**EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS**
Vol. 17, No. 4, 2024, 3492-3516
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

# Orthogonal Decompositions of Regular Graphs and Designing Tree-Hamming Codes

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**Abstract.** The paper introduces the concept of graph decomposition. That is orthogonal decompositions. Orthogonal decompositions of a graph $H$ are a partitioning $H$ into subgraphs of $H$ such that any two subgraphs intersect in at most one edge. These decompositions are called $G-$orthogonal decompositions of $H$ if and only if every subgraph in such decompositions is isomorphic to the graph $G$. Such decomposition appears in a lot of applications; statistics, information theory, in the theory of experimental design, and many others. An approach of constructing orthogonal decompositions of regular graphs is introduced here. Application to this approach for constructing tree – orthogonal decompositions of complete bipartite graphs is considered. Further, the use of orthogonal decompositions for designing tree hamming codes is also discussed along with examples. The study shows that such codes have efficient properties when used to detect and correct the errors that may occur during the transmission of data through a network. Furthermore, we present a method for the recursive construction of orthogonal decompositions.

**2020 Mathematics Subject Classifications**: 05B30, 0570, 94A60, 94A62

**Key Words and Phrases**: Graph decomposition; Orthogonal cover; Bipartite graph; Hamming codes

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## 1. Introduction

All graphs being discussed here are undirected, finite, and do not have loops or multiple edges. For standard graph-theoretic terminology, we refer to [1]. A decomposition of a graph $H$ is a set of edge-disjoint subgraphs of $H$ whose union gives the graph $H$. Thus, we say that the set $\mathcal{G} = \{G_1, G_2, \cdots, G_k\}$ of $k$ subgraphs of $H$ decompose $H$ if and only if $\bigcup_{i=1}^{k} G_i = H$ (ignoring isolated vertices) & $\bigcap_{i=1}^{k} G_i = \phi$(empty graph). If $G_i \approxeq G$ for each $i \in \{1, 2, \cdots k\}$ , then $\mathcal{G}$ is called a decomposition of $H$ by $G$. Throughout the paper we use

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DOI: https://doi.org/10.29020/nybg.ejpam.v17i4.5353

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$V(X)$, $E(X)$ for the vertex set and edge set of the graph $X$ respectively, $P_k$ for a path of $k$ vertices, $mG$ for $m$ disjoint copies of $G$, and $G \cup H$ for disjoint union of graphs $G$ and $H$. Let $\mathcal{G}_1 = \{G_1, G_2, \cdots, G_k\}$ and $\mathcal{G}_2 = \{F_1, F_2, \cdots, F_k\}$ be two distinct edge decompositions of $H$ by $G$ such that for each $i \in \{1, 2, \cdots k\}$, $G_i \approxeq G \approxeq F_i$. Such two decompositions are called *orthogonal decompositions* if $|E(G_i) \cap E(F_j)| = 1$ for all $i, j \in \{1, 2, \cdots k\}$. Whence, the collection $\{\mathcal{G}_1 \cup \mathcal{G}_2\}$ is equivalent to an *orthogonal double cover* (ODC) of $H$ by $G$. An ODC of $H$ by $G$ is a collection $\mathcal{F} = \{\phi(x) : x \in V(H)\}$ of subgraphs of $H$ all isomorphic to the graph $G$, such that every edge of $H$ belongs to exactly two elements from $\mathcal{F}$ and any two elements $\phi(x_1)$ and $\phi(x_2)$ from $\mathcal{F}$ have a common edge if and only if the edge $(x_1, x_2) \in E(H)$. Not all graphs $H$ have an ODC. The necessary condition to find an ODC of $H$ is that the graph $H$ is regular. For the complete graph $K_n$, ODC is extensively studied, we refer the reader to a survey [14]. ODC of Cayley graphs was studied in [6, 7, 10]. An ODC $\mathcal{F}$ of $H$ is cyclic (CODC) if the cyclic group of order $|V(H)|$ is a subgroup of the automorphism group of $H$ ( the group of automorphism of the graph $H$ which preserves the covering) [16]. Sampathkumar et al. [20] investigated CODCs of circulant graphs by 4-regular circulant graphs. In [11], the authors attacked the problem of the existence of ODCs of 2-regular graphs and 3-regular graphs. Sampathkumar et al. [21] presented $\sigma-$labeling, as a special category of orthogonal labeling. Using $\sigma-$labeling, they constructed CODCs of circulant graphs by some caterpillars of diameters 4. Higazy et al. [22] gave a complete classification for circulant graphs of degree five which lead to an ODC by some graphs.

The present paper is interested in studying ODC for balanced complete bipartite graph $K_{n,n}$. Given a positive integer $n$, the balanced bipartite graph $K_{n,n}$ is a bipartite graph with a $2n-$element vertex set $V$. This set $V$ is divided into two partite sets of vertices, each containing n elements. The vertices of $K_{n,n}$ are labeled by the elements of $\mathbb{Z}_n \times \{0, 1\}$ where $\mathbb{Z}_n = \{0, 1, 2, \cdots, n-1\}$ represents all residue classes modulo $n$. The edge set of $K_{n,n}$ is defined as pairs $E(K_{n,n}) = \{((u, 0), (v, 1))\, ;\ u, v \in \mathbb{Z}_n\}$. For simplicity and if there is no danger of ambiguity,we use $v_r$ for the vertex $(v, r) \in \mathbb{Z}_n \times \{0, 1\}$, and $(u_0, v_1)$ for the edge $((u, 0), (v, 1))$. We aim to construct an ODC of $K_{n,n}$ by $G$ where $G$ is isomorphic to certain trees with $n$ edges. In the next section, we introduce the fundamentals of our approach for constructing an ODC of $K_{n,n}$. We call this approach a *base-generated approach* (BGA). Section 3, shows the construction of ODCs of $K_{n,n}$ by certain trees based on BGA introduced in Section 2. An application of BGA in designing graph error detecting and correcting codes is presented in Section 4. Section 5 introduces a recursive construction of ODC of higher order balanced complete bipartite graph by disjoint trees. The conclusion of the paper and future work are presented in Section 6.

## 2. Fundamentals of base-generated approach

To construct an ODC of $K_{n,n}$ we have to find two orthogonal decompositions of $K_{n,n}$. In base-generated approach (BGA) we first seek to find the base for each decomposition.

Then ODC is generated from such two bases. Let us introduce the principals of this approach. Assume $e = (a_0, b_1)$ be an edge belonging to $E(K_{n,n})$.The length of the edge $e$ is defined by $d(e) = b - a$, where addition and subtraction are calculated modulo $n$.

