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Periodic Solutions of Strongly Nonlinear Oscillators Using He's Frequency Formulation

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Abstract. In this paper, we address several scientific and technological challenges with a novel He's non-perturbative approach (NPA), it simplifying processing time compared to traditional methods. The proposed approach transforms nonlinear ordinary differential equations (ODEs) into linear ones, analogous to simple harmonic motion, and producing a new frequency. Studying the periodic solutions leads to enhanced design, performance, reliability, and efficiency across these fields. This new approach is based mainly on the He's frequency formulation (HFF). This method yields highly accurate outcomes, surpassing well-known approximate methodologies, as validated through numerical comparisons in the Mathematical Software (MS). The congruence between numerical solution tests and theoretical predictions further supports our findings. While classical perturbation methods rely on Taylor expansions to simplify restoring forces, the NPA also enables stability analysis. Consequently, for analyzing approximations of highly non-linear oscillators in MS, the NPA serves as a more reliable tool.

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Key Words and Phrases: Nonlinear Oscillators, Non-perturbative Approach, Stability Analysis, Numerical Techniques

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1. Introduction

Differential equations, both linear and nonlinear, are used in many domains to characterize concerns appropriate to mathematics, physics, biology, chemistry, and engineering. The solutions of a linear ODE can be readily found using some of the well-established techniques, in comparison with nonlinear ODE, which are typically considered to have approximate solutions via several perturbation techniques. Moreover, nonlinear oscillations have drawn an interest of the increasing number of scientists since most vibration-related

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issues are of nonlinear features. Therefore, the nonlinear ODE is very useful in describing scientific and engineering phenomena, which often took the form of nonlinear types. For this reason, nonlinear ODEs are fundamental in engineering, physics, and applied mathematics [17]. It is important to highlight the role of mathematical computations in numerous research works of literature that deal with nonlinear ODEs that arise in diverse scientific and engineering areas [13]. While many nonlinear ODEs have numerical approximations, only insufficient of them have direct solutions. Many approximation analytical methods are applied in the literature to determine the connection between the amplitude and frequency of the non-linear oscillators. The perturbation approach, which is widely used to obtain approximate analytical solutions to nonlinear ODEs, is the most multipurpose tool for analyzing nonlinear engineering problems [37]. The explosive growth of nonlinear sciences over the past two decades led to a growing interest among scientists and engineers in analytical techniques for nonlinear challenges. It is created to investigate how these nonlinear ODEs behaved through the use of approximation techniques, both numerical and otherwise [18, 19, 21, 23, 24, 28, 39]. Recently, several new methods have been developed for solving the nonlinear ODEs analytically [9–11, 36]. Consequently, numerous researchers created a few unique techniques. To achieve analytical responses that are approximately near the exact solutions, a number of scientists were explored a range of innovative and unique approaches. The Lindstedt-Poincaré methodology was conducted [14]. Among these methods is the homotopy perturbation method (HPM) [6].

