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Optimality and Duality for Second-order Multiobjective Variational Problems

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Abstract. In this work, we first modify a converse duality theorem for second-order dual of a scalar variational problem. We then consider its multiobjective analogue and obtain necessary optimality conditions and duality relations. At the end, the static case of our problems has also been discussed.

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1. Introduction

Consider the variational problem:

$$(\mathbf{CP})\quad \text{Minimize} \quad \int_a^b f(t,x,\dot{x})dt$$
$$\text{Subject to} \quad x(a)=0=x(b),$$
$$g(t,x,\dot{x}) \leqq 0,\ t \in I,$$

where $I=[a,b]$ is a real interval, $f: I \times R^n \times R^n \to R$, $g: I \times R^n \times R^n \to R^m$, $x(t)$ is an n-dimensional piecewise smooth function of t and $\dot{x}(t)$ is the derivative of $x(t)$ with respect to t in I. For notational simplicity, we write $x(t)$ and $\dot{x}(t)$ as x and $\dot{x}$, respectively.

Mond and Hanson [11] studied duality for the above problem under convexity. The work in [11] was generalized to invex functions in [10, 12] and to multiobjective variational problems by Bector and Husain [2], Bhatia and Mehra [3], Kim and Kim [9], Ahmad and Gulati [1] among others. In the work mentioned above, the boundary conditions are

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$x(a)=\alpha,\ x(b)=\beta.$

Recently, Husain et al. [8] formulated the following second-order dual (CD) for (CP):

$$(\mathbf{CD})\quad \text{Maximize} \quad \int_a^b \{f(t,u,\dot{u})-\frac{1}{2}\beta(t)^T F\beta(t)\}dt$$

$$\text{Subject to}\quad u(a)=0=u(b), \tag{1}$$

$$f_u+y(t)^T g_u - D(f_{\dot{u}}+y(t)^T g_{\dot{u}})+(F+H)\beta(t)=0,\ t\in I, \tag{2}$$

$$\int_a^b \{y(t)^T g(t,u,\dot{u})-\frac{1}{2}\beta(t)^T H\beta(t)\}dt \geqq 0 \tag{3}$$

$$y(t)\geqq 0,\ t\in I, \tag{4}$$

where the symbols are as defined in [8].

They established the following converse duality theorem :

Theorem 1. *[Converse duality] Suppose that f and g are thrice continuously differentiable. Let $(\bar{x}(t),\bar{y}(t),\bar{\beta}(t))$ be an optimal solution of (CD) at which*

(A1) the Hessian matrices F and H are not the multiple of each other,

(A2) $\bar{y}(t)^T g_x - D\bar{y}(t)^T g_{\dot{x}} \neq 0$,

(A3) (i) $\int_a^b \bar{\beta}(t)^T(\bar{y}(t)^T g_x - D\bar{y}(t)^T g_{\dot{x}})dt \geqq 0$ and $\int_a^b \bar{\beta}(t)^T H\bar{\beta}(t)dt > 0$, or

(ii) $\int_a^b \bar{\beta}(t)^T(\bar{y}(t)^T g_x - D\bar{y}(t)^T g_{\dot{x}})dt \leqq 0$ and $\int_a^b \bar{\beta}(t)^T H\bar{\beta}(t)dt < 0$.

If, for all feasible $(x(t),y(t),\beta(t))$, $\int_a^b f(t,.,.)dt$ is second-order pseudoinvex and $\int_a^b y(t)^T g(t,.,.)dt$ is second-order quasiinvex with respect to the same η, then $\bar{x}(t)$ is an optimal solution of (CP).

This result has been established by first proving that $\bar{\beta}(t)=0,\ t\in I$. It is obvious that the assumption $\int_a^b \bar{\beta}(t)^T H\bar{\beta}(t)dt > 0$ (or <0) and the conclusion $\bar{\beta}(t)=0,\ t\in I$, are inconsistent. Moreover, assumption (A1) and equation (3.11) in [8], namely
$(\lambda(t)+\alpha\bar{\beta}(t))^T F+(\lambda(t)+\gamma\bar{\beta}(t))^T H=0,\ t\in I$, do not imply $\lambda(t)+\alpha\bar{\beta}(t)=0,\ t\in I$ and $\lambda(t)+\gamma\bar{\beta}(t)=0,\ t\in I$. One needs to assume that the rows of F and H are linearly independent.

This paper is organized as follows. In Section 2, we prove a converse duality theorem modifying Theorem 1. In Section 3, we consider the multiobjective analogue of problem (CP). This section also contains notations and preliminaries. The necessary optimality conditions for an efficient solution are obtained in Section 4. The duality results are established in Section 5. The static case of our problems has been given in the last section.

2. Converse Duality

We denote the first partial derivatives of f with respect to t, x and $\dot{x}$, respectively, by f_t, f_x and $f_{\dot{x}}$ such that $f_x = (\frac{\partial f}{\partial x^1}, \frac{\partial f}{\partial x^2}, \ldots, \frac{\partial f}{\partial x^n})^T$ and $f_{\dot{x}} = (\frac{\partial f}{\partial \dot{x}^1}, \frac{\partial f}{\partial \dot{x}^2}, \ldots, \frac{\partial f}{\partial \dot{x}^n})^T$. The matrices f_{xx} and g_x are of order $n \times n$ and $n \times m$, respectively. Similarly $f_{x\dot{x}}, f_{\dot{x}\dot{x}}$ and the partial derivatives of g^j are also defined. All derivative of x, and all partial and total derivatives of f and g used in this section are assumed to be continuous. Let the set $M = \{1, 2, \ldots, m\}$.

Remark 1. *It may be noted that if we write*

$$f_x(t,x,\dot{x}) - \frac{d}{dt} f_{\dot{x}}(t,x,\dot{x}) = L(t,x,\dot{x},\ddot{x}),$$

then the function F should be

$$\begin{aligned}
F &= L_x(t,x,\dot{x},\ddot{x}) - \frac{d}{dt} L_{\dot{x}}(t,x,\dot{x},\ddot{x}) + \frac{d^2}{dt^2} L_{\ddot{x}}(t,x,\dot{x},\ddot{x}) \\
&= \frac{\partial}{\partial x}(f_x(t,x,\dot{x}) - \frac{d}{dt} f_{\dot{x}}(t,x,\dot{x})) - \frac{d}{dt}(\frac{\partial}{\partial \dot{x}}(f_x(t,x,\dot{x}) - \frac{d}{dt} f_{\dot{x}}(t,x,\dot{x}))) \\
&\quad + \frac{d^2}{dt^2}(\frac{\partial}{\partial \ddot{x}}(f_x(t,x,\dot{x}) - \frac{d}{dt} f_{\dot{x}}(t,x,\dot{x}))) \\
&= f_{xx} - Df_{\dot{x}x} - Df_{x\dot{x}} + D^2 f_{\dot{x}\dot{x}} - D^3 f_{\dot{x}\ddot{x}} \\
&= f_{xx} - 2Df_{x\dot{x}} + D^2 f_{\dot{x}\dot{x}} - D^3 f_{\dot{x}\ddot{x}}.
\end{aligned}$$

Thus F is a function of $t, x(t), \dot{x}(t), \ddot{x}(t), \dddot{x}(t), \ddddot{x}(t)$ and is given by

$$\begin{aligned}
F(t,x,\dot{x},\ddot{x},\dddot{x},\ddddot{x}) &= f_{xx}(t,x,\dot{x}) - 2Df_{x\dot{x}}(t,x,\dot{x}) \\
&+ D^2 f_{\dot{x}\dot{x}}(t,x,\dot{x}) - D^3 f_{\dot{x}\ddot{x}}(t,x,\dot{x}),\ t \in I.
\end{aligned}$$

Therefore, it seems to us, that the function F should be as given above, while in Chen [6] and Husain et al. [8], F has been taken as $f_{xx} - 2Df_{x\dot{x}} + D^2 f_{\dot{x}\dot{x}}$ and $f_{xx} - Df_{x\dot{x}} + D^2 f_{\dot{x}\dot{x}}$, respectively.

We formulate the following dual problem for (CP) :

$$(\widehat{\mathbf{CD}}) \quad \text{Maximize} \quad \int_a^b (f(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T F \beta(t)) dt$$

$$\text{Subject to} \quad u(a) = 0 = u(b), \tag{5}$$

$$f_x(t,u,\dot{u}) + g_x(t,u,\dot{u})y(t) - D(f_{\dot{x}}(t,u,\dot{u}) + g_{\dot{x}}(t,u,\dot{u})y(t)) + (F+H)\beta(t) = 0,\ t \in I, \tag{6}$$

$$y(t)^T g(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T H\beta(t) \geqq 0,\ t \in I, \tag{7}$$

$$y(t) \geqq 0,\ t \in I, \tag{8}$$

where

$$H(t,u,\dot{u},\ddot{u},\dddot{u},\ddddot{u},y(t),\dot{y}(t),\ddot{y}(t),\dddot{y}(t)) = (g_x(t,u,\dot{u})y(t))_x - 2D(g_x(t,u,\dot{u})y(t))_{\dot{x}} + D^2(g_{\dot{x}}(t,u,\dot{u})y(t))_{\dot{x}} - D^3(g_{\dot{x}}(t,u,\dot{u})y(t))_{\ddot{x}},\ t \in I,$$

and $F(t,u,\dot{u},\ddot{u},\dddot{u},\ddddot{u}),\ t \in I$, is as given in Remark 1.