**Definition 1.** *Let $G$ to be a subgraph of $K_{n,n}$, and $s \in \mathbb{Z}_n$. Then the graph $G+s$ (or $G_s$) with $E(G+s) = \{((a+s)_0, (b+s)_1);\ (a_0, b_1) \in E(G)\}$ is called $s$- translation of $G$.*

**Definition 2.** *A subgraph $G$ of $K_{n,n}$ is called a base of an edge decomposition of $K_{n,n}$ by $G$ if and only if $\bigcup_{s=0}^{n-1} \{E(G+s)\} = E(K_{n,n})$.*

The next theorem proves the validity of the method by which we build a base.

**Theorem 1.** *Let $G$ be a subgraph of $K_{n,n}$ such that $|E(G)| = n$. Then $G$ is a base of an edge decomposition of $K_{n,n}$ by $G$ if all the edges of $G$ are mutually different in lengths, i.e.$\{d(e);\ e \in E(G)\} = \mathbb{Z}_n$.*

*Proof.* Let $X = \{(a_i, b_i); i \in \mathbb{Z}_n\}$ be the set of all edges of $G$, such that the edge $e_i = (a_i, b_i)$. Since all the edges of $G$ are mutually different in lengths, then the set $X$ satisfies $\{b_i - a_i\ ; i \in \mathbb{Z}_n\} = \mathbb{Z}_n$. Let $d_i = b_i - a_i$ that is a unique for every edge $(a_i, b_i) \in E(G); i \in \mathbb{Z}_n$. For any $s, t \in \mathbb{Z}_n$ and $s \neq t$, let $(a_i + s, b_i + s) \in E(G_s)$ and $(a_i + t, b_i + t) \in E(G_t)$. Assume that $|E(G_s) \cap E(G_t)| \neq 0$, that is there is at least one edge $(a_i, b_i) \in |E(G_s) \cap E(G_t)|$. From the definition of $s$- translation of $G$, we know that

$(a_i + s - s, b_i + s - s) = (a_i, b_i)$. Also $(a_i + t - t, b_i + t - t) = (a_i, b_i)$. Since $b_i - a_i = d_i$ is unique for every edge $(a_i, b_i) \in E(G)$, then we have a contradiction. Therefore, $|E(G_s) \cap E(G_t)| = 0$ for any $s, t \in \mathbb{Z}_n$ and $s \neq t$. Moreover, $\bigcup_{i=0}^{n-1} \{E(G_i)\} = E(K_{n,n})$, thus $G$ is a base of an edge decomposition of $K_{n,n}$ by $G$.

## 2.1. Construction of an ODC of $K_{n,n}$ using two orthogonal bases

Let $G_1$ and $G_2$ be two bases of two decompositions of $K_{n,n}$. Such two bases are orthogonal if $|E(G_1) \cap E(G_2)| = 1$.

If two bases $G_1$ and $G_2$ are orthogonal, then the collection $\mathcal{G} = \{G_a^i : a \in \mathbb{Z}_n, i \in \{1, 2\}\}$ with $G_a^i = (G_i + a)$ is equivalent to an ODC of $K_{n,n}$. Moreover, $\mathcal{G}$ represents an ODC of $K_{n,n}$ by $G$ if $G_1 \approxeq G \approxeq G_2$.

**Definition 3.** *Let $G$ to be be a subgraph of $K_{n,n}$, the subgraph $G'$ of $K_{n,n}$ with $E(G') = \{\{a_0, b_1\} : \{b_0, a_1\} \in E(G)\}$ is called the symmetric graph of $G$.*

**Remark 1.** *The graph $G'$ is a base of a decomposition of $K_{n,n}$ by $G$ if and only if $G$ is also a base of a decomposition of $K_{n,n}$ by $G$.*

in the next section, we use an algebraic representation for a base $G$ of a decomposition of $K_{n,n}$.

## 2.2. Bases representation

The representation of the base G is given by the ordered n-tuple $v(G) = (v_0, v_1, \cdots, v_{n-1}) \in \underbrace{\mathbb{Z}_n \times \mathbb{Z}_n \times \cdots \times \mathbb{Z}_n}_{n \text{ times}}$ such that $v_i \in \mathbb{Z}_n$ for all $i \in \{0, 1, 2, \cdots, n-1\}$ and $(v_i)_0$ is the unique vertex $((v_i, 0) \in \mathbb{Z}_n \times \{0\})$ that belongs to the unique edge of length $i$ in $G$. The edge set of $G$ is $E(G) = \{((v_i)_0, (v_i + i)_1); 0 \leq i \leq n-1\}$. For instance, the base in Figure 1 is represented by the vector $v(P_4) = (0, 1, 1)$, (*e.g.* $(1_0, 0_1)$ is the unique edge of length 2, thus $v_2 = 1$). If $G$ is a base, we will call also $v(G)$ a base.

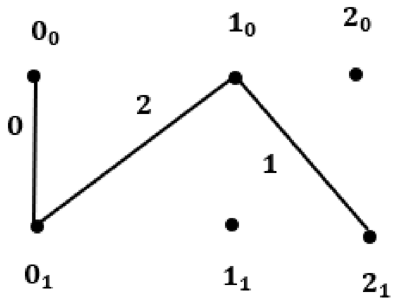


Figure 1: The base $P_4$ in $K_{3,3}$ with the length of each edge, where $v(P_4) = (0, 1, 1)$.

**Definition 4.** *Let $v(G_1)$ and $v(G_2)$ be two different bases in $K_{n,n}$. Then, $G_1$ and $G_2$ are orthogonal if and only if $\{v_i(G_1) - v_i(G_2) : i \in \mathbb{Z}_n\} = \mathbb{Z}_n$.*

**Theorem 2.** *If $G_0^1$ and $G_0^2$ are the orthogonal bases in $K_{n,n}$, then the collections $\mathcal{G}_1$=$\{G_{0+x}^1 : x \in \mathbb{Z}_n\}$ and $\mathcal{G}_2$=$\{G_{0+y}^2 : y \in \mathbb{Z}_n\}$ with $\left(G_{0+x}^i = (G_0^i + x)\right)$ and $\left(G_{0+y}^i = (G_0^i + y)\right)$ for $i \in \{1, 2\}$ represent an ODC of $K_{n,n}$.*

*Proof.* Since $G_0^1$ and $G_0^2$ are bases then each collection from the collections $\mathcal{G}_1$=$\{G_{0+x}^1 : x \in \mathbb{Z}_n\}$ and $\mathcal{G}_2$=$\{G_{0+y}^2 : y \in \mathbb{Z}_n\}$ forms an edge decomposition of $K_{n,n}$. Assume that the edge $\{a_0, b_1\} \in E(K_{n,n})$. Then there are exactly two graphs $G_x^1$ and $G_y^2$ from the collections $\mathcal{G}_1$ and $\mathcal{G}_2$ such that $\{a_0, b_1\} \in E(G_x^1)$ and $\{a_0, b_1\} \in E(G_y^2)$. Moreover, for any $x, y \in \mathbb{Z}_n$ and $s, t \in \{1, 2\}$, $\left|E(G_{0+x}^s) \cap E(G_{0+y}^t)\right| = 0$ whenever $s = t$. Besides that, there is a unique $v_i$ satisfies $v_i(G_0^1) - v_i(G_0^2) = b - a$ and $v_i(G_0^1) + a = v_i(G_0^2) + b$. Thus, there is exactly one edge $l = \{(v_i(G_0) + a)_0, (v_i(G_0) + a + v_i)_1\} \in E(G_{0+x}^1)$ and $l = \{(v_i(G_1) + b)_0, (v_i(G_1) + b + v_i)_1\} \in E(G_{0+y}^2)$. Thus, $\left|E(G_{0+x}^s) \cap E(G_{0+y}^t)\right| = 1$ whenever $s \neq t$.