Because nonlinearity exists in many practical aspects, it has been difficult for investigators and physicists to arrive at a precise solution, or even one that was very close to the exact one. Weak non-linear ODEs were demonstrated using the averaging method and the least factor methodology, respectively [12]. In the situation of low-intensity noises, the exact asymptotic equations were calculated using the HPM. The method of multiple time scales was applied to determine the solutions of the oscillation systems. Finding the minor parameter necessary to describe the fundamental equations in a more realistic and useful manner was a crucial first step in any perturbation or asymptotic method. Many HPM-based methods were become more popular recently for estimating a wide range of nonlinear ODEs with the initial and arbitrary constants and getting them somewhat closer to their closed-form solutions [3, 7]. Consequently, the procedure would diverge and fail to produce the intended results if the original guess did not match the approach employed to solve the problem. These techniques are dependent on the initial approximation solution. Analytical approximations for magnetic spherical pendulums were analyzed using the HPM [20]. In the case of non-conservative oscillators, the HPM proved to be difficult to operate even with the seeming many advancements. To solve the non-linear oscillators, Prof. He proposed a simple but effective approach named HFF [4]. The HFF was used by many authors with great success. When dealing with nonlinear oscillator problems, the HFF method was a simple and efficient way to create a conservation nonlinear oscillator [40]. Under generalized beginning conditions, an analysis of the Duffing oscillator (DO) in vibration periodic behavior was conducted [15]. Numerical testing showed that the HFF was practically beneficial, physically perceptive, and mathematically simple. Engineers can use the novel method in quickly and accurately analyze nonlinear vibration systems utilizing the HFF. It was proposed to separate the oscillators into two extreme situations in order to change the HFF as a new method [16]. When the approximate and exact frequencies for various amplitudes were compared, there was good agreement. The Hamiltonian-based on the HFF was addressed as a remarkable accomplishment since it made a complex nonlinear vibration system easily comprehensible through computing. The cubic-quintic DO was given as an example to show how remarkably accurate and straightforward calculation. A simple approach to the cubic-quintic DO was described [5]. The method provided a very efficient and reasonably accurate means of estimating the frequency of a nonlinear conservative oscillator. The simplified HFF of nonlinear oscillators was presented and demonstrated [2]. A fractal vibration was shown in a porous medium, and its low-frequency characteristic was explained by the HFF. In a fractal space, it was discovered that the inertia force was equivalent to a few damping and inertia forces. A modification was suggested, and the most straightforward frequency formulation for nonlinear oscillators was presented and validated [1]. It was demonstrated that the inertia force for the standard differential model is equal to the total of the damping force and the inertia force in a fractal space. In the event of any non-linearity, a modification to the HFF was recommended [38]. A simple frequency formula for fractal systems, obtained from the HFF, was examined [22]. The coupled simple calculation with reliable results produced a valuable instrument of detailed study of fractal vibration phenomena. The non-Newtonian fluids were essential in numerous fields, including manufacture and technology. Investigation into these fluids was therefore quite appealing. The work of nonlinear stability assessment was directed towards some of the non-Newtonian fluids. A representation was constructed for the composition of a flat disturbed interface as well as some dynamical systems. Using the NPA, as an addition performance, was the main goal of the theoretical inquiry [25–27, 29–35]. In the current study, we will suggest an alternative form of the HFF to analyze some highly non-linear oscillators.

The NPA is a key technique in the current study, for locating the analytical approximation solutions of some nonlinear ODEs. Comparing the numerical solution (NS) and the present results to show the effectiveness and precision of the NPA. It is evident that the present method yields more precise results than similar the approximations for the previously discussed problems. The NPA has wonderful potential and may be applied to address additional substantially nonlinear situations, as is proven at the conclusion. When combined, the NPA resolves several real-world situations. Previously, these real-world cases were resolved with other well-known analytical methods that were previously documented in the literature. However, the best results are obtained faster using our existing methods. On the other hand, the calculations using MS assistance are even more straightforward with the NPA than they are with other analytical techniques, and the procedures for determining the analytical solutions are fully illustrated. Other calculation methods were extensive to use or required a lengthy time to analyze the solution. Regarding the unique approach used or notable results, the following details have been emphasized:

(i) The current non-linear ODE is equivalent to the alternate comparable linear one.

(ii) These two equations match each other exactly when using the NS.

(iii) Every traditional method employs expansion of Taylor to simplify the challenge of restoring forces. The current NPA eliminates this issue

(iv) In contrast to other traditional methods, the NPA enables us to investigate the stability analysis of the problem.

(v) The new technique seems to be a simple, convenient, and intriguing instrument. It can be utilized for studying various categories of nonlinear oscillators. Finally, the NPA is beneficial in the fields of science, technology, and applied research as it can easily be adjusted to address various nonlinear issues.

There are five sections in the current paper, which help readers to clarify how it is accessible. We demonstrate and briefly review the NPA description in § 2. Using the NPA, few nonlinear ODEs from the actual world are examined in § 3. In § 4, a summary is given of the dissections of the current study. Finally, § 5 provides a summary of the final explanations.