Like [8], we shall use Fritz John necessary optimality conditions [4] for the dual problem to establish the converse duality theorem. Since the constraints in the problem considered in [4] do not involve integrals, we have not taken integral in the dual constraint (7). Moreover, some terms in $(\widehat{CD})$ are in different form than in (CD). It has been done so to make all the terms in an expression to be of the same dimension. However, the weak duality theorem given in [8] holds for problems (CP) and $(\widehat{CD})$.

Theorem 2. *[Converse duality] Let $(\bar{u}(t),\bar{y}(t),\bar{\beta}(t))$ be an optimal solution of $(\widehat{CD})$. If for each $t \in I$,*

(B1) the vectors $\{F_i, H_i, i = 1,2,\dots,n\}$ are linearly independent, where F_i and H_i are the ith rows of $F(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}},\dddot{\bar{u}},\ddddot{\bar{u}})$ and $H(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}},\dddot{\bar{u}},\ddddot{\bar{u}},\bar{y}(t),\dot{\bar{y}}(t),\ddot{\bar{y}}(t),\dddot{\bar{y}}(t))$, respectively,

(B2) $g_x(t,\bar{u},\dot{\bar{u}})\bar{y}(t) - D(g_{\dot{x}}(t,\bar{u},\dot{\bar{u}})\bar{y}(t)) \neq 0$, and

(B3) either

(i) the $n \times n$ matrix $H(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}},\dddot{\bar{u}},\ddddot{\bar{u}},\bar{y}(t),\dot{\bar{y}}(t),\ddot{\bar{y}}(t),\dddot{\bar{y}}(t)) + (g_x(t,\bar{u},\dot{\bar{u}})\bar{y}(t))_x$ is positive definite and $\bar{\beta}(t)^T(g_x(t,\bar{u},\dot{\bar{u}})\bar{y}(t)) \geqq 0$, or

(ii) the $n \times n$ matrix $H(t,\bar{u},\dot{\bar{u}},\ddot{\bar{u}},\dddot{\bar{u}},\ddddot{\bar{u}},\bar{y}(t),\dot{\bar{y}}(t),\ddot{\bar{y}}(t),\dddot{\bar{y}}(t)) + (g_x(t,\bar{u},\dot{\bar{u}})\bar{y}(t))_x$ is negative definite and $\bar{\beta}(t)^T(g_x(t,\bar{u},\dot{\bar{u}})\bar{y}(t)) \leqq 0$,

then $\bar{u}(t)$ is feasible for (CP) and the two objective functionals have same value. Also, if the weak duality theorem [8] holds for all feasible solution of (CP) and $(\widehat{CD})$, then $\bar{u}(t)$ is an optimal solution of (CP).

Proof. Since $(\bar{u}(t),\bar{y}(t),\bar{\beta}(t))$ is an optimal solution of $(\widehat{CD})$, there exist $\alpha \in R$ and piecewise smooth functions $\lambda : I \to R^n$, $\gamma : I \to R$ and $\mu : I \to R^m$ such that the following Fritz John conditions [4] are satisfied at $(\bar{u}(t),\bar{y}(t),\bar{\beta}(t))$ (for brevity, $f_x \equiv f_x(t,\bar{u},\dot{\bar{u}})$, $g^j \equiv g^j(t,\bar{u},\dot{\bar{u}})$, $g^j_x \equiv g^j_x(t,\bar{u},\dot{\bar{u}})$ etc.):

$$-\alpha(f_x - Df_{\dot{x}} - \frac{1}{2}(\bar{\beta}(t)^T F\bar{\beta}(t))_x + \frac{1}{2}D(\bar{\beta}(t)^T F\bar{\beta}(t))_{\dot{x}} - \frac{1}{2}D^2(\bar{\beta}(t)^T F\bar{\beta}(t))_{\ddot{x}}$$

$$+\frac{1}{2}D^3(\bar{\beta}(t)^T F\bar{\beta}(t))_{\dddot{x}} - \frac{1}{2}D^4(\bar{\beta}(t)^T F\bar{\beta}(t))_{\ddddot{x}}) + (f_{xx} - Df_{x\dot{x}} + (g_x\bar{y}(t))_x - D(g_x\bar{y}(t))_{\dot{x}}$$
$$-(D(f_{\dot{x}x} + (g_{\dot{x}}\bar{y}(t))_x) - D(D(f_{\dot{x}\dot{x}} + (g_{\dot{x}}\bar{y}(t))_{\dot{x}})) + D^2(D(f_{\dot{x}\ddot{x}} + (g_{\dot{x}}\bar{y}(t))_{\ddot{x}})))$$
$$+((F+H)\bar{\beta}(t))_x - D((F+H)\bar{\beta}(t))_{\dot{x}} + D^2((F+H)\bar{\beta}(t))_{\ddot{x}} - D^3((F+H)\bar{\beta}(t))_{\dddot{x}}$$
$$+D^4((F+H)\bar{\beta}(t))_{\ddddot{x}})\lambda(t) - \gamma(t)(g_x\bar{y}(t) - D(g_{\dot{x}}\bar{y}(t)) - \frac{1}{2}(\bar{\beta}(t)^T H\bar{\beta}(t))_x$$
$$+\frac{1}{2}D(\bar{\beta}(t)^T H\bar{\beta}(t))_{\dot{x}} - \frac{1}{2}D^2(\bar{\beta}(t)^T H\bar{\beta}(t))_{\ddot{x}} + \frac{1}{2}D^3(\bar{\beta}(t)^T H\bar{\beta}(t))_{\dddot{x}}$$
$$-\frac{1}{2}D^4(\bar{\beta}(t)^T H\bar{\beta}(t))_{\ddddot{x}}) = 0,\ t \in I, \tag{9}$$

$$\lambda(t)^T(g_x^j + g_{xx}^j\bar{\beta}(t)) - \gamma(t)(g^j - \frac{1}{2}\bar{\beta}(t)^T g_{xx}^j\bar{\beta}(t)) - \mu^j(t) = 0,\ t \in I,\ j \in M, \tag{10}$$
$$(\lambda(t) + \alpha\bar{\beta}(t))^T F + (\lambda(t) + \gamma(t)\bar{\beta}(t))^T H = 0,\ t \in I, \tag{11}$$
$$\gamma(t)(\bar{y}(t)^T g - \frac{1}{2}\bar{\beta}(t)^T H\bar{\beta}(t)) = 0,\ t \in I, \tag{12}$$
$$\mu(t)^T\bar{y}(t) = 0,\ t \in I, \tag{13}$$
$$(\alpha, \gamma(t), \mu(t)) \geqq 0,\ t \in I, \tag{14}$$
$$(\alpha, \lambda(t), \gamma(t), \mu(t)) \neq 0,\ t \in I. \tag{15}$$

By Hypothesis (B1), equation (11) yields

$$\lambda(t) + \alpha\bar{\beta}(t) = 0,\ t \in I \text{ and} \tag{16}$$
$$\lambda(t) + \gamma(t)\bar{\beta}(t) = 0,\ t \in I. \tag{17}$$

Using (6), (16) and (17) in (9), we have

$$(\alpha - \gamma(t))(g_x\bar{y}(t) - D(g_{\dot{x}}\bar{y}(t)) + H\bar{\beta}(t)) + \frac{1}{2}\alpha((\bar{\beta}(t)^T F\bar{\beta}(t))_x - D(\bar{\beta}(t)^T F\bar{\beta}(t))_{\dot{x}}$$
$$+D^2(\bar{\beta}(t)^T F\bar{\beta}(t))_{\ddot{x}} - D^3(\bar{\beta}(t)^T F\bar{\beta}(t))_{\dddot{x}} + D^4(\bar{\beta}(t)^T F\bar{\beta}(t))_{\ddddot{x}}) + (((F+H)\bar{\beta}(t))_x -$$
$$D((F+H)\bar{\beta}(t))_{\dot{x}} + D^2((F+H)\bar{\beta}(t))_{\ddot{x}} - D^3((F+H)\bar{\beta}(t))_{\dddot{x}} + D^4((F+H)\bar{\beta}(t))_{\ddddot{x}})\lambda(t) +$$
$$\frac{1}{2}\gamma(t)((\bar{\beta}(t)^T H\bar{\beta}(t))_x - D(\bar{\beta}(t)^T H\bar{\beta}(t))_{\dot{x}} + D^2(\bar{\beta}(t)^T H\bar{\beta}(t))_{\ddot{x}} - D^3(\bar{\beta}(t)^T H\bar{\beta}(t))_{\dddot{x}} +$$
$$D^4(\bar{\beta}(t)^T H\bar{\beta}(t))_{\ddddot{x}}) = 0,\ t \in I. \tag{18}$$

Let $\gamma(t) = 0$ for some t. Suppose $t_0 \in I$ and $\gamma(t_0) = 0$. Then by (17), we get $\lambda(t_0) = 0$ and so $\alpha\bar{\beta}(t_0) = 0$, by (16). Thus (18) yields $\alpha(g_x\bar{y}(t_0) - D(g_{\dot{x}}\bar{y}(t_0))) = 0$, which by Hypothesis (B2) implies $\alpha = 0$. Since $\lambda(t_0) = 0$ and $\gamma(t_0) = 0$, from (10), we obtain $\mu^j(t_0) = 0$, $j \in M$. Therefore $(\alpha, \lambda(t_0), \gamma(t_0), \mu(t_0)) = 0$, contradicting (15). Hence $\gamma(t) > 0$, $t \in I$.