Hereafter, we show a method by which we can construct an ODC of $K_{n,n}$ using only one base instead of two bases.

**Definition 5.** *A base $G$ is called a symmetric base with respect to $\mathbb{Z}_n$ if $v(G)$ and $v(G')$ are orthogonal.*

The following theorem introduces the condition for a symmetric base.

**Theorem 3.** *A base $G$ in $K_{n,n}$ represented by the vector $v(G) = (v_0, v_1, \cdots, v_{n-1})$ is called a symmetric base if it satisfies $\{v_i - v_{-i} + i\} = \mathbb{Z}_n$ for all $i \in \mathbb{Z}_n$.*

*Proof.* Since $G$ is a base in $K_{n,n}$.The graph $G^{'}$ is also a base in $K_{n,n}$ represented by $v(G^{'})$. Let the edge $e = \left\{v_i(G^{'}), v_i(G^{'}) + i\right\} \in E\left(G^{'}\right)$ such that the length of $e$ equals $i$. Following the definition of a symmetric graph, the edge $\left\{v_i(G^{'}) + i, v_i(G^{'})\right\}$ of length $-i$ will belong to $E(G)$. Whence, $v_{-i}(G) = v_i(G^{'}) + i$, which means that $v_i(G^{'}) = v_{-i}(G) - i$. Following Definition 4, $v(G)$ and $v(G^{'})$ are orthogonal if $\left\{v_i(G) - v_i(G^{'});\ i \in \mathbb{Z}_n\right\} = \mathbb{Z}_n$. Then the necessary and sufficient condition for the orthogonality of the bases $G$ and $G'$ is $\{v_i(G) - v_{-i}(G) + i; i \in \mathbb{Z}_n\} = \mathbb{Z}_n$. Since $G^{'}$ is the symmetric graph of $G$, then the base $G$ is a symmetric base with respect to $\mathbb{Z}_n$ if and only if $\{v_i(G) - v_{-i}(G) + i;\ i \in \mathbb{Z}_n\} = \mathbb{Z}_n$.

Theorem 3 yields the next theorem.

**Theorem 4.** *There is an ODC of $K_{n,n}$ by $G$ if there exists a symmetric base that is isomorphic to the graph $G$.*

In the next section we use the symmetric base in constructing an ODC of $K_{n,n}$ by different classes of graphs.

**Definition 6.** *Let $r \geq 1$, and $n_1, n_2, n_3, \cdots, n_r$ be positive integers such that $n_i \geq 0$ for $i \in \{1, 3, \cdots, r\}$.The caterpillar tree $C_r(n_1, n_2, n_3, \cdots, n_r)$ is the tree obtained from the path $P_r = x_1x_2x_3 \cdots x_r$ by joining vertex $x_i$ to $n_i$ new vertices, $i \in \{1, 2, \cdots, r\}$.*

**Definition 7.** *Let $\delta, \alpha$ be positive integers and the parameter $x_{\delta,} \geq 0$. The rooted tree $\tau^{\alpha}(x_1, x_2, x_3, \cdots, x_\delta)$ is the tree with a root $\alpha\, (\alpha \in V\,(K_{n,n}))$ and for all $i \in \{1, 2, \cdots, \delta\}$, $x_i$ is the number of leaves of level $i$.*

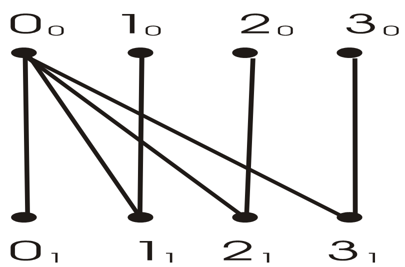


Figure 2: Tree $\tau^{0_0}(1, 3)$ as a subgraph of $K_{4,4}$.

For example, the graph in Figure 2, represents a tree $\tau^{\alpha}(x_1, x_2) \approxeq \tau^{0_0}(1, 3)$ as a subgraph of $K_{4,4}$. Such tree $\tau^{0_0}(1, 3)$ is a rooted tree with a root $\alpha = 0_0$, $x_1 = 1$ (one leaf in level 1), while $x_2 = 3$ (3 leaves with level 2).

**Remark 2.** *A Path $P_r$ is caterpillar tree $C_r(\underbrace{0, 0, 0, \cdots, 0}_{r})$*

## 3. Designing orthogonal decompositions of $K_{n,n}$ by certain trees

In this section, we claim to construct ODC of $K_{n,n}$ by certain trees based on BGA approach introduced in Section 2.

### 3.1. Orthogonal decompositions of $K_{n,n}$ by rooted trees

**Theorem 5.** *Let $n \geq 3$ be a positive integer and $x_1 \in \{1, 2\}$, then there is a symmetric base of an ODC of $K_{n,n}$ by $G \approxeq \tau^{0_1}(x_1, \lfloor \frac{n-1}{2} \rfloor)$.*

*Proof.* **case 1.** Let $x_1 = 1$, $m \in \mathbb{Z}^+$ and $n = 2m+1$. Then the vector $v(G)$ of the base $G$, is defined as

$$v_i(G) = \begin{cases} i & \text{if} \quad 0 \leq i \leq m \\ -i & \text{if} \quad m+1 \leq i \leq 2m \end{cases}$$

Therefore,

$$v_{-i}(G) = \begin{cases} i & \text{if} \quad 0 \leq i \leq m \\ -i & \text{if} \quad m+1 \leq i \leq 2m \end{cases}$$

For all $i \in \mathbb{Z}_{2m+1},\ v_i - v_{-i} + i = i$. By Theorem 3, $G$ is a symmetric base. By definition of $v(G)$, the graph $G \approxeq \tau^{0_1}(1, m)$.

$$E(G) = \{(0_0, 0_1)\} \cup \{(i_0, 0_1) : 1 \leq i \leq m\} \cup \{(i_0, (2i)_1 : 1 \leq i \leq m)\}$$

**Case 2.** Let $x_1 = 2$, $m \in \mathbb{Z}^+$ and $n = 2m$, then the vector $v(G)$ of the base $G$ is defined as:

$$v_i(G) = v_{-i}(G) = \begin{cases} i & \text{if} \quad 0 \leq i \leq m \\ -i & \text{if} \quad m+1 \leq i \leq 2m-1 \end{cases}$$