2. Explanation of the NPA

Consider a highly non-linear ODE, up to the third order, of the form:

$$\eta'' + F(\eta, \eta', \eta'') + G(\eta, \eta', \eta'') = H(\eta, \eta', \eta''), \tag{1}$$

where both $F(\eta, \eta', \eta'')$ and $G(\eta, \eta', \eta'')$ are functions of the third order. Simultaneously, $H(\eta, \eta', \eta'')$ is a quadratic function. Generally, these functions may be represented as:

$$\left.\begin{array}{r} F(\eta, \eta', \eta'') = a_1\eta' + b_1\eta\eta'\eta'' + c_1\eta^2\eta'' + d_1\eta'^3 + e_1\eta''\eta'^2 \\ G(\eta, \eta', \eta'') = \omega^2\eta + b_2\eta^2\eta' + c_2\eta\eta'^2 + d_2\eta^3 + e_2\eta''\eta^2 \\ H(\eta, \eta', \eta'') = a_3\eta\eta' + b_3\eta'^2 + c_3\eta^2 + d_3\eta'\eta'' + e_3\eta\eta'' \end{array}\right\}, \tag{2}$$

where $a_j,\ b_j,\ c_j,\ d_j,\ e_j\ (j = 1, 2, 3)$ are constant coefficients, and ω represents the natural frequency of the given structure.

As previously demonstrated throughout the conventional perturbation techniques [37], the first two functions yield secular terms, but the third function does not generate secular terms. Now, as well-known, the NPA aims mainly to obtaining an alternative linear ODE. Three constants will be established in order to create the necessary linear differential equation for this objective. It should be noted that the intention in generating the linear ODE is to behave similar to the simple harmonic motion. To accomplish this aim, following He [4], where a guessing solution of the given nonlinear ODE in the form:

$$x = A\ \cos\Omega t, \tag{3}$$

the initial conditions (ICs) are given as: $x(0) = A$, and $x'(0) = 0$.

The parameter Ω represents the overall frequency, which will be specified at a later time.

The required linear differential equation may be expressed as:

$$x'' + \sigma_{eqv} x' + \omega_{eqv}^2 x = \Lambda, \tag{4}$$

As previously shown [1], the above three parameters may be evaluated as follows:

$$\sigma_{eqv} = \int_0^{2\pi/\Omega} x' F\left(x, x', x''\right) dt \Big/ \int_0^{2\pi/\Omega} x'^2 dt = \sigma_{eqv}(\Omega). \tag{5}$$

Consider a comparable frequency ω_{eqv}^2 , which can be calculated using the total frequency according to the following function:

$$\omega_{eqv}^2 = \int_0^{2\pi/\Omega} x \; G\left(x, x', x''\right) dt \Big/ \int_0^{2\pi/\Omega} x^2 dt = \omega_{eqv}^2(\Omega). \tag{6}$$

It should be understood that the non-secular portion includes the quadratic formula. Therefore, the inhomogeneity will be computed by replacing: $\eta \to kA, \eta' \to kA\Omega, \; and \; \eta'' \to kA\Omega^2$ in the even non-secular function $H\left(\eta, \eta', \eta''\right)$. As shown by El-Dib [1], the parameter k is defined as $k = 1/2\sqrt{n-r}$, where n indicates the order of the system and r signifies the degree of freedom of the system. Therefore, in the present case, one gets $n = 2$ and, $r = 1$ then the value of k becomes $k = 1/2$. It follows that the value of the quadratic (non-secular term). Consequently, the inhomogeneity part will be computed by replacing: $\eta \to \frac{A}{2}, \; \eta' \to \frac{A\Omega}{2}, and \;\; \eta'' \to \frac{A\Omega^2}{2}$.

For simplicity, Eq. (4) can be stated in the ordinary normal form using the following substitution:

$$x(t) = f(t) Exp\left(-\sigma_{eqv} t/2\right). \tag{7}$$

Putting Eq. (7) into Eq. (4) yields

$$f'' + \left(\omega_{eqv}^2 - \frac{1}{4}\sigma_{eqv}^2\right) f = \Lambda \; Exp\left(\sigma_{eqv} t/2\right). \tag{8}$$

Finally, the overall frequency is expressed as $\Omega^2 = \omega_{eqv}^2 - \frac{1}{4}\sigma_{eqv}^2$.

3. Applications

In this section, we will test the validity of the NPA as previously indicated to analyze some different illustrations that characterize some highly nonlinear constructions. The physical interpretation with plot structure agrees well with the proposed technique.