Now, multiplying (10) by $\bar{y}^j(t)$, $t \in I$, summing over j, and then using (12), (13), (17) and $\gamma(t) > 0$, $t \in I$, we get

$$2\bar{\beta}(t)^T(g_x\bar{y}(t)) + \bar{\beta}(t)^T(H + (g_x\bar{y}(t))_x)\bar{\beta}(t) = 0,\ t \in I,$$

which contradicts Hypothesis (B3) unless

$$\bar{\beta}(t) = 0, \ t \in I. \tag{19}$$

Thus from (16), $\lambda(t) = 0$, $t \in I$. Therefore for $j \in M$, equation (10) gives

$$g^j = -\frac{\mu^j(t)}{\gamma(t)} \leqq 0, \ t \in I.$$

Thus $\bar{u}(t)$ is feasible for (CP). Also, in view of (19), the two objectives are equal.

Now, assume that $\bar{u}(t)$ is not an optimal solution of (CP). Then, there exists $\hat{u}(t) \in X$ such that

$$\int_a^b f(t, \hat{u}, \dot{\hat{u}})dt < \int_a^b f(t, \bar{u}, \dot{\bar{u}})dt.$$

As $\bar{\beta}(t) = 0$, $t \in I$, we have

$$\int_a^b f(t, \hat{u}, \dot{\hat{u}})dt < \int_a^b (f(t, \bar{u}, \dot{\bar{u}}) - \frac{1}{2}\bar{\beta}(t)^T F \bar{\beta}(t))dt,$$

a contradiction to the weak duality theorem [8]. Hence $\bar{u}(t)$ is an optimal solution for (CP).

3. Multiobjective Variational Problem

We consider the following multiobjective variational problem (P):

$$(\mathbf{P}) \quad \text{Minimize} \quad \left(\int_a^b f^1(t, x, \dot{x})dt, \int_a^b f^2(t, x, \dot{x})dt, \ldots, \int_a^b f^k(t, x, \dot{x})dt\right)$$

$$\text{Subject to} \quad x(a) = \alpha, x(b) = \beta, \tag{20}$$

$$g(t, x, \dot{x}) \leqq 0, \ t \in I, \tag{21}$$

where $f^i : I \times S \times S \to R (i \in K)$, $g = (g^1, g^2, \ldots, g^m) : I \times S \times S \to R^m$ and S is an open set in R^n. We denote the first partial derivatives of f^i, with respect to t, x and $\dot{x}$ respectively by f^i_t, f^i_x and $f^i_{\dot{x}}$ such that $f^i_x = (\frac{\partial f^i}{\partial x^1}, \frac{\partial f^i}{\partial x^2}, \ldots, \frac{\partial f^i}{\partial x^n})^T$ and $f^i_{\dot{x}} = (\frac{\partial f^i}{\partial \dot{x}^1}, \frac{\partial f^i}{\partial \dot{x}^2}, \ldots, \frac{\partial f^i}{\partial \dot{x}^n})^T$. The Hessian matrix f^i_{xx}, is an $n \times n$ symmetric matrix. Similarly $f^i_{x\dot{x}}, f^i_{\dot{x}\dot{x}}$ and the partial derivatives of g^j are also defined. All the partial and total derivatives of f^i and g used here onwards are assumed to be continuous. We shall use X for the set of all feasible solutions of (P). For the sake of convenience, we shall not write the limits a and b in the integrals, i.e., $\int f^i dt$ shall mean $\int_a^b f^i dt$. Let $K = \{1, 2, \dots, k\}$ and for $r \in K$, the set $K_r = K - r$. For a and b in R^n, we shall use the following three inequalities :

$$a \geqq b \Leftrightarrow a^i \geqq b^i \ (i = 1, 2, \dots, n)$$
$$a \geq b \Leftrightarrow (a \geqq b, \ a \neq b)$$
$$a > b \Leftrightarrow a^i > b^i \ (i = 1, 2, \dots, n).$$

Definition 1. *A point $\bar{x}(t) \in X$ is said to be an efficient solution of (P) if there exists no $x(t) \in X$ such that*

$$\int f^r(t, x, \dot{x})dt < \int f^r(t, \bar{x}, \dot{\bar{x}})dt \text{ for some } r \in K$$
$$\text{and } \int f^i(t, x, \dot{x})dt \leqq \int f^i(t, \bar{x}, \dot{\bar{x}})dt \text{ for } i \in K_r.$$

Let $p : I \to R^n$ and $A^i(t, x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x}) = f^i_{xx}(t, x, \dot{x}) - 2Df^i_{x\dot{x}}(t, x, \dot{x}) + D^2 f^i_{\dot{x}\dot{x}}(t, x, \dot{x}) - D^3 f^i_{\dot{x}\ddot{x}}(t, x, \dot{x})$, $t \in I$, $i \in K$.

Definition 2. *A functional $G : I \times S \times S \times R^n \to R$ is said to be sublinear, if for all $x(t), u(t) \in S$,*

$$G(t, x, u; \xi_1 + \xi_2) \leqq G(t, x, u; \xi_1) + G(t, x, u; \xi_2) \text{ for all } \xi_1, \xi_2 \in R^n \tag{22}$$

and

$$G(t, x, u; a\xi) = aG(t, x, u; \xi) \text{ for all } a \in R, \ a \geqq 0 \text{ and } \xi \in R^n. \tag{23}$$

From (23), it follows that $G(t, x, u; 0) = 0$.

Let $\rho \in R$ and $d : I \times S \times S \to R^n$ be a pseudometric on R^n.

Definition 3. *The functional $\int f^i(t, x, \dot{x})dt$ is said to be second-order (G, ρ)-convex at $u(t) \in S$, if there exists a sublinear functional $G : I \times S \times S \times R^n \to R$ such that for all $x(t) \in S$, $p(t) \in R^n$,*

$$\int f^i(t, x, \dot{x})dt - \int f^i(t, u, \dot{u})dt + \frac{1}{2}\int p(t)^T A^i p(t)dt \geqq$$
$$\int G(t, x, u; f^i_x(t, u, \dot{u}) - Df^i_{\dot{x}}(t, u, \dot{u}) + A^i p(t))dt + \rho \int d^2(t, x, u)dt.$$

If in the above definition, inequality is satisfied as strict inequality, then we say that the functional $\int f^i(t, x, \dot{x})dt$ is second-order strictly (G, ρ)-convex at $u(t) \in S$.

Definition 4. *The functional $\int f^i(t, x, \dot{x})dt$ is said to be second-order (G, ρ)-pseudoconvex at $u(t) \in S$, if there exists a sublinear functional $G : I \times S \times S \times R^n \to R$ such that for all $x(t) \in S$, $p(t) \in R^n$,*

$$\int G(t, x, u; f^i_x(t, u, \dot{u}) - Df^i_{\dot{x}}(t, u, \dot{u}) + A^i p(t))dt + \rho \int d^2(t, x, u)dt \geqq 0$$

$$\Rightarrow \quad \int f^i(t,x,\dot{x})dt \geqq \int f^i(t,u,\dot{u})dt - \frac{1}{2}\int p(t)^T A^i p(t)dt.$$

Definition 5. *The functional $\int f^i(t,x,\dot{x})dt$ is said to be second-order (G,ρ)-quasiconvex at $u(t) \in S$, if there exists a sublinear functional $G : I \times S \times S \times R^n \to R$ such that for all $x(t) \in S$, $p(t) \in R^n$,*

$$\begin{aligned}\int f^i(t,x,\dot{x})dt \quad &\leqq \quad \int f^i(t,u,\dot{u})dt - \frac{1}{2}\int p(t)^T A^i p(t)dt \\ &\Rightarrow \quad \int G(t,x,u; f_x^i(t,u,\dot{u}) - Df_{\dot{x}}^i(t,u,\dot{u}) + A^i p(t))dt + \rho\int d^2(t,x,u)dt \leqq 0.\end{aligned}$$

4. Necessary optimality conditions

We now prove three results. The first gives Fritz John type necessary conditions for (P) and the remaining two are Kuhn-Tucker type necessary conditions for (P). To establish these necessary conditions, we shall use the corresponding result for single objective variational problems obtained by Chandra et al. [4] in their Theorem 1. As stated in the remarks after Theorem 1 in [4], to use their Kuhn-Tucker conditions, we shall assume Slater's or Robinson condition.