For $i \in \mathbb{Z}_{2m},\ v_i - v_{-i} + i = i$. By Theorem 3, $G$ is a symmetric base. By definition of $v(G)$, for any $i \in \mathbb{Z}_{2m,}$ the graph $G \approxeq \tau^{0_1}(2, m-1)$.

$$E(G) = \{(0_0, 0_1)\} \cup \{(i_0, 0_1) : 1 \leq i \leq m\} \cup \{(i_0, (2i)_1 : 1 \leq i \leq m-1)\}$$

**Example 1.** *There is an ODC of $K_{5,5}$ by $\tau^{0_1}(1, 2)$, since $v(G) = (0, 1, 2, 2, 1)$ is a symmetric base, see Figure 3.*

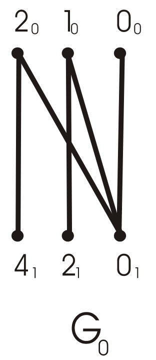


Figure 3: Symmetric base of an ODC of $K_{5,5}$ by $\tau^{0_1}(1,2)$.

**Theorem 6.** *Let $n \geq 5$ be a positive integer, then there is an ODC of $K_{n,n}$ by $\tau^{0_0}(n-4,2)$*

*Proof.* For any positive integer $n \geq 5$, the vectors $v(G)$ and $u(F)$ of the bases $G$ and $F$ are represented as :

$$v_i(G) = \begin{cases} (n-2)i & \text{if} \quad i = 0, n-2 \\ -2i-1 & \text{if} \quad i = 1, n-1 \\ -i & \quad otherwise \end{cases}$$

$$u_i(F) = \begin{cases} (n-1)i & \text{if} \quad i = 0, n-2 \\ -i-1 & \text{if} \quad i = 1, n-1 \\ 0 & \quad otherwise \end{cases}$$

From the representation of $v(G)$ and $u(F)$, $v_i(G)-u_i(F) = 0$ at $i = 0$, $v_i(G)-u_i(F) = 2$ at $i = n-2$, and $v_i(G) - u_i(F) = -i$ otherwise. Consequently, $\{v_i(G) - u_i(F); i \in \mathbb{Z}_n\} = \mathbb{Z}_n$. By theorem 4, the two bases $v(G)$ and $u(F)$ are orthogonal. Moreover, the edges set $E(G)$ and $E(F)$ of the bases $G$ and $F$ respectively can be represented by:

$$E(G) = \{(((n-2)i)_0\,, ((n-1)i)_1\,;\; i = 0, n-2\}$$

$$\cup\{((-2i-1)_0\,, (-i-1)_1)\,; i = 1, n-1\}$$

$$\cup\{(-i_0, 0_1); i \in \mathbb{Z}_n \backslash \{0, 1, n-2, n-1\}\}$$

and

$$E(F) = \{(((n-1)i)_0\,, 0_1);\; i = 0, n-2\}$$

$$\cup\{((-i-1)_0\,, (n-1)_1); i = 1, n-1\}$$

$$\cup\{(0_0, i); i \in \mathbb{Z}_n \backslash \{0, 1, n-2, n-1\}\}$$

Therefore, $G \cong F \cong \tau^{0_0}(n-4, 2)$.

**Theorem 7.** *Let* $n > 3$ *to be an odd integer and* $x_1, x_2$ *and* $\gamma \in \mathbb{Z}_n$. *Then there is a symmetric base of an ODC of* $K_{n,n}$ *by* $G \approxeq \tau^{0_0}(x_1, x_2) \cup \gamma K_2$.

*Proof.* **Case 1.** Let $m, n$ be positive integers such that $n = 2m+1$, $n > 3$ and $n \not\equiv 0 \bmod 3$. The vector $v(G)$ of the base $G$ is defined as:

$$v_i(G) = \begin{cases} 2i & \text{if} \quad 0 \le i \le m \\ 0 & \text{if} \quad m+1 \le i \le 2m \end{cases}$$

Hence,

$$v_{-i}(G) = \begin{cases} 0 & \text{if} \quad 0 \le i \le m \\ -2i & \text{if} \quad m+1 \le i \le 2m \end{cases}$$

For $i \in \mathbb{Z}_{2m+1}$, $v_i - v_{-i} + i = 3i$,since $\gcd(3, 2m+1) = 1$, By Theorem 3, $G$ is a symmetric base. By definition of $v(G)$, the graph $G \approxeq \tau^{0_0}(\lceil \frac{n+1}{3} \rceil, \lceil \frac{n-3}{6} \rceil) \cup \lfloor \frac{n-1}{3} \rfloor K_2$, where $x_1 = \lceil \frac{n+1}{3} \rceil$, $x_2 = \lceil \frac{n-3}{6} \rceil$ and $\gamma = \lfloor \frac{n-1}{3} \rfloor$.

**Case 2.** For an odd integer $n > 3$, $n \equiv 0 \bmod 3$, the vector $v(G)$ of the base $G$ is represented as :

$$v_i(G) = \begin{cases} 2i & \text{if} \quad 0 \le i \le \frac{n-1}{2} \\ 0 & \text{if} \quad \frac{n+1}{2} \le i \le n-1 \end{cases}$$

Hence,

$$v_{-i}(G) = \begin{cases} 0 & \text{if} \quad 0 \le i \le \frac{n-1}{2} \\ -2i & \text{if} \quad \frac{n+1}{2} \le i \le n-1 \end{cases}$$

For $i \in \mathbb{Z}_n$, $v_i - v_{-i} + i = 3i$. By Theorem 3, $G$ is a symmetric base. By the definition of $v(G)$, the graph $G \approxeq \tau^{0_0}(\frac{n}{3}, \lfloor \frac{n+1}{5} \rfloor) \cup \lfloor \frac{n-4}{5} \rfloor K_2$, where $x_1 = \frac{n}{3}$, $x_2 = \lfloor \frac{n+1}{5} \rfloor$ and $\gamma = \lfloor \frac{n-4}{5} \rfloor$.

**Theorem 8.** *Let* $n > 3$,*and* $n \not\equiv 0 \bmod 3$ *be an even integer, then there is an ODC of* $K_{n,n}$ *by* $G \approxeq \tau^{0_0}(\lceil \frac{n+2}{3} \rceil, \lceil \frac{n-3}{6} \rceil) \cup \lfloor \frac{n-2}{3} \rfloor K_2$.