3.1. Application 1

Consider the motion of a particle of mass m moving without friction along a circle of radius R and center, which is located at the origin 0 and rotating with uniform angular velocity Ω about its vertical diameter [37]. The forces acting on the particle are the gravitational force mg , the centrifugal force becomes $m\Omega^2 R\sin\theta$, where θ is the central angle between the radius that containing the particle and the vertically downwards. The moments about the origin and equation their sum to the rate of change of the angular momentum of the particle about the origin 0 are reflected. Following Nayfeh and Mook [37], the equation that governs motion can be simplified as follows:

$$\ddot{\theta} - \Omega^2 \sin\theta\cos\theta + g\sin\theta/R = 0. \tag{9}$$

In the previous example, the nonlinearity comes due to both the inertia and large deformation.

Eq. (9) can be written as follows:

$$\ddot{\theta} + f(\theta) = 0, \tag{10}$$

where $f(\theta) = -\Omega^2 \sin\theta\cos\theta + g\sin\theta/R$.

It should be noted that the previous function represent an odd function without any damping terms. As previously seen, assuming a predicting solution, where the guessing solution is given by

$$u = A\cos\Omega t, \quad ICs \quad u(0) = A, \ and \ \ \dot{u}(0) = 0. \tag{11}$$

The equivalent frequency may be determined from the following integration:

$$\omega_{eqv}^2 = \int_0^{2\pi/\Omega} u f(u)\, dt \ \Big/ \int_0^{2\pi/\Omega} u^2 dt. \tag{12}$$

By means of the MS, Eq. (12) yields

$$\omega_{eqv}^2 = \frac{1}{AR}\left(2gJ_1(A) - R\Omega^2 J_1(2A)\right), \tag{13}$$

where $J_1(A)$, and $J_1(2A)$ are the Bessel function in the arguments A, and $2A$ respectively.

Since there is no equivalent damping, in considered problem, it follows that the total frequency becomes

$$\Omega^2 = \frac{1}{AR}\left(2gJ_1(A) - R\Omega^2 J_1(2A)\right). \tag{14}$$

The equivalent linear differential equation is then given as:

$$\ddot{u} \ + \Omega^2 u = 0. \tag{15}$$

The stability criterion requires that $\Omega^2 > 0$. The following figure displays this criterion for various values of the radius of the circle.

Fig. 1 shows that the increasing of the radius has a destabilizing influence. As seen, the small values of the radius of the circle improvement a larger stability zones. For more convenience, with the aid of the MS, a matching between the original nonlinear differential equation as given in Eq. (9) with the equivalent linear ODE as given in Eq. (15) can be drawn for the sample chosen system $\Omega = 0.5,\ g = 12,\ R = 14$, and $A = 0.5$:
From the NS in Fig. 2, it is found that the absolute error between the two solutions is 0.00367. It should be noted that, in contrast with the traditional perturbation techniques, the present NPA does not use the Taylor expansion for the restoring forces. Additionally, it enables us to discuss the stability criterion.

3.2. Application 2

A simple pendulum with viscous damping, the governing equation of motion is given by [37]:

$$\ddot{\theta} + 2\mu\dot{\theta} + \omega^2 \sin\theta = 0. \tag{16}$$

Eq. (16) can be rewritten as follows:

$$\ddot{\theta} + f_1(\dot{\theta}) + f_2(\theta) = 0, \tag{17}$$

where $f_1(\dot{\theta}) = 2\mu\dot{\theta}$, and $f_2(\theta) = \omega^2 \sin\theta$

Assuming that the guessing solution is given by

$$w = A\cos\Omega t, \quad ICs \quad w(0) = A, \quad and \quad \dot{w}(0) = 0. \tag{18}$$

The equivalent frequency may be determined from the following integration:

$$\omega_{eqv}^2 = \int\limits_0^{2\pi/\Omega} w f_2(w)\, dt \;/ \int\limits_0^{2\pi/\Omega} w^2 dt. \tag{19}$$

By means of the MS, Eq. (19) yields

$$\omega_{eqv}^2 = \frac{2\omega^2}{A} J_1(A). \tag{20}$$

The equivalent damping may be evaluated from the integral form:

$$\sigma_{eqv} = \int\limits_0^{2\pi/\Omega} \dot{w}\ f_1(\dot{w})\, dt / \int\limits_0^{2\pi/\Omega} \dot{w}^2 dt = 2\mu. \tag{21}$$

