < \prod_{i=1}^{k} p_i$ where p_{k+1} is the prime next to p_k."*

Also, if $k = 4$, then $p_k = 7$ and $p_{k+1} = 11$ and its clear $11^2 < (2)(3)(5)(7)$. So, we have the following corollary.

Corollary 1. *Let $k \geq 4$ and $p_1, p_2, ..., p_k$ be the first k primes. Then $p_{k+1}^2 < \prod_{i=1}^{k} p_i$ where p_{k+1} is the prime next to p_k.*

Theorem 8. *Let G_m be a graph with maximum size such that $K_n \cup G_m$ be a PG and $\pi(n+m) \geq n$. Then G_m is connected.*

Proof. Since $\pi(n+m) \geq n$, then the number of primes less than or equal to $n+m$ is greater than or equal to the number of vertices of K_n and these primes are mutually relatively prime. So, we can use a subset of these primes to label the vertices of K_n and hence one of the vertices of G_m will be labeled by 1. This vertex is adjacent to all other vertices of G_m, because G_m is a graph with maximum size such that $K_n \cup G_m$ is a PG. Thus, G_m is connected.

Theorem 9. *Let G_m be a graph with maximum size such that $K_n \cup G_m$ be a PG and $\pi(n+m) = n-1$. Then*

(i) G_m is the trivial graph ($m=1$) whenever $n+m=4$ or 5.

(ii) G_m is disconnected whenever $6 \leq n+m < 25$ or $30 \leq n+m < 49$.

(iii) G_m is connected whenever $25 \leq n+m < 30$ or $n+m \geq 49$.

Proof. By Remark 1, label the vertices of K_n by the primes less than or equal to $n+m$ together with 1 and label the vertices of G_m by the composite numbers less than or equal to $n+m$.

(i) If $n+m=4$, then $\pi(n+m)=2$. So $n=\pi(n+m)+1=3$. Thus $m=1$. Similarly, if $n+m=5$.

(ii) If $6 \leq n+m < 25$, then the vertex of G_m whose label is 6 must be an isolated vertex in G_m because any composite number less than 25 is not relatively prime to 6. Thus G_m is disconnected. If $30 \leq n+m < 49$, then any composite number less than 49 is not relatively prime to 30 So, 30 is isolated and thus G_m is disconnected.

(iii) Let $p_1, p_2, ..., p_k$ be the primes less than or equal $\sqrt{n}$ in ascending order. We refer to the vertices of G_m by their labels. We partition the vertices of G_m into the following sets
$$A_0 = \{p_1^2, p_2^2, ..., p_k^2\}$$
and
$$A_i = \{s : \ p_i \text{ does not divide } s\} - \bigcup_{j=0}^{j=i-1} A_j \quad \text{for all } i=1,2,...k.$$
Notice that $A_0, A_1, A_2, ..., A_k$ are mutually disjoint sets. We want to show that $G_m = \bigcup_{i=0}^{i=k} A_i$. Suppose there is a composite number t less than or equal to $n+m$ such that p_i divides t for all $i=1,2,...k$. If $25 \leq n+m < 30$, then 2 divides t, 3 divides t and 5 divides t. So, $t \geq 30$ which is a contradiction. If $n+m \geq 49$, then by Corollary 1 we

get $p_{k+1}^2 < \prod_{i=1}^{k} p_i$ where p_{k+1} is the prime next to p_k. So, $n+m < p_{k+1}^2 < \prod_{i=1}^{k} p_i < t$ because p_i divides t for all $i = 1, 2, ..., k$ which is a contradiction. Now, Let $u, v \in G_m$. We want to find a path between u and v and this shows that G_m is connected. We have the following cases:

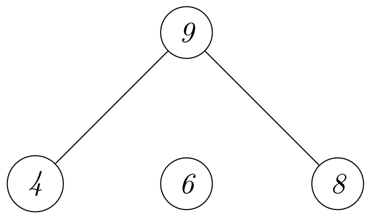
(a) If $u, v \in A_0$, then $u - v$ is a path in G_m.

(b) If $u, v \in A_i$ for some $i = 1, 2, ...k$, then $u - p_i^2 - v$ is a path in G_m.

(c) If $u \in A_i$ for some $i = 1, 2, ...k$ and $v \in A_j$ for some $j = 1, 2, ...k$ such that $i \neq j$, then $u - p_i^2 - p_j^2 - v$ is a path in G_m.

(d) If $u \in A_0$ and $v \in A_j$ for some $j = 1, 2, ...k$, then $u - p_j^2 - v$ is a path in G_m whenever $u \neq p_j^2$ and $u - v$ is a path in $G_m$whenever $u = p_j^2$.
Therefore, G_m is connected.

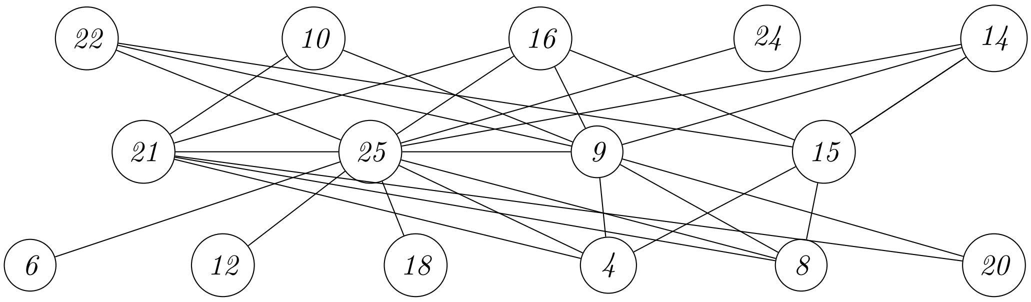
Example 1. *(i) Consider the complete graph K_5 and let G_4 be a graph with maximum size such that $K_5 \cup G_4$ is a PG.*
Then, $\pi(9) = 4 = 5 - 1$ and since $K_5 \cup G_4$ is a PG, we can label the vertices of K_5 by the numbers $1, 2, 3, 5, 7$ and hence G_4 is the following graph



G_4

So, G_4 is disconnected.

(ii) Consider the complete graph K_{10} and let G_{15} be a graph with maximum size such that $K_{10} \cup G_{15}$ is a PG.
Then, $\pi(25) = 9 = 10 - 1$ and since $K_{10} \cup G_{15}$ is a PG, we can label the vertices of K_{10} by the numbers $1, 2, 3, 5, 7, 11, 13, 17, 19, 23$ and hence G_{15} is the following graph



G_{15}

So, G_{15} is connected.

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