*Proof.* For any even integer $n = 2m$, $m \in \mathbb{Z}^+$, the vector $v(G)$ of the base $G$ is defined as :

$$v_i(G) = \begin{cases} 2i & \text{if} \quad 0 \le i \le \lceil \frac{n-1}{2} \rceil \\ 0 & \text{if} \quad \lceil \frac{n+1}{2} \rceil \le i \le n-1 \end{cases}$$

Hence,

$$v_{-i}(G) = \begin{cases} 0 & \text{if} \quad 0 \le i \le \lceil \frac{n-1}{2} \rceil \\ -2i & \text{if} \quad \lceil \frac{n+1}{2} \rceil \le i \le n-1 \end{cases}$$

For $i \in \mathbb{Z}_{2m}$, $v_i - v_{-i} + i = 3i$. Since $\gcd(3, 2m) = 1$, By Theorem 3, $G$ is a symmetric base. Moreover, the edges set $E(G)$ of the basis $G$ can be represented by:

$$E(G) = \left\{ ((2i)_0, (3i)_1) : \ 0 \le i \le \lceil \frac{n-1}{2} \rceil \right\} \cup \{(0_0, i_1) : \ \lceil \frac{n+1}{2} \rceil \le i \le n-1\}.$$

By the definition of $v(G)$, the graph $G \approxeq \tau^{0_0}(\lceil \frac{n+2}{3} \rceil, \lceil \frac{n-3}{6} \rceil) \cup \lfloor \frac{n-2}{3} \rfloor K_2$.

**Theorem 9.** *Let* $m > 2$*,* $m \not\equiv 0 \bmod 3$*, and* $m+3$ *to be an odd integer. Then there is a symmetric base of an ODC of* $K_{m+3,m+3}$ *by* $G \cong \tau^{0_1}(1,2) \cup (n-5)K_2$.

*Proof.* For all odd integer $n > 5, n = m+3$ such that $m > 2$, $m \not\equiv 0 \bmod 3$, the vector $v(G)$ of the symmetric base $G$ can be defined as:

$$v_i(G) = \begin{cases} 0 & \text{if} \quad i = 0 \\ n-1 & \text{if} \quad i = 1 \\ 3 & \text{if} \quad i = n-1 \\ x_j - i & \text{if} \quad i \in \{2, 3, \cdots, m+1\}, \, j = i-2 \end{cases},$$

Where $x_j = 1-j$, for $j \in \{0, 1, 2, \cdots, m-1\}$. From the definition of $v(G)$, we find that $v_i - v_{-i} + i = 0$, for $i = 0$, for $i = 1$, $v_i - v_{-i} + i = n-3$, for $i = n-1$, $v_i - v_{-i} + i = 3$, and for $i = j+2$, $j \in \{0, 1, 2, \cdots, m-1\}$, $v_i - v_{-i} + i = x_j - x_{m-(j+1)} - i$. By Theorem 3, the base $v(G)$ is symmetric. Moreover, the edges set $E(G)$ of the base $G$ can be represented by:

$$\begin{aligned} E(G) &= \{(0_0, 0_1), ((n-1)_0, 0_1), (3_0, 2_1)\} \cup \{((x_j - i)_0, (x_j)_1)\} \\ \text{for all } i &= j+2, \; j \in \{0, 1, 2, \cdots, m-1\}. \end{aligned}$$

Then the graph $G \cong \tau^{0_1}(1,2) \cup (n-5)K_2$.

**Theorem 10.** *Let* $n \geq 3$ *to be a positive integer, then there is a symmetric base of an ODC of* $K_{n,n}$ *by* $C_4(0,0,0,0) \cup \tau^{(n-1)_1}(n-3)$.

*Proof.* For a positive integer $n \geq 3$, the vector $v(G)$ of the base $G$ can be written as:

$$v_i(G) = \begin{cases} 0 & \text{if} \quad i = 0 \\ n-1 & \text{if} \quad i = 1, n-1 \\ -i-1 & \text{otherwise} \end{cases},$$

therefore,

$$v_{-i}(G) = \begin{cases} 0 & \text{if} \quad i = 0 \\ n-1 & \text{if} \quad i = 1, n-1 \\ i-1 & \text{otherwise} \end{cases},$$

For $i \in \{0, 1, n-1\}$, $v_i - v_{-i} + i = i$. For any $i \in \mathbb{Z}_n \setminus \{0, 1, n-1\}$, $v_i - v_{-i} + i = -i$. Consequently, $\{v_i - v_{-i} + i;\ i \in \mathbb{Z}_n\} = \mathbb{Z}_n$. By Theorem 3, the base $v(G)$ is symmetric. Moreover the edges set $E(G)$ of the base $G$ can be represented by:

$$\begin{aligned} E(G) &= \{(0_0, 0_1), (0_1, (n-1)_0), ((n-1)_0, (n-2)_1)\} \\ \cup\{(\beta_0, (n-1)_1) &: 1 \leq \beta \leq n-3\} \end{aligned}$$

.

Then the graph $G \cong C_4(0,0,0,0) \cup \tau^{(n-1)_1}(n-3)$.

### 3.2. ODCs of $K_{n,n}$ by a combination of caterpillar trees

**Theorem 11.** *Let $\alpha$, $\beta$ and $\gamma$ be elements of $\mathbb{Z}_n$. For all positive integers $n \geq 7$, and $n \not\equiv 0 \bmod 3$, there is a symmetric base of an ODC of $K_{n,n}$ by $\alpha C_4(0,0,0,0) \cup \beta C_3(0,0,0) \cup \gamma C_2(0,0)$.*

*Proof.* **Case 1.** For an even integer $n$, let $\alpha = 2$, $\beta = \frac{n-8}{2}$, and $\gamma = 2$. The vector $v(G)$ of a base $G$ can be defined as:

$$v_i(G) = \begin{cases} -2i-1 & \text{if} \quad i = 1, n-1 \\ -2i & otherwise \end{cases},$$

Hence,

$$v_{-i}(G) = \begin{cases} 2i-1 & \text{if} \quad i = 1, n-1 \\ 2i & otherwise \end{cases}$$

For any $i \in \mathbb{Z}_n$, $v_i - v_{-i} + i = -3i$. By Theorem 3, the base $v(G)$ is symmetric. Moreover, the edges set $E(G)$ of the base $G$ can be represented by:

$$E(G) = \{1_0, 0_1, 0_0, (\tfrac{n}{2})_1, 1_0\} \cup \{(n-3)_0, (n-2)_1, (\tfrac{n}{2})_0, (\lfloor\tfrac{n-3}{2}\rfloor)_1\}$$

$$\cup\{(2, \tfrac{n+2}{2}), (n-2, \tfrac{n-2}{2})\} \cup \{(\tfrac{n+4}{2})_1, 4_0, 2_1\} \cup \{(\tfrac{n+6}{2})_1, 6_0, 3_1\}$$

$$\cup\{(\tfrac{n+8}{2})_1, 8_0, 4_1\}\cup, ..., \cup\{(n-3)_1, (n-6)_0, (\tfrac{n-6}{2})_1\}$$

then the graph $G \approxeq 2C_4(0,0,0,0) \cup (p-8)C_3(0,0,0) \cup 2C_2(0,0)$.