From Eqs. (20) and (21), it follows that the equivalent linear ODE may be written as follows:

$$\ddot{w} + 2\mu\dot{w} + \omega_{eqv}^2 = 0. \tag{22}$$

From the standard normal form, it follows that the equivalent linear ODE is then given as

$$\ddot{w} + \Omega^2 w = 0. \tag{23}$$

The stability criterion requires that $\Omega^2 = \frac{2\omega^2}{A} J_1(A) - \mu^2$. This means that $\frac{2\omega^2}{A} J_1(A) - \mu^2 > 0$, or $\mu^2 < \frac{2\omega^2}{A} J_1(A)$. For more convenience, with the aid of the MS, a matching between the original non-linear ODE as given in Eq. (16) with the equivalent linear differential equation as given in Eq. (22) can be drawn for the sample chosen system $\omega = 2.0,\ \mu = 0.1$,and $A = 0.5$:

From the NS in Fig. 3, it is found that the absolute error between the two solutions is 0.0345. It should be noted that, in contrast with the traditional perturbation techniques, the present NPA enables us to discuss the stability criterion.

The stability criterion may be plotted as follows:

From Fig. 4, it is found that the stable regions increase as the frequency ω increases. Therefore, this parameter has a stabilizing influence.

3.3. Application 3

A system composed of a mass on a spring with cubic and quantic nonlinearity is described by a strongly non-linear oscillator with a cubic and harmonic restoring force equation, as illustrated in Figure 5, where k is the linear stiffness coefficients, M is the mass, $b \sin x$ is the driving force and $x(t)$ is the system response.

The strongly non-linear oscillator with cubic/quantic and harmonic restoring force is modeled mathematically by the following non-linear ODE [8]:

$$\ddot{x} + x + ax^3 + cx^5 + b \sin x = 0, \tag{24}$$

where $a,\ b$, and c are constants and the dot denotes to the time derivative.

The ICs are assumed as follows: $x(0) = A$, and $\dot{x}(0) = 0$.

Eq. (24) can be rewritten as follows:

$$\ddot{x} + f(x) = 0, \tag{25}$$

where $f(x) = x + ax^3 + cx^5 + b \sin x$

Assuming that the guessing solution is given by

$$h = A \cos \Omega t, \quad ICs \quad h(0) = A, \quad and \quad \dot{h}(0) = 0. \tag{26}$$

The equivalent frequency may be determined from the following integration:

$$\omega_{eqv}^2 = \int_0^{2\pi/\Omega} h f(h) dt \ / \int_0^{2\pi/\Omega} h^2 dt. \tag{27}$$

By means of the MS, Eq. (27) yields

$$\omega_{eqv}^2 = 1 + \frac{3}{4} a A^2 + \frac{5}{8} c A^4 + \frac{2b}{A} J_1(A). \tag{28}$$

Since the original nonlinear differential is independent of damping terms, the forgoing equivalent frequency represents the same total frequency. Therefore, the stability criterion becomes:

$$\Omega^2 = 1 + \frac{3}{4}aA^2 + \frac{5}{8}cA^4 + \frac{2b}{A}J_1(A) > 0 \tag{29}$$

The equivalent linear ODE becomes:

$$\ddot{h} + \Omega^2 h = 0. \tag{30}$$

For more convenience, with the aid of the MS, a matching between the original non-linear ODE as given in Eq. (24) with the equivalent linear ODE as given in Eq. (30) can be drawn for the sample chosen system $a = 1,\ b = 1,\ c = 1$, and $A = 0.5$:

From the NS in Fig. 6, it is found that the absolute error between the two solutions is 0.00948421. It should be noted that, in contrast with the traditional perturbation techniques, the present NPA enables us to discuss the stability criterion.

Table 1: validates the convergence of the numerical and NPA solutions.

t	Numerical	Approximate	Absolute error
0	0.5	0.5	0
5	0.21334	0.215293	0.00215861
10	-0.31025	-0.314596	0.00434554
15	-0.486305	-0.486214	0.00009056
20	-0.105488	-0.104118	0.00137032
25	0.391676	0.396551	0.00487468
30	0.446169	0.445617	0.000552448
35	-0.0072927	-0.0127984	0.00550568
40	-0.452727	-0.456639	0.0039112
45	-0.382264	-0.380446	0.00181781
50	0.119721	0.129009	0.00928869

Additionally, Table 1 displays the absolute error between the numerical and approximate solutions. As seen, this Table demonstrates that the changes between the two solutions are very small.