The following result relates an efficient solution of (P) with an optimal solution of k-scalar objective variational problems.

Lemma 1 (Chankong and Haimes [5]). *A point $\bar{x}(t) \in X$ is an efficient solution of (P) if and only if $\bar{x}(t)$ is an optimal solution of (P_r) for each $r \in K$.*

$$\begin{aligned} (\boldsymbol{P}_{\mathbf{r}}) \quad & \textit{Minimize} \quad \int f^r(t,x,\dot{x})dt \\ & \textit{Subject to} \quad x(a) = \alpha, x(b) = \beta, \\ & \qquad g(t,x,\dot{x}) \leqq 0,\ t \in I, \\ & \qquad \int f^i(t,x,\dot{x})dt \leqq \int f^i(t,\bar{x},\dot{\bar{x}})dt,\ i \in K_r. \end{aligned}$$

The k-problems (P_r) involve integral in the constraints, while to obtain necessary optimality conditions for (P) via (P_r) we need necessary optimality conditions for scalar variational problem [4]. Since the variational problem in [4] does not involve the integral in the constraints, we first derive the result relating an efficient solution of (P) with the optimal solution of the following k-single objective problems:

$$\begin{aligned} (\hat{\mathbf{P}}_{\mathbf{r}}) \quad & \text{Minimize} \quad \int f^r(t,x,\dot{x})dt \\ & \text{Subject to} \quad x(a) = \alpha, x(b) = \beta, \\ & \qquad g(t,x,\dot{x}) \leqq 0,\ t \in I, \\ & \qquad f^i(t,x,\dot{x}) \leqq f^i(t,\bar{x},\dot{\bar{x}}),\ t \in I,\ i \in K_r. \end{aligned}$$

Lemma 2. *Let $\bar{x}(t) \in X$ be an efficient solution of (P). Then $\bar{x}(t)$ is an optimal solution of ($\hat{P}_r$) for each $r \in K$.*

Proof. Let $\bar{x}(t)$ be an efficient solution of (P) and suppose, to the contrary, that $\bar{x}(t)$ is not an optimal solution of ($\hat{\mathrm{P}}_r$), for some $r \in K$. Then there exists an $\hat{x}(t) \in X$ such that

$$g(t,\hat{x},\dot{\hat{x}}) \leqq 0,\ t \in I, \tag{24}$$

$$f^i(t,\hat{x},\dot{\hat{x}}) \leqq f^i(t,\bar{x},\dot{\bar{x}}),\ t \in I,\ i \in K_r, \tag{25}$$

$$\text{and} \int f^r(t,\hat{x},\dot{\hat{x}})dt < \int f^r(t,\bar{x},\dot{\bar{x}})dt. \tag{26}$$

Inequality (25) implies

$$\int f^i(t,\hat{x},\dot{\hat{x}})dt \leqq \int f^i(t,\bar{x},\dot{\bar{x}})dt,\ i \in K_r. \tag{27}$$

Inequalities (26) and (27) contradict the fact that $\bar{x}(t)$ is an efficient solution of (P). Hence $\bar{x}(t)$ is an optimal solution of ($\hat{\mathrm{P}}_r$) for each $r \in K$.

Theorem 3 (Fritz John type necessary conditions). *Let $\bar{x}(t)$ be an efficient solution of (P). Then there exist $\bar{\lambda}^i \in R$, $i \in K$ and a piecewise smooth function $\bar{y} : I \to R^m$ such that*

$$\sum_{i=1}^{k} \bar{\lambda}^i(f_x^i(t,\bar{x},\dot{\bar{x}}) - Df_{\dot{x}}^i(t,\bar{x},\dot{\bar{x}})) + g_x(t,\bar{x},\dot{\bar{x}})\bar{y}(t) - D(g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})\bar{y}(t)) = 0,\ t \in I, \tag{28}$$

$$\bar{y}(t)^T g(t,\bar{x},\dot{\bar{x}}) = 0,\ t \in I, \tag{29}$$

$$(\bar{\lambda},\bar{y}(t)) \geq 0,\ t \in I. \tag{30}$$

Proof. Since $\bar{x}(t)$ is an efficient solution of (P), by Lemma 2, $\bar{x}(t)$ is an optimal solution of ($\hat{\mathrm{P}}_r$) for each $r \in K$ and hence in particular of ($\hat{P}_1$). Therefore, by [4], there exist $\bar{\lambda}^i$, $i \in K$ and a piecewise smooth function $\bar{y}(t) \in R^m$ such that

$$g_x(t,\bar{x},\dot{\bar{x}})\bar{y}(t) - D(g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})\bar{y}(t)) = 0,\ t \in I,$$

$$\bar{y}(t)^T g(t,\bar{x},\dot{\bar{x}}) = 0,\ t \in I,$$

$$(\bar{\lambda}^1,\bar{\lambda}^2,\ldots,\bar{\lambda}^k,\bar{y}(t)) \geq 0,\ t \in I,$$

which give (28) to (30).

Theorem 4 (Kuhn-Tucker type necessary conditions). *Let $\bar{x}(t)$ be an efficient solution of (P) and let for some $r \in K$, the constraints of ($\hat{P}_r$) satisfy Slater's or Robinson condition at $\bar{x}(t)$. Then there exist $\bar{\lambda} \in R^k$ and a piecewise smooth function $\bar{y} : I \to R^m$ such that*

$$\sum_{i=1}^{k} \bar{\lambda}^i(f_x^i(t,\bar{x},\dot{\bar{x}}) - Df_{\dot{x}}^i(t,\bar{x},\dot{\bar{x}})) + g_x(t,\bar{x},\dot{\bar{x}})\bar{y}(t) - D(g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})\bar{y}(t)) = 0,\ t \in I,$$

$$\bar{y}(t)^T g(t,\bar{x},\dot{\bar{x}}) = 0,\ t \in I,$$

$$\bar{\lambda} \geq 0$$

$$\bar{y}(t) \geqq 0,\ t \in I.$$

Proof. Since $\bar{x}(t)$ is an efficient solution of (P), by Lemma 2, $\bar{x}(t)$ is an optimal solution of ($\hat{P}_r$) for each r. As for some r, the constraints of ($\hat{P}_r$) satisfy Slater's or Robinson condition at $\bar{x}(t)$, by the Kuhn-Tucker necessary conditions in [4], there exist $0 < \bar{\lambda}^r \in$R, $0 \leqq \bar{\lambda}^i \in R$, $i \in K_r$ and a piecewise smooth function $\bar{y}(t) \in R^m$ such that

$$\bar{\lambda}^r(f_x^r(t,\bar{x},\dot{\bar{x}}) - Df_{\dot{x}}^r(t,\bar{x},\dot{\bar{x}})) + \sum_{i\in K_r} \bar{\lambda}^i(f_x^i(t,\bar{x},\dot{\bar{x}}) - Df_{\dot{x}}^i(t,\bar{x},\dot{\bar{x}}))+$$
$$g_x(t,\bar{x},\dot{\bar{x}})\bar{y}(t) - D(g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})\bar{y}(t)) = 0,\ t \in I,$$

$$\bar{y}(t)^T g(t,\bar{x},\dot{\bar{x}}) = 0,\ t \in I,$$
$$\bar{\lambda}^r > 0, 0 \leqq \bar{\lambda}^i \in R,\ i \in K_r,$$
$$\bar{y}(t) \geqq 0,\ t \in I.$$

Or equivalently

$$\sum_{i=1}^{k} \bar{\lambda}^i(f_x^i(t,\bar{x},\dot{\bar{x}}) - Df_{\dot{x}}^i(t,\bar{x},\dot{\bar{x}})) + g_x(t,\bar{x},\dot{\bar{x}})\bar{y}(t) - D(g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})\bar{y}(t)) = 0,\ t \in I,$$
$$\bar{y}(t)^T g(t,\bar{x},\dot{\bar{x}}) = 0,\ t \in I,$$
$$\bar{\lambda} \geq 0,\ \bar{\lambda} \in R^k,$$
$$\bar{y}(t) \geqq 0,\ t \in I.$$

In Theorem 4, we assumed Slater's or Robinson condition for some ($\hat{P}_r$), which gave us $\bar{\lambda} \geq 0$. In the following Theorem, we assume Slater's or Robinson condition for every ($\hat{P}_r$) and obtain $\bar{\lambda} > 0$.