**Case 2.** For an $n$ odd integer $n$, let $\alpha = 1$, $\beta = 1$, and $\gamma = n-5$. The vector $v(G)$ of a base $G$ can be defined as:

$$v_i(G) = \begin{cases} -2i-1 & \text{if} \quad i = 1, n-1 \\ -2i & otherwise \end{cases}$$

Hence,

$$v_{-i}(G) = \begin{cases} 2i-1 & \text{if} \quad i = 1, n-1 \\ 2i & otherwise \end{cases}$$

For any $i \in \mathbb{Z}_n$, $v_i - v_{-i} + i = -3i$. By Theorem 3, the base $v(G)$ is symmetric. Moreover the edges set $E(G)$ of the base $G$ can be represented by:

$$E(G) = \{(-2i-1)_0, (-i-1)_1; i \in \{1, n-1\}\} \cup \{(-2i)_0, (-i)_1; i \in \mathbb{Z}_n \setminus \{1, n-1\}\}$$

then the graph $G \approxeq C_4(0,0,0,0) \cup C_3(0,0,0) \cup (n-5)C_2(0,0)$.

**Theorem 12.** *Let $p \geq 13$ to be a prime integer, then there an ODC of $K_{p,p}$ by $G = C_4(0,0,0,0) \cup C_3(0,0,0) \cup (p-5)C_2(0,0)$.*

*Proof.* For any prime integer $p \geq 13$, the vectors $v(G)$ and $u(F)$ of the bases $G$ and $F$ are:

$$v_i(G) = \begin{cases} (\frac{p-5}{2})i & \text{if} \quad i = 0,1 \\ (\frac{p-3}{2})i & \text{if} \quad 2 \leq i \leq p-3 \\ (\frac{p-5}{2})i - 2 & \text{if} \quad i = p-2, p-1 \end{cases},$$

and

$$u_i(F) = \begin{cases} (\lceil \frac{p-2}{2} \rceil)i & \text{if} \quad i = 0,1 \\ (\lceil \frac{p}{2} \rceil)i & \text{if} \quad 2 \leq i \leq p-3 \\ \lceil \frac{p-2}{2} \rceil)i - 2 & \text{if} \quad i = p-2,\ p-1 \end{cases},$$

Hence, $v_i(G) - u_i(F) = (\frac{p-5}{2})i - \left(\lceil \frac{p-2}{2} \rceil\right) i$ if $i \in \{0,1\}$, $v_i(G) - u_i(F) = (\frac{p-3}{2})i - (\lceil \frac{p}{2} \rceil)\, i$ if $i \in \{2,3,\cdots,p-3\}$, and $v_i(G) - u_i(F) = (\frac{p-5}{2})i - \lceil \frac{p-2}{2} \rceil)i$ if $i \in \{p-2, p-1\}$. By theorem 4, then the two bases $v(G)$ and $u(F)$ are orthogonal. Moreover, the edges set $E(G)$ and $E(F)$ of the bases $G$ and $F$ respectively can be represented by:

$$E(G) = \left\{ \left( \left( (\tfrac{p-5}{2})i \right)_0, \left( (\tfrac{p-3}{2})i \right)_1 \right);\ \ i \in \{0,1\} \right\}$$

$$\cup \left\{ \left( \left( (\tfrac{p-3}{2})i \right)_0, \left( (\tfrac{p-1}{2})i \right)_1 \right);\ \ i \in \{2,3,\cdots,p-3\} \right\}$$

$$\cup \left\{ \left( \left( (\tfrac{p-5}{2})i - 2 \right)_0, \left( (\tfrac{p-3}{2})i - 2 \right)_1 \right);\ \ i \in \{p-2, p-1\} \right\}.$$

And

$$E(F) = \left\{ \left( \left( \lceil \tfrac{p-3}{2} \rceil i \right)_0, \left( \lceil \tfrac{p-1}{2} \rceil i \right)_1 \right);\ \ i \in \{0,1\} \right\}$$

$$\cup \left\{ \left( \left( \lceil \tfrac{p-5}{2} \rceil i \right)_0, \left( \lceil \tfrac{p-3}{2} \rceil i \right)_1 \right);\ \ i \in \{2,3,\cdots,p-3\} \right\}$$

$$\cup \left\{ \left( \left( \left( \lceil \tfrac{p-3}{2} \rceil i \right) - 2 \right)_0, \left( \left( \lceil \tfrac{p-1}{2} \rceil i \right) - 2 \right)_1 \right);\ \ i \in \{p-2, p-1\} \right\}.$$

Hence $G \approxeq F \approxeq C_4(0,0,0,0) \cup C_3(0,0,0) \cup (p-5)C_2(0,0)$.

**Theorem 13.** *Let $n \geq 8$ to be a positive integer, then there is an ODC of $K_{n,n}$ by $C_5(0,0,(n-4),0,0)$.*

*Proof.* For all positive integers $n \geq 8$, the vectors $v(G)$ and $v(F)$ of the bases $G$ and

$F$ are defined as:

$$v_i(G) = \begin{cases} -3i & \text{if} \quad i = 0, 1 \\ -i & \text{if} \quad 2 \le i \le n-3 \\ -2-3i & \text{if} \quad i = n-2, n-1 \end{cases}$$

and

$$v_i(F) = \begin{cases} -2i & \text{if} \quad i = 0, 1 \\ 0 & \text{if} \quad 2 \le i \le n-3 \\ -2i-2 & \text{if} \quad i = n-2, n-1 \end{cases}$$

Since $\{v_i(G) - v_i(F); i \in \mathbb{Z}_n\} = \mathbb{Z}_n$, following Theorem 4, the bases $G$ and $F$ are orthogonal. From the above definitions of $v_i(G)$ and $v_i(F)$ with the two relations $E(G) = \{(v_i(G))_0, (v_i(G)+i)_i\}$ and $E(F) = \{(v_i(F))_0, (v_i(F)+i)_i\}$ for all $i \in \mathbb{Z}_n$, then $G \approxeq C_5(0, 0, (n-4), 0, 0) \approxeq F$.

**Theorem 14.** *Let $n$ to be a positive integer such that $n \ge 5$, then there is a symmetric base of an ODC of $K_{n,n}$ by $G = C_3(1, 0, n-3)$.*

*Proof.* For $n \ge 5$, the vector $v(G)$ of the base $G$ can be defined by:

$$v_i(G) = v_{-i}(G) = \begin{cases} n-1 & \text{if} \quad i = 1, n-1 \\ 1 & \quad \text{otherwise} \end{cases}$$

For any $i \in \mathbb{Z}_n$, $v_i - v_{-i} + i = i$. By Theorem 3, the base $v(G)$ is a symmetric. Moreover, the edges set $E(G)$ of the base $G$ can be represented by:

$$E(G) = \{0_1, (n-1)_0, (n-2)_1\} \cup^{1_0} \{(1_0, \beta_1) : \beta \in \{1, 3, \cdots, n-1\}\}$$

Thus, the graph $G \cong C_3(1, 0, n-3)$.