3.4. Application 4

Consider the system generated by Nayfeh and Mook [37], the governing equation of motion is may be simplified as follows:

$$\ddot{u} + \mu \sin \dot{u} + u = 0. \tag{31}$$

In the abovementioned example, the nonlinearity arises from inertia and significant deformation.

Eq. (31) can be written as follows:

$$\ddot{u} + f_1(\dot{u}) + f_2(u) = 0, \tag{32}$$

where $f_1(\dot{u}) = \mu \sin \dot{u}$, and $f_2(u) = u$

Assuming that the guessing solution is given by

$$p = A\cos \Omega t, \quad ICs \quad p(0) = A, \ and \quad \dot{p}(0) = 0. \tag{33}$$

The equivalent frequency may be determined from the following integration:

$$\omega_{eqv}^2 = \int_0^{2\pi/\Omega} p f_2(p) dt \ / \int_0^{2\pi/\Omega} p^2 dt = 1. \tag{34}$$

The equivalent damping may be determined from the following integral

$$\sigma_{eqv} = \int_0^{2\pi/\Omega} f_1(\dot{p}) dt \ / \int_0^{2\pi/\Omega} \dot{p}^2 dt. \tag{35}$$

By means of the MS, Eq. (35) yields

$$\sigma_{eqv} = \frac{2\mu}{A\Omega} J_1(A\Omega). \tag{36}$$

As previously seen, the equivalent linear equation may be written as follows:

$$\ddot{p} + \sigma_{eqv}\dot{p} + p = 0. \tag{37}$$

By making use of the standard normal form, the total frequency of the considered system can be written as:

$$\Omega^2 = 1 - \left(\frac{\mu}{A\Omega} J_1(A\Omega)\right)^2. \tag{38}$$

The stability criterion requires that $\Omega^2 > 0$ or $\left(\frac{\mu}{A\Omega} J_1(A\Omega)\right) < 1$. It should be noted that Eq. (30) is a transcendental equation in the total frequency Ω. By making use of the MS, through the command FindRoot, for the data $\mu = 0.05$ and, $A = 0.5$ utilizing the NS, it follows that the approximate root of the frequency becomes: 0.999706. It should be noted that the total frequency is very close with nature frequency of the system. For more convenience, with the aid of the MS, a matching between the original non-linear ODE as given in Eq. (31) with the equivalent linear ODE as given in Eq. (37) can be drawn for the previous data system:

From the NS in Fig. 7, it is found that the absolute error between the two solutions is 0.00347. It should be renowned that, in contrast with the traditional perturbation techniques, the present NPA does not use the Taylor expansion for the restoring forces. Additionally, it enables us to discuss the stability criterion.

4. Discussion and Results

The solutions from the NPA are compared to the fourth order Runge-Kutta approximations illustrated in Figs. 2, 3, 6 and 7. The precisions of all derived analytical approximations using the NPA and RK4 are remarkably compatible. Based on these comparisons, the current study gives assurance in the approach used to discover a quick and efficient analytical solution of the previous ODEs. Furthermore, the accuracy of the NPA is sometimes better than that of traditional perturbation techniques, and there is a high degree of harmony and agreement between the analytical and numerical approximations, which contributes to the exceptional accuracy of all derived analytic approximations.

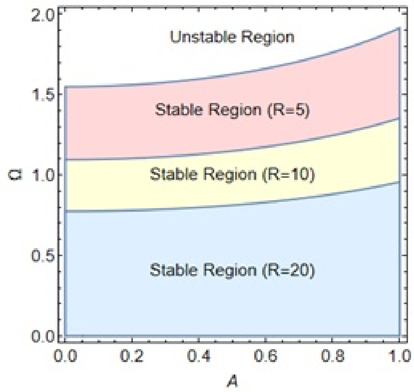

Figure 1: Displays the stability profile for Ω via A at $g = 12$.