Theorem 5 (Kuhn-Tucker type necessary conditions)**.** *Let $\bar{x}(t)$ be an efficient solution of (P) and let for each $r \in K$, the constraints of ($\hat{P}_r$) satisfy Slater's or Robinson condition at $\bar{x}(t)$. Then there exist $\bar{\lambda} \in R^k$ and a piecewise smooth function $\bar{y} : I \to R^m$ such that*

$$\sum_{i=1}^{k} \bar{\lambda}^i(f_x^i(t,\bar{x},\dot{\bar{x}}) - Df_{\dot{x}}^i(t,\bar{x},\dot{\bar{x}})) + g_x(t,\bar{x},\dot{\bar{x}})\bar{y}(t) - D(g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})\bar{y}(t)) = 0,\ t \in I,$$
$$\bar{y}(t)^T g(t,\bar{x},\dot{\bar{x}}) = 0,\ t \in I,$$
$$\bar{\lambda} > 0, \sum_{i=1}^{k} \bar{\lambda}^i = 1,$$
$$\bar{y}(t) \geqq 0,\ t \in I.$$

Proof. Since $\bar{x}(t)$ is an efficient solution of (P), by Lemma 2, $\bar{x}(t)$ is an optimal solution of ($\hat{P}_r$) for each $r \in K$. As for each r, the constraints of ($\hat{P}_r$) satisfy Slater's or Robinson condition at $\bar{x}(t)$, by the Kuhn-Tucker necessary conditions in [4], for each $r \in K$, there exist $\bar{v}_r^i \in R$ ($i \in K_r$) and piecewise smooth functions $\bar{\mu}_r^j(t) \in R$ ($j \in M$) such that

$$f_x^r(t,\bar{x},\dot{\bar{x}}) - Df_{\dot{x}}^r(t,\bar{x},\dot{\bar{x}}) + \sum_{i\in K_r} \bar{v}_r^i(f_x^i(t,\bar{x},\dot{\bar{x}}) - Df_{\dot{x}}^i(t,\bar{x},\dot{\bar{x}})) + \sum_{j=1}^{m}(g_x^j(t,\bar{x},\dot{\bar{x}})\bar{\mu}_r^j(t) - D(g_{\dot{x}}^j(t,\bar{x},\dot{\bar{x}})\bar{\mu}_r^j(t))) = 0,\ t\in I,$$

$$\sum_{j=1}^{m}\bar{\mu}_r^j(t)g^j(t,\bar{x},\dot{\bar{x}}) = 0,\ t\in I,$$
$$\bar{v}_r^i \geqq 0,\ i\in K_r,$$
$$\bar{y}(t)\geqq 0,\ t\in I.$$

Summing over $r\in K$, we get

$$\sum_{i=1}^{k}(\bar{v}_1^i + \bar{v}_2^i + ... + \bar{v}_k^i)(f_x^i(t,\bar{x},\dot{\bar{x}}) - Df_{\dot{x}}^i(t,\bar{x},\dot{\bar{x}})) + \sum_{j=1}^{m}(g_x^j(t,\bar{x},\dot{\bar{x}})(\bar{\mu}_1^j(t) + \bar{\mu}_2^j(t) + \ldots + \bar{\mu}_k^j(t)) - D(g_{\dot{x}}^j(t,\bar{x},\dot{\bar{x}})(\bar{\mu}_1^j(t) + \bar{\mu}_2^j(t) + \ldots + \bar{\mu}_k^j(t)))) = 0,\ t\in I,$$

$$\sum_{j=1}^{m}(\bar{\mu}_1^j(t) + \bar{\mu}_2^j(t) + \ldots + \bar{\mu}_k^j(t))g^j(t,\bar{x},\dot{\bar{x}}) = 0,\ t\in I,$$

where $\bar{v}_i^i = 1$ for each $i\in K$.

Equivalently,

$$\sum_{i=1}^{k}\bar{v}^i(f_x^i(t,\bar{x},\dot{\bar{x}}) - Df_{\dot{x}}^i(t,\bar{x},\dot{\bar{x}})) + \sum_{j=1}^{m}(g_x^j(t,\bar{x},\dot{\bar{x}})\bar{\mu}^j(t) - D(g_{\dot{x}}^j(t,\bar{x},\dot{\bar{x}})\bar{\mu}^j(t))) = 0,\ t\in I, \quad (31)$$

$$\sum_{j=1}^{m}\bar{\mu}^j(t)g^j(t,\bar{x},\dot{\bar{x}}) = 0,\ t\in I, \quad (32)$$

where $\bar{v}^i = 1 + \sum_{r\in K_i}\bar{v}_r^i > 0, i\in K$, and $\bar{\mu}^j(t) = \sum_{r=1}^{k}\bar{\mu}_r^j(t)\geqq 0,\ t\in I, j\in M$.

Dividing (31) and (32) by $\sum_{i=1}^{k}\bar{v}^i$ and setting

$$\bar{\lambda}^i = \frac{\bar{v}^i}{\sum_{i=1}^{k}\bar{v}^i},\ i\in K,\ \bar{y}^j(t) = \frac{\bar{\mu}^j(t)}{\sum_{i=1}^{k}\bar{v}^i},\ j\in M,$$

we get

$$\sum_{i=1}^{k}\bar{\lambda}^i(f_x^i(t,\bar{x},\dot{\bar{x}}) - Df_{\dot{x}}^i(t,\bar{x},\dot{\bar{x}})) + \sum_{j=1}^{m}(g_x^j(t,\bar{x},\dot{\bar{x}})\bar{y}^j(t) - D(g_{\dot{x}}^j(t,\bar{x},\dot{\bar{x}})\bar{y}^j(t))) = 0,\ t\in I,$$

$$\sum_{j=1}^{m} \bar{y}^j(t)g^j(t,\bar{x},\dot{\bar{x}}) = 0,\ t \in I,$$

or

$$\sum_{i=1}^{k} \bar{\lambda}^i(f_x^i(t,\bar{x},\dot{\bar{x}}) - Df_{\dot{x}}^i(t,\bar{x},\dot{\bar{x}})) + g_x(t,\bar{x},\dot{\bar{x}})\bar{y}(t) - D(g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})\bar{y}(t)) = 0,\ t \in I,$$

$$\bar{y}(t)^T g(t,\bar{x},\dot{\bar{x}}) = 0,\ t \in I,$$

$$\bar{\lambda} = (\bar{\lambda}^1, \bar{\lambda}^2, \ldots, \bar{\lambda}^k) > 0,\ \sum_{i=1}^{k} \bar{\lambda}^i = 1,$$

$$\bar{y}(t) = (\bar{y}^1(t), \bar{y}^2(t), \ldots, \bar{y}^m(t)) \geqq 0,\ t \in I.$$

5. Second-order Mond-Weir Type Duality

We present the following multiobjective variational dual problem for (P):

(**MWD**) Maximize
$$\Big(\int (f^1(t,u,\dot{u}) - \frac{1}{2}p(t)^T A^1 p(t))dt, \ldots,$$
$$\int (f^k(t,u,\dot{u}) - \frac{1}{2}p(t)^T A^k p(t))dt\Big)$$

Subject to
$$u(a) = \alpha, u(b) = \beta, \tag{33}$$
$$\sum_{i=1}^{k} \lambda^i(f_x^i(t,u,\dot{u}) - Df_{\dot{x}}^i(t,u,\dot{u}) + A^i p(t)) +$$
$$g_x(t,u,\dot{u})y(t) - D(g_{\dot{x}}(t,u,\dot{u})y(t)) + Bp(t) = 0,\ t \in I, \tag{34}$$
$$y(t)^T g(t,u,\dot{u}) - \frac{1}{2}p(t)^T Bp(t) \geqq 0,\ t \in I, \tag{35}$$
$$\lambda \geq 0, \tag{36}$$
$$y(t) \geqq 0,\ t \in I, \tag{37}$$

where $y : I \to R^m$, $p : I \to R^n$, $\lambda = (\lambda^1, \lambda^2, \ldots, \lambda^k) \in R^k$, $A^i(t,u,\dot{u},\ddot{u},\dddot{u},\ddddot{u})$, $t \in I$, $i \in K$ (as defined earlier) and

$$B(t,u,\dot{u},\ddot{u},\dddot{u},\ddddot{u},y(t),\dot{y}(t),\ddot{y}(t),\dddot{y}(t)) = (g_x y(t))_x - 2D(g_x y(t))_{\dot{x}} + D^2(g_{\dot{x}} y(t))_{\dot{x}} - D^3(g_{\dot{x}} y(t))_{\ddot{x}},$$

$t \in I$, are $n \times n$ symmetric matrices. Let Y be the set of all feasible solutions of the above problem.