**Theorem 15.** *Let $p \ge 5$ to be prime integer, then there is a symmetric base of an ODC of $K_{p,p}$ by $C_5(0, 0, 0, 0, 0) \cup (p-4)C_2(0, 0)$.*

*Proof.* For a prime integer $p \ge 5$, the vector $v(G)$ of the base $G$ is defined as:

$$v_i(G) = \begin{cases} i-1 & \text{if} \quad i = 1, p-1 \\ 3+i & \quad \textit{otherwise} \end{cases}$$

hence,

$$v_{-i}(G) = \begin{cases} -i-1 & \text{if} \quad i = 1, p-1 \\ 3-i & \quad \textit{otherwise} \end{cases}$$

For any $i \in \mathbb{Z}_p$, $v_i - v_{-i} + i = 3i$, Since $\gcd(3, p) = 1$, By Theorem 3, $G$ is a symmetric base. By the definition of $v(G)$, the base $G \cong C_5(0, 0, 0, 0, 0) \cup (p-4)C_2(0, 0)$.

**Theorem 16.** *Let* $p \geq 5$ *to be prime integer such that* $p = 5 + 6n$, *and* $n$ *be a non negative integer, such that* $n \neq 0 \bmod 5$, *then there is a symmetric base of an ODC of* $K_{p,p}$ *by* $C_3(0,0,0) \cup C_{p-1}(\underbrace{0,0,\cdots,0}_{p-1\ times})$.

*Proof.* For all prime integers $p \geq 5$, $p = 5 + 6n$, and a non negative integer $n \neq 0 \bmod 5$.The vector $v(G)$ of the base $G$ can be defined as:

$$v_i(G) = \begin{cases} 0 & \text{if} \quad i = 0 \\ i^2 + i + 1 & \quad otherwise \end{cases},$$

hence,

$$v_{-i}(G) = \begin{cases} 0 & \text{if} \quad i = 0 \\ i^2 - i + 1 & \quad otherwise \end{cases}$$

From the definition of $v_i(G)$ and $v_{-i}(G)$, $\{v_i - v_{-i} + i; i \in \mathbb{Z}_p\} = \mathbb{Z}_p$. By Theorem 3, the base $v(G)$ is symmetric. Moreover, the edges set $E(G)$ of the base $G$ can be represented by:

$$E(G) = \{(0_0, 0_1), (1_0, 0_1)\} \cup \{\left((i^2+i+1)_0, \left((i+1)^2\right)_1\right); 1 \leq i \leq p-2\}$$

thus the graph $G \cong C_3(0,0,0) \cup C_{p-1}(\underbrace{0,0,\cdots,0}_{p-1\text{ times}})$.

**Theorem 17.** *Let* $m \geq 2$, *then there is a symmetric base of an ODC of* $K_{m+3,m+3}$ *by* $G \cong C_4(0,0,1,m-1)$.

*Proof.* For a positive integer $m \geq 2$,the vector $v(G)$ of the base $G$ in $K_{m+3,m+3}$ can be defined as:

$$v_i(G) = \begin{cases} 0 & \text{if} \quad i = 0,1 \\ 2 & \text{if} \quad i = m+2 \\ x_j & \text{if} \quad i \in \{2,3,\cdots,m+1\}, j = i-2 \end{cases},$$

Where $x_j = 1 - j$, for $j \in \{0,1,2,\cdots,m-1\}$. Moreover,

$$v_i - v_{-i} + i = \begin{cases} 0 & \text{if} \quad i = 0 \\ -i & \text{if} \quad i \in \{1, m+2\} \\ x_j - x_{m-(j+1)} + i & \text{if} \quad i = j+2; \ j \in \{0,1,2,\cdots,m-1\}. \end{cases}$$

Hence, $\{v_i - v_{-i} + i; i \in \mathbb{Z}_{m+3}\} = \mathbb{Z}_{m+3}$. By Theorem 3, the base $v(G)$ is symmetric. Moreover, the edges set $E(G)$ of the base $G$ can be represented by:

$$E(G) = \{(0_0, 0_1), (1_0, 3_1)\} \cup (2_0, 1_1), (1_1, 0_0), (0_0, 3_1)\}$$

$$\cup\{((3_1, (3-i)_0) : 4 \le i \le m+1\}$$

That yields $G \cong C_4(0, 0, 1, m-1)$.

**Theorem 18.** *Let $p$ to be a prime integer s.t $p \ge 13$, and $m = n - 3$. Then there is an ODC of $K_{p,p}$ by $C_4(0,0,0,0) \cup 2C_3(0,0,0) \cup (p-7)C_2(0,0,)$.*

*Proof.* For a prime integer $p \ge 13$, the vectors $v(G)$ and $u(F)$ of the bases $G$ and $F$ are defined as :

$$v_i(G) = \begin{cases} i & \text{if} \quad i = 0, p-1 \\ 3 & \text{if} \quad i = 1 \\ x_j + 3i & \text{if} \quad i = j+2,\ j \in \{0, 1, 2, \cdots, m-1\} \end{cases},$$

and

$$u_i(F) = \begin{cases} 0 & \text{if} \quad i = 0 \\ (n-2) & \text{if} \quad i = 1 \\ 2-2i & \text{if} \quad i = p-1 \\ x_j - 2i & \text{if} \quad i = j+2,\ j \in \{0, 1, 2, \cdots, m-1\} \end{cases},$$

For all $i \in \mathbb{Z}_p$, $v_i(G) - u_i(F) = 5i$. By theorem 4, then the two bases $v(G)$ and $u(F)$ are orthogonal. Moreover the edges set $E(G)$ and $E(F)$ of the bases $G$ and $F$ respectively can be represented by:

$$E(G) = \{(0_0, 0_1), ((p-1)_0, (p-2)_1), (3_0, 4_1)\}$$

$$\cup\{\left((x_j + 3i)_0, (x_j + 4i)_1\right);\ i = j+2,\ j \in \{0, 1, 2, \cdots, m-1\}\}$$

and

$$E(F) = \{(0_0, 0_1), ((p-2)_0, (p-1)_1), (4_0, 3_1)\}$$

$$\cup\{\left((x_j - 2i)_0, (x_j - i)\right);\ i = j+2,\ j \in \{0, 1, 2, \cdots, m-1\}\}$$

which yields $G \cong F \cong C_4(0,0,0,0) \cup 2C_3(0,0,0) \cup (p-7)C_2(0,0,)$.