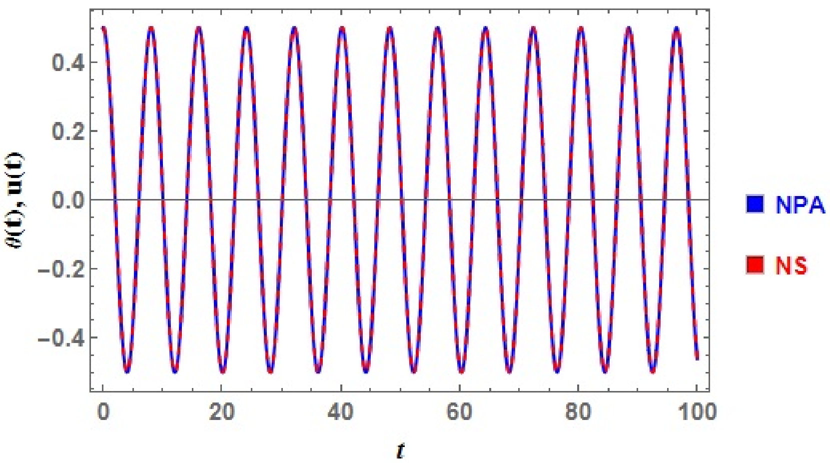

Figure 2: A matching between the nonlinear/linear ODEs as given in Eqs. (9) and (15), respectively.

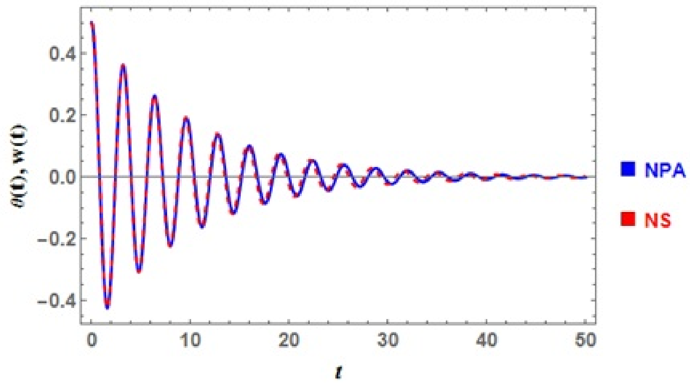

Figure 3: A matching between the two nonlinear/linear ODEs as given in Eqs. (16) and (22), respectively.

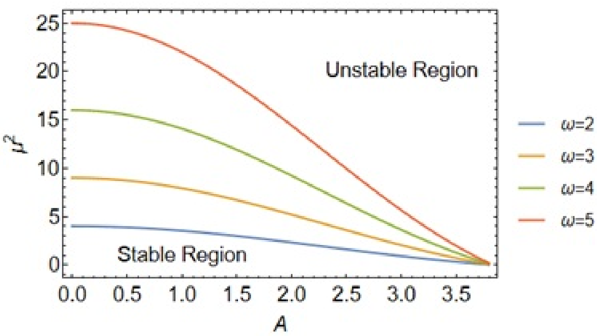

Figure 4: Sketches the stability/instability zones.

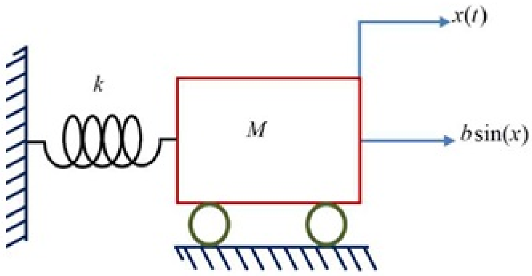

Figure 5: Geometric structure of the problem.