Theorem 6. *(Weak duality) Let $x(t) \in X$ and $(u(t), \lambda, y(t), p(t)) \in Y$ such that* *(i)* $\int \sum_{i=1}^{k} \lambda^i f^i(t,.,.)dt$ *is second-order* (G,ρ_1)*-pseudoconvex at* $u(t)$,

(ii) $\int y(t)^T g(t,.,.)dt$ *is second-order* (G,ρ_2)*-quasiconvex at* $u(t)$,

(iii) $\lambda > 0$ *and* $\rho_1 + \rho_2 \geqq 0$.

Then

$$\int f^r(t,x,\dot{x})dt < \int (f^r(t,u,\dot{u}) - \frac{1}{2}p(t)^T A^r p(t))dt \text{ for some } r \in K \tag{38}$$

$$\text{and} \int f^i(t,x,\dot{x})dt \leqq \int (f^i(t,u,\dot{u}) - \frac{1}{2}p(t)^T A^i p(t))dt, \; i \in K_r \tag{39}$$

cannot hold.

Proof. Suppose, to the contrary, that (38) and (39) hold. Since $\lambda > 0$, the above inequalities give

$$\int \sum_{i=1}^{k} \lambda^i f^i(t,x,\dot{x})dt < \int \sum_{i=1}^{k} \lambda^i (f^i(t,u,\dot{u}) - \frac{1}{2}p(t)^T A^i p(t))dt. \tag{40}$$

Now, by the constraints (21), (35) and (37), we have

$$\int y(t)^T g(t,x,\dot{x})dt \leqq \int (y(t)^T g(t,u,\dot{u}) - \frac{1}{2}p(t)^T Bp(t))dt$$

As $\int y(t)^T g(t,.,.)dt$ is second-order (G,ρ_2)-quasiconvex at $u(t)$, we get

$$\int (G(t,x,u; g_x(t,u,\dot{u})y(t) - D(g_{\dot{x}}(t,u,\dot{u})y(t)) + Bp(t)) + \rho_2 d^2(t,x,u))dt \leqq 0. \tag{41}$$

From the constraint (34) and the fact that $G(t,x,u;0) = 0$, we have

$$\begin{aligned}\int (G(t,x,u; \sum_{i=1}^{k} \lambda^i (f_x^i(t,u,\dot{u}) \quad &- \quad Df_{\dot{x}}^i(t,u,\dot{u}) + A^i p(t)) + g_x(t,u,\dot{u})y(t) \\ &- \quad D(g_{\dot{x}}(t,u,\dot{u})y(t)) + Bp(t)))dt = 0,\end{aligned}$$

which on using inequality (41), Hypothesis (iii) and the sublinearity of G, yield

$$\int G(t,x,u; \sum_{i=1}^{k} \lambda^i (f_x^i(t,u,\dot{u}) - Df_{\dot{x}}^i(t,u,\dot{u}) + A^i p(t)))dt + \int \rho_1 d^2(t,x,u)dt \geqq 0.$$

By Hypothesis (i), it implies

$$\int \sum_{i=1}^{k} \lambda^i f^i(t,x,\dot{x})dt \geqq \int \sum_{i=1}^{k} \lambda^i (f^i(t,u,\dot{u}) - \frac{1}{2}p(t)^T A^i p(t))dt,$$

a contradiction to (40). Hence the result.

Remark 2. *If we drop the assumption $\lambda > 0$ from Theorem 6, then we need to replace Hypothesis (i) by :* $\int \sum_{i=1}^{k} \lambda^i f^i(t,.,.)dt$ *is second-order strictly (G, ρ_1)-pseudoconvex at $u(t)$.*

Theorem 7 (Strong duality). *Let $\bar{x}(t)$ be normal and is an efficient solution of (P). Then, there exist $\bar{\lambda} \in R^k$, a piecewise smooth function $\bar{y} : I \to R^m$ such that $(\bar{x}(t), \bar{\lambda}, \bar{y}(t), \bar{p}(t) = 0)$ is feasible for (MWD) and the two objective functionals are equal. Furthermore, if the weak duality holds for all feasible solutions of (P) and (MWD), then $(\bar{x}(t), \bar{\lambda}, \bar{y}(t), \bar{p}(t) = 0)$ is an efficient solution of the problem (MWD).*

Proof. Since $\bar{x}(t)$ is normal and an efficient solution of (P), therefore by Theorem 4, there exist $\bar{\lambda} \in R^k$ and a piecewise smooth function $\bar{y}(t) \in R^m$ satisfying

$$\sum_{i=1}^{k} \bar{\lambda}^i (f_x^i(t, \bar{x}, \dot{\bar{x}}) - D f_{\dot{x}}^i(t, \bar{x}, \dot{\bar{x}})) + g_x(t, \bar{x}, \dot{\bar{x}})\bar{y}(t) - D(g_{\dot{x}}(t, \bar{x}, \dot{\bar{x}})\bar{y}(t)) = 0, \ t \in I,$$

$$\bar{y}(t)^T g(t, \bar{x}, \dot{\bar{x}}) = 0, \ t \in I,$$

$$\bar{\lambda} \geq 0, \ t \in I,$$

$$\bar{y}(t) \geqq 0, \ t \in I.$$

Hence $(\bar{x}(t), \bar{\lambda}, \bar{y}(t), \bar{p}(t) = 0)$ satisfies the constraints of (MWD) and thus the two objective functionals have the same value.

Now, we claim that $(\bar{x}(t), \bar{\lambda}, \bar{y}(t), \bar{p}(t) = 0)$ is an efficient solution of (MWD). If not, then there exists $(\hat{u}(t), \hat{\lambda}, \hat{y}(t), \hat{p}(t)) \in Y$, such that

$$\begin{aligned} &\Big(\int (f^1(t, \hat{u}, \dot{\hat{u}}) - \frac{1}{2}\hat{p}(t)^T A^1(t, \hat{u}, \dot{\hat{u}}, \ddot{\hat{u}}, \dddot{\hat{u}}, \ddddot{\hat{u}})\hat{p}(t))dt, \ldots, \\ &\qquad \int (f^k(t, \hat{u}, \dot{\hat{u}}) - \frac{1}{2}\hat{p}(t)^T A^k(t, \hat{u}, \dot{\hat{u}}, \ddot{\hat{u}}, \dddot{\hat{u}}, \ddddot{\hat{u}})\hat{p}(t))dt\Big) \\ &\geq \Big(\int (f^1(t, \bar{x}, \dot{\bar{x}}) - \frac{1}{2}\bar{p}(t)^T A^1(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}, \dddot{\bar{x}}, \ddddot{\bar{x}})\bar{p}(t))dt, \ldots, \\ &\qquad \int (f^k(t, \bar{x}, \dot{\bar{x}}) - \frac{1}{2}\bar{p}(t)^T A^k(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}, \dddot{\bar{x}}, \ddddot{\bar{x}})\bar{p}(t))dt\Big). \end{aligned}$$

As $\bar{p}(t) = 0$, we have

$$\begin{aligned} &\Big(\int (f^1(t, \hat{u}, \dot{\hat{u}}) - \frac{1}{2}\hat{p}(t)^T A^1(t, \hat{u}, \dot{\hat{u}}, \ddot{\hat{u}}, \dddot{\hat{u}}, \ddddot{\hat{u}})\hat{p}(t))dt, \ldots, \\ &\int (f^k(t, \hat{u}, \dot{\hat{u}}) - \frac{1}{2}\hat{p}(t)^T A^k(t, \hat{u}, \dot{\hat{u}}, \ddot{\hat{u}}, \dddot{\hat{u}}, \ddddot{\hat{u}})\hat{p}(t))dt\Big) \geq \Big(\int f^1(t, \bar{x}, \dot{\bar{x}})dt, \ldots, \int f^k(t, \bar{x}, \dot{\bar{x}})dt\Big), \end{aligned}$$

which contradicts the weak duality theorem. Hence $(\bar{x}(t), \bar{\lambda}, \bar{y}(t), \bar{p}(t) = 0)$ is an efficient solution of (MWD).