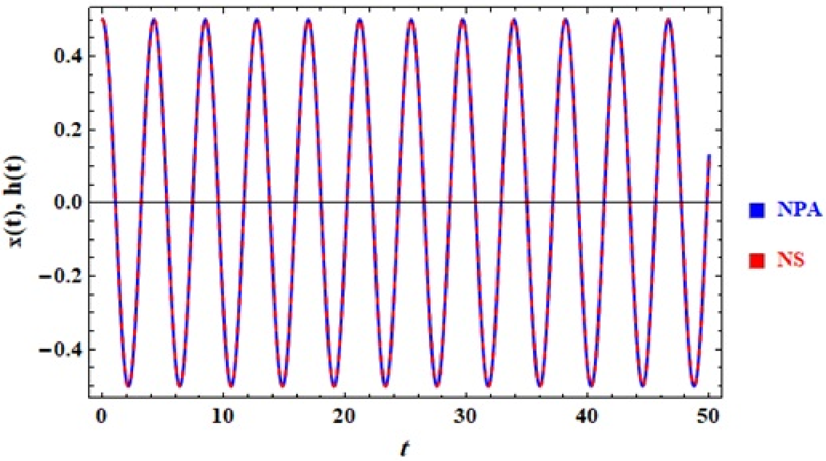

Figure 6: A matching between the two solutions of nonlinear/linear ODEs as given in Eqs. (24) and (30), respectively.

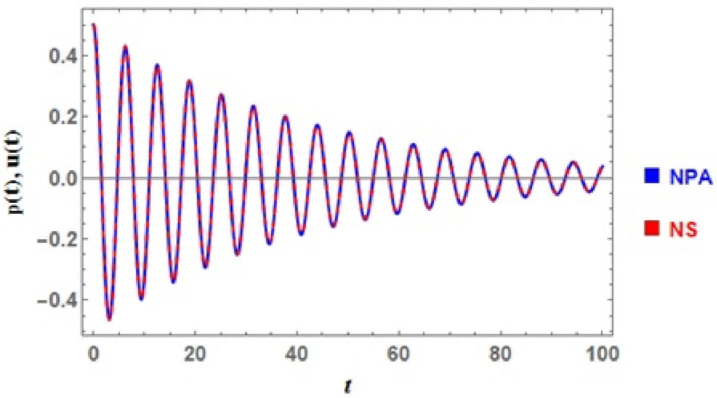

Figure 7: A matching between the two solutions nonlinear/linear ODEs as given in Eqs. (31) and (37), respectively.

Finally, we can say that the NPA is appropriate for obtaining an accurate solution for the strong nonlinear oscillator.

5. Conclusion

The primary goal of the current study is to employ the NPA in order to realize the analytical explanations for many different types of exceedingly nonlinear oscillators, where the nonlinear oscillators are growing more and more corporate. We scrutinize the relationship involving elastic forces and the solution of a certain type of oscillators with large nonlinear damping both mathematically and numerically. It is supposed that the suitable sum of the trigonometric functions equals the solution of a powerful nonlinear ODE that represents motion. We give several instances from diverse arenas of science and technology. The novel approach clearly requires less processing time and is less difficult than the conventional perturbation methods that were widely employed in the field of the dynamical systems. This new method, which is essentially a linear transformation of the nonlinear ODE, is characterized by the NPA. This process yields a new frequency that is analogous to a linear ODE, similar to a situation involving simple harmonic motion. When evaluated for physiologically significant expert examples, this simple methodology produces results that not only correspond well with the NS, but also prove to be more accurate than the results obtained with many well-known traditional approximate methodologies. For the advantage of the readers, a detailed description of the NPA is provided. The theoretical conclusions are validated by a numerical comparison with the MS. The NS test results and the theories were remarkably consistent. It is commonly known that all traditional perturbation techniques use the Taylor expansion to enlarge the restoring forces when they exist, therefore simplifying the current situation. Using the NPA eliminates this susceptibility. Additionally, the stability analysis of the problems may be fully examined agrees to the

NPA, which was not achievable with previous conventional methodologies. Consequently, the NPA is a more helpful accountability tool when analyzing approximations for very nonlinear oscillators in MS. Because the NPA is easily adaptable to address a wide range of nonlinear challenges, it is a valuable tool in the fields of science and technology as well as applied studies.

It should be noted that the understanding the periodic solutions of strongly nonlinear oscillators have important applications in various fields: Improve stability and comfort in buildings, automobiles, and marine engines. Prevent resonance and potential damage in turbines and engines. Protect buildings and bridges during earthquakes. Mitigate impacts of wind and traffic loads on structures. Ensure effective operation of devices like pacemakers and respiratory machines.

As a progress work, the NPA can be developed to handling the coupled dynamical system. Coupled dynamic systems, characterized by the mutual interaction of numerous interacting subsystems, find extensive applications across many fields: Improves the efficiency of design and maintenance processes by analyzing the interactions within machines.

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