Theorem 8 (Converse duality)**.** *Let $(\bar{u}(t), \bar{\lambda}, \bar{y}(t), \bar{p}(t))$ be an efficient solution of (MWD) for which*

(C1) the vectors $\{A_m^i, B_m, t \in I, i \in K, m = 1, 2, \ldots, n\}$ are linearly independent, where A_m^i is the mth row of $A^i(t, \bar{u}, \dot{\bar{u}}, \ddot{\bar{u}}, \dddot{\bar{u}}, \ddddot{\bar{u}})$ and B_m is the mth row of $B(t, \bar{u}, \dot{\bar{u}}, \ddot{\bar{u}}, \dddot{\bar{u}}, \ddddot{\bar{u}}, \bar{y}(t), \dot{\bar{y}}(t), \ddot{\bar{y}}(t), \dddot{\bar{y}}(t))$,

(C2) $f_x^i(t, \bar{u}, \dot{\bar{u}}) - Df_{\dot{x}}^i(t, \bar{u}, \dot{\bar{u}})$, $t \in I$, $i \in K$, are linearly independent, and

(C3) for $t \in I$, either

(a) the $n \times n$ matrix $B(t, \bar{u}, \dot{\bar{u}}, \ddot{\bar{u}}, \dddot{\bar{u}}, \ddddot{\bar{u}}, \bar{y}(t), \dot{\bar{y}}(t), \ddot{\bar{y}}(t), \dddot{\bar{y}}(t)) + (g_x(t, \bar{u}, \dot{\bar{u}})\bar{y}(t))_x$ is positive definite and $\bar{p}(t)^T(g_x(t, \bar{u}, \dot{\bar{u}})\bar{y}(t)) \geqq 0$, or

(b) the $n \times n$ matrix $B(t, \bar{u}, \dot{\bar{u}}, \ddot{\bar{u}}, \dddot{\bar{u}}, \ddddot{\bar{u}}, \bar{y}(t), \dot{\bar{y}}(t), \ddot{\bar{y}}(t), \dddot{\bar{y}}(t)) + (g_x(t, \bar{u}, \dot{\bar{u}})\bar{y}(t))_x$ is negative definite and $\bar{p}(t)^T(g_x(t, \bar{u}, \dot{\bar{u}})\bar{y}(t)) \leqq 0$.

Then $\bar{u}(t)$ is feasible for (P) and the two objective functionals have same value. Also, if the weak duality theorem holds for all feasible solutions of (P) and (MWD), then $\bar{u}(t)$ is an efficient solution of (P).

Proof. Since $(\bar{u}(t), \bar{\lambda}, \bar{y}(t), \bar{p}(t))$ is an efficient solution of (MWD), there exist $\alpha, \eta \in R^k$ and piecewise smooth functions $\beta : I \to R^n$, $\gamma : I \to R$, $\mu : I \to R^m$, such that the following Fritz John conditions (Theorem 3) are satisfied at $(\bar{u}(t), \bar{\lambda}, \bar{y}(t), \bar{p}(t))$ (for brevity, $f_x^i \equiv f_x^i(t, \bar{u}, \dot{\bar{u}})$, $g^j \equiv g^j(t, \bar{u}, \dot{\bar{u}})$, $g_x^j \equiv g_x^j(t, \bar{u}, \dot{\bar{u}})$, $A^i \equiv A^i(t, \bar{u}, \dot{\bar{u}}, \ddot{\bar{u}}, \dddot{\bar{u}}, \ddddot{\bar{u}})$, $B \equiv B(t, \bar{u}, \dot{\bar{u}}, \ddot{\bar{u}}, \dddot{\bar{u}}, \ddddot{\bar{u}}, \bar{y}(t), \dot{\bar{y}}(t), \ddot{\bar{y}}(t), \dddot{\bar{y}}(t))$ etc.):

$$\sum_{i=1}^{k} \alpha^i (f_x^i - Df_{\dot{x}}^i - \frac{1}{2}(\bar{p}(t)^T A^i \bar{p}(t))_x + \frac{1}{2}D(\bar{p}(t)^T A^i \bar{p}(t))_{\dot{x}} - \frac{1}{2}D^2(\bar{p}(t)^T A^i \bar{p}(t))_{\ddot{x}} +$$
$$\frac{1}{2}D^3(\bar{p}(t)^T A^i \bar{p}(t))_{\dddot{x}} - \frac{1}{2}D^4(\bar{p}(t)^T A^i \bar{p}(t))_{\ddddot{x}}) - (\sum_{i=1}^{k} \bar{\lambda}^i (A^i + (A^i \bar{p}(t))_x - D(A^i \bar{p}(t))_{\dot{x}} +$$
$$D^2(A^i \bar{p}(t))_{\ddot{x}} - D^3(A^i \bar{p}(t))_{\dddot{x}} + D^4(A^i \bar{p}(t))_{\ddddot{x}}) + B + (B\bar{p}(t))_x - D(B\bar{p}(t))_{\dot{x}} +$$
$$D^2(B\bar{p}(t))_{\ddot{x}} - D^3(B\bar{p}(t))_{\dddot{x}} + D^4(B\bar{p}(t))_{\ddddot{x}})\beta(t) + \gamma(t)(g_x y(t) - D(g_{\dot{x}}\bar{y}(t)) -$$
$$\frac{1}{2}(\bar{p}(t)^T B\bar{p}(t))_x + \frac{1}{2}D(\bar{p}(t)^T B\bar{p}(t))_{\dot{x}} - \frac{1}{2}D^2(\bar{p}(t)^T B\bar{p}(t))_{\ddot{x}} + \frac{1}{2}D^3(\bar{p}(t)^T B\bar{p}(t))_{\dddot{x}} -$$
$$\frac{1}{2}D^4(\bar{p}(t)^T B\bar{p}(t))_{\ddddot{x}}) = 0, \ t \in I, \tag{42}$$

$$\beta(t)^T (f_x^i - Df_{\dot{x}}^i + A^i \bar{p}(t)) - \eta^i = 0, \ t \in I, \ i \in K, \tag{43}$$

$$\beta(t)^T (g_x^j + g_{xx}^j \bar{p}(t)) - \gamma(t)(g^j - \frac{1}{2}\bar{p}(t)^T g_{xx}^j \bar{p}(t)) - \mu^j(t) = 0, \ t \in I, \ j \in M, \tag{44}$$

$$\sum_{i=1}^{k}\alpha^i A^i \bar{p}(t) + (\sum_{i=1}^{k}\bar{\lambda}^i A^i + B)\beta(t) + \gamma(t)B\bar{p}(t) = 0,\ t \in I, \tag{45}$$

$$\gamma(t)(\bar{y}(t)^T g - \frac{1}{2}\bar{p}(t)^T B\bar{p}(t)) = 0,\ t \in I, \tag{46}$$

$$\mu(t)^T \bar{y}(t) = 0,\ t \in I, \tag{47}$$

$$\eta^T \bar{\lambda} = 0, \tag{48}$$

$$(\alpha, \beta(t), \gamma(t), \mu(t), \eta) \neq 0,\ t \in I, \tag{49}$$

$$(\alpha, \gamma(t), \mu(t), \eta) \geq 0,\ t \in I. \tag{50}$$

On rearranging (45), we get

$$\sum_{i=1}^{k} A^i(\alpha^i \bar{p}(t) + \bar{\lambda}^i \beta(t)) + B(\beta(t) + \gamma(t)\bar{p}(t)) = 0,\ t \in I,$$

which by Hypothesis (C1), yield

$$\alpha^i \bar{p}(t) + \bar{\lambda}^i \beta(t) = 0,\ t \in I,\ i \in K, \tag{51}$$

and

$$\beta(t) + \gamma(t)\bar{p}(t) = 0,\ t \in I. \tag{52}$$

Now, suppose $\gamma(t) = 0$ for some t, i.e., $\gamma(t_0) = 0$ for some $t_0 \in I$. Then equation (52) gives $\beta(t_0) = 0$ and so equation (51) implies $\alpha^i \bar{p}(t_0) = 0$. Therefore for $t = t_0$, equation (42) reduces to

$$\sum_{i=1}^{k}\alpha^i(f_x^i - Df_{\dot{x}}^i) = 0.$$

On using Hypothesis (C2) in the above equation, we obtain

$$\alpha^i = 0,\ i \in K.$$

Also, equations (43) and (44) give $\eta^i = 0$, $i \in K$ and $\mu^j(t_0) = 0$, $j \in M$, respectively. Hence $(\alpha, \beta(t_0), \gamma(t_0), \mu(t_0), \eta) = 0$, which contradicts (49). Therefore

$$\gamma(t) > 0,\ t \in I. \tag{53}$$

Multiplying (44) by $\bar{y}^j(t)$, summing over j and then using (46), (47), (52) and $\gamma(t) > 0$, we have

$$2\bar{p}(t)^T(g_x \bar{y}(t)) + \bar{p}(t)^T(B + (g_x \bar{y}(t))_x)\bar{p}(t) = 0,\ t \in I,$$

which contradicts Hypothesis (C3) unless

$$\bar{p}(t) = 0,\ t \in I. \tag{54}$$

And thus relation (52) gives $\beta(t) = 0$, $t \in I$. For $j \in M$, equation (44) yield

$$g^j = \frac{-\mu_j(t)}{\gamma(t)} \leqq 0, \ t \in I, \ j \in M.$$

Thus $\bar{u}(t)$ is feasible for (P).

As $\bar{p}(t) = 0$, $t \in I$, the two objectives functionals are equal.

Now, suppose that $\bar{u}(t)$ is not an efficient solution of (P). Then, there exists $\hat{u}(t) \in X$ such that

$$\int f^r(t, \hat{u}, \dot{\hat{u}})dt < \int f^r(t, \bar{u}, \dot{\bar{u}})dt \text{ for some } r \in K$$

and

$$\int f^i(t, \hat{u}, \dot{\hat{u}})dt \leqq \int f^i(t, \bar{u}, \dot{\bar{u}})dt, \ i \in K_r.$$

Using (54), we get

$$\int f^r(t, \hat{u}, \dot{\hat{u}})dt < \int (f^r(t, \bar{u}, \dot{\bar{u}}) - \frac{1}{2}\bar{p}(t)^T A^r(t, \bar{u}, \dot{\bar{u}}, \ddot{\bar{u}}, \dddot{\bar{u}}, \ddddot{\bar{u}})\bar{p}(t))dt \text{ for some } r \in K$$

and

$$\int f^i(t, \hat{u}, \dot{\hat{u}})dt \leqq \int (f^i(t, \bar{u}, \dot{\bar{u}}) - \frac{1}{2}\bar{p}(t)^T A^i(t, \bar{u}, \dot{\bar{u}}, \ddot{\bar{u}}, \dddot{\bar{u}}, \ddddot{\bar{u}})\bar{p}(t))dt, \ i \in K_r,$$

which contradicts the weak duality theorem. Hence $\bar{u}(t)$ is an efficient solution for (P).

Remark 3. *For the single objective problems in Section 2, the Hypothesis (C2) reduces to (C2′)* $f_x - Df_{\dot{x}} \neq 0$, $t \in I$.
Therefore following the above proof, Theorem 2 can also be proved if Hypothesis (B2) is replaced by (C2′). This also makes the proof simpler as it does not require equation (18). However, we proved Theorem 2 assuming (B2), which is same as Hypothesis (A2) in [8].

Theorem 9 (Strict converse duality)**.** *Let* $\bar{x}(t)$ *and* $(\bar{u}(t), \bar{\lambda}, \bar{y}(t), \bar{p}(t))$ *be efficient solutions of (P) and (MWD) respectively, such that*

$$\int \sum_{i=1}^{k} \bar{\lambda}^i f^i(t, \bar{x}, \dot{\bar{x}})dt = \int \sum_{i=1}^{k} \bar{\lambda}^i (f^i(t, \bar{u}, \dot{\bar{u}}) - \frac{1}{2}\bar{p}(t)^T A^i(t, \bar{u}, \dot{\bar{u}}, \ddot{\bar{u}}, \dddot{\bar{u}}, \ddddot{\bar{u}})\bar{p}(t))dt \tag{55}$$

If

(i) $\int \sum_{i=1}^{k} \bar{\lambda}^i f^i(t, ., .)dt$ *is second-order strictly* (G, ρ_1)*-convex at* $\bar{u}(t)$*,*

(ii) $\int \bar{y}(t)^T g(t, ., .)dt$ *is second-order* (G, ρ_2)*-convex at* $\bar{u}(t)$*, and*

(iii) $\rho_1 + \rho_2 \geqq 0$.

Then $\bar{x}(t) = \bar{u}(t)$.

Proof. Suppose $\bar{x}(t_0) \neq \bar{u}(t_0)$ for some $t_0 \in I$. By Hypothesis (i), we have

$$\int \sum_{i=1}^{k} \bar{\lambda}^i f^i(t_0, \bar{x}, \dot{\bar{x}})dt - \int \sum_{i=1}^{k} \bar{\lambda}^i f^i(t_0, \bar{u}, \dot{\bar{u}})dt > \int (G(t_0, \bar{x}, \bar{u}; \sum_{i=1}^{k} \bar{\lambda}^i (f_x^i(t_0, \bar{u}, \dot{\bar{u}}) - D f_{\dot{x}}^i(t_0, \bar{u}, \dot{\bar{u}})$$
$$+ A^i(t_0, \bar{u}, \dot{\bar{u}}, \ddot{\bar{u}}, \dddot{\bar{u}}, \ddddot{\bar{u}})\bar{p}(t_0))) - \frac{1}{2}\sum_{i=1}^{k} \bar{\lambda}^i \bar{p}(t_0)^T A^i(t_0, \bar{u}, \dot{\bar{u}}, \ddot{\bar{u}}, \dddot{\bar{u}}, \ddddot{\bar{u}})\bar{p}(t_0) + \rho_1 d^2(t_0, \bar{x}, \bar{u}))dt. \tag{56}$$

By Hypothesis (ii), we get

$$\int \bar{y}(t_0)^T g(t_0, \bar{x}, \dot{\bar{x}})dt - \int \bar{y}(t_0)^T g(t_0, \bar{u}, \dot{\bar{u}})dt \geqq \int (G(t_0, \bar{x}, \bar{u}; g_x(t_0, \bar{u}, \dot{\bar{u}})\bar{y}(t_0) -$$
$$D(g_{\dot{x}}(t_0, \bar{u}, \dot{\bar{u}})\bar{y}(t_0)) + B(t_0, \bar{u}, \dot{\bar{u}}, \ddot{\bar{u}}, \dddot{\bar{u}}, \ddddot{\bar{u}}, \bar{y}(t_0), \dot{\bar{y}}(t_0), \ddot{\bar{y}}(t_0), \dddot{\bar{y}}(t_0))\bar{p}(t_0)) -$$
$$\frac{1}{2}\bar{p}(t_0)^T B(t_0, \bar{u}, \dot{\bar{u}}, \ddot{\bar{u}}, \dddot{\bar{u}}, \ddddot{\bar{u}}, \bar{y}(t_0), \dot{\bar{y}}(t_0), \ddot{\bar{y}}(t_0), \dddot{\bar{y}}(t_0))\bar{p}(t_0) + \rho_2 d^2(t_0, \bar{x}, \bar{u}))dt. \tag{57}$$

Adding (56), (57) and using Hypothesis (iii), (21), (35), (37), (55), we obtain

$$\int G(t_0, \bar{x}, \bar{u}; \sum_{i=1}^{k} \bar{\lambda}^i (f_x^i(t_0, \bar{u}, \dot{\bar{u}}) - D f_{\dot{x}}^i(t_0, \bar{u}, \dot{\bar{u}}) + A^i(t_0, \bar{u}, \dot{\bar{u}}, \ddot{\bar{u}}, \dddot{\bar{u}}, \ddddot{\bar{u}})\bar{p}(t_0)) + g_x(t_0, \bar{u}, \dot{\bar{u}})\bar{y}(t_0) -$$
$$D(g_{\dot{x}}(t_0, \bar{u}, \dot{\bar{u}})\bar{y}(t_0)) + B(t_0, \bar{u}, \dot{\bar{u}}, \ddot{\bar{u}}, \dddot{\bar{u}}, \ddddot{\bar{u}}, \bar{y}(t_0), \dot{\bar{y}}(t_0), \ddot{\bar{y}}(t_0), \dddot{\bar{y}}(t_0))\bar{p}(t_0))dt < 0,$$

which by (34) implies $\int G(t_0, \bar{x}, \bar{u}; 0)dt < 0$, a contradiction to the fact that $G(t, \bar{x}, \bar{u}; 0) = 0$, $t \in I$. Hence $\bar{x}(t) = \bar{u}(t)$, $t \in I$.

Remark 4. *The above Theorem also holds true if we replace "efficient solutions" by "feasible solutions".*

6. Related Problem

If the time dependency of problems (P) and (MWD) is removed, then these problems reduce to the following second-order multiobjective nonlinear problems studied in Mond and Zhang [13] and Gulati and Agarwal [7]. The assumptions in our converse duality theorem are similar to the assumptions in [7].

(**NP**) Minimize $(f^1(x), f^2(x), \ldots, f^k(x))$

Subject to $g(x) \leqq 0$,

$$(\mathbf{ND})\quad \text{Minimize}\quad (f^1(u)-\frac{1}{2}p^T\nabla^2 f^1(u)p, f^2(u)-\frac{1}{2}p^T\nabla^2 f^2(u)p,\dots,f^k(u)-\frac{1}{2}p^T\nabla^2 f^k(u)p)$$

$$\text{Subject to}\quad \sum_{i=1}^{k}\lambda^i(\nabla f^i(u)+\nabla^2 f^i(u)p)+\sum_{j=1}^{m}y^j(\nabla g^j(u)+\nabla^2 g^j(u)p)=0,$$

$$y^T g(u)-\frac{1}{2}p^T\nabla^2(y^T g(u))p \geqq 0,\ \lambda \geq 0,\ y \geqq 0.$$

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