**Theorem 19.** *Let $p \ge 7$ to be a prime integer, then there is an ODC of $K_{n,n}$ by $2C_3(0,0,0) \cup (p-4)C_2(0,0)$.*

*Proof.* For any prime integer $p \ge 7, \alpha_1 = \lfloor \frac{3p-7}{10} \rfloor, \alpha_2 = \lfloor \frac{2p}{3} \rfloor - 1$, the vectors $v(G)$ , $v(M)$ and $v(F)$ of the starters $G, M$ and $F$ are defined as:

**Case 1.** let $p$ be a prime integer such that $p \equiv 1 \bmod 6$ and $\alpha_1$be a positive integer $\alpha_1 = \lfloor \frac{3p-7}{10} \rfloor$. The vectors $v(G)$ , $u(M)$ of the bases $G$ and $M$ are defined as:

$$v_i(G) = \begin{cases} 0 & \text{if} \quad i = 0, p-2 \\ p-1 & \text{if} \quad i = 1, p-1 \\ i & \quad otherwise \end{cases}$$

and

$$u_i(M) = \begin{cases} \alpha_1 i & \text{if} \quad i = 0, p-2 \\ \alpha_1 i - 1 & \text{if} \quad i = 1, p-1 \\ (\alpha_1 + 1)i & \quad otherwise \end{cases}$$

Therefore, $v_i(G) - u_i(M) = -\alpha_1 i$ for any $i \in \mathbb{Z}_p$. Consequently, $\{v_i(G) - u_i(M)\} = \mathbb{Z}_p$. By theorem 4, then the two bases $v(G)$ and $u(M)$ are orthogonal. Moreover, the edges set $E(G)$ and $E(M)$ of the bases $G$ and $M$ respectively, can be represented by:

$$\begin{gathered} E(G) = \{(0_0, 0_1), (0_0, -2_1)\} \\ \cup\{((p-1)_0, (i-1)_1); i = 1, p-1\} \\ \cup\{(i_0, 2i_1); i \in \mathbb{Z}_n \setminus \{0, 1, p-2, p-1\}\} \end{gathered}$$

$$\begin{gathered} E(M) = \{((\alpha_1 i)_0, ((\alpha_1 + 1)i)_1); \ i = 0, p-2\} \\ \cup\{((\alpha_1 i - 1)_0, (((\alpha_1 + 1)i) - 1)_1); i = 1, p-1\} \\ \cup\{(((\alpha_1 + 1)i)_0, ((\alpha_1 + 2)i)_1); \ i \in \mathbb{Z}_n \setminus \{0, 1, p-2, p-1\}\} \end{gathered}$$

and hence $G \cong M \cong 2C_3(0,0,0) \cup (p-4)C_2(0,0)$.

**Case 2.** let $p$ to be a prime integer such that $p \equiv 5 \bmod 6$ and $\alpha_2$ be a positive integer $\alpha_2 = \lfloor \frac{2p}{3} \rfloor - 1$. The vectors $v(G)$ , $h(F)$ of the bases $G$ and $F$ are defined as:

$$v_i(G) = \begin{cases} 0 & \text{if} \quad i = 0, p-2 \\ p-1 & \text{if} \quad i = 1, p-1 \\ i & \quad otherwise \end{cases}$$

and

$$h_i(F) = \begin{cases} \alpha_2 i & \text{if} \quad i = 0, p-2 \\ \alpha_2 i - 1 & \text{if} \quad i = 1, p-1 \\ (\alpha_2 + 1)i & \quad otherwise \end{cases}$$

Therefore, $v_i(G) - h_i(F) = -\alpha_2 i$ for any $i \in \mathbb{Z}_p$. Consequently, $\{v_i(G) - h_i(F)\} = \mathbb{Z}_p$. By theorem 4, then the two bases $v(G)$ and $h(F)$ are orthogonal. Moreover, the edges set $E(F)$ of the basis $F$ can be represented by:

$$\begin{gathered} E(F) = \{((\alpha_2 i)_0, ((\alpha_2 + 1)i)_1); \ i = 0, p-2\} \\ \cup\{((\alpha_2 i - 1)_0, (((\alpha_2 + 1)i) - 1)_1); i = 1, p-1\} \\ \cup\{(((\alpha_2 + 1)i)_0, ((\alpha_2 + 2)i)_1); \ i \in \mathbb{Z}_n \setminus \{0, 1, p-2, p-1\}\} \end{gathered}$$

and hence $F \cong 2C_3(0,0,0) \cup (p-4)C_2(0,0)$.

**Theorem 20.** *Let* $p > 7$ *to be a prime integer, then there is an ODC of* $K_{n,n}$ *by* $C_3(1, p-4, 1) \cup (p-5)C_2(0, 0)$.

*Proof.* **Case 1.** For any prime integer $p > 7$ such that $p \equiv 5 \bmod 6$ and a positive integer $\alpha_1 = \lfloor \frac{p}{3} \rfloor$, the vectors $v(G)$ and $u(M)$ of the bases $G$ and $M$ are defined as:

$$v_i(G) = \begin{cases} 0 & \text{if} \quad i = 0, p-2 \\ p-1 & \text{if} \quad i = 1, p-1 \\ i & \quad otherwise \end{cases}$$

and

$$u_i(M) = \begin{cases} \alpha_1 i & \text{if} \quad i = 0, p-2 \\ \alpha_1 i - 1 & \text{if} \quad i = 1, p-1 \\ (\alpha_1 + 1)i & \quad otherwise \end{cases} ,$$

Therefore, $v_i(G) - u_i(M) = -\alpha_1 i$ for every $i \in \mathbb{Z}_p$. Consequently, $\{v_i(G) - u_i(M)\} = \mathbb{Z}_p$. By theorem 4, then the two bases $v(G)$ and $u(M)$ are orthogonal. Moreover, the edges set $E(G)$ and $E(M)$ of the bases $G$ and $M$ respectively, can be represented by:

$$\begin{gathered} E(G) = \{((p-1)_0, (i-1)_1); i = 1, p-1\} \\ \cup\{(0_0, 0_1), (0_0, -2_1)\} \\ \cup\{(i_0, 2i_1); i \in \mathbb{Z}_n \setminus \{0, 1, p-2, p-1\}\} \end{gathered}$$

and

$$\begin{gathered} E(M) = \{((\alpha_1 i)_0, ((\alpha_1 + 1)i)_1);\ i = 0, p-2\} \\ \cup\{((\alpha_1 i - 1)_0, (((\alpha_1 + 1)i) - 1)_1); i = 1, p-1\} \\ \cup\{(((\alpha_1 + 1)i)_0, ((\alpha_1 + 2)i)_1);\ i \in \mathbb{Z}_n \setminus \{0, 1, p-2, p-1\}\} \end{gathered}$$

and hence $G \cong M \cong C_3(1, p-4, 1) \cup (p-5)C_2(0, 0)$.

**Case 2.** For any prime integer $p > 7$ such that $p \equiv 1 \bmod 6$ and a positive integer $\alpha_2 = \lfloor \frac{3p-1}{5} \rfloor$ , the vectors $v(G)$ and $h(F)$ of the bases $G$ and $F$ are defined as:

$$v_i(G) = \begin{cases} 0 & \text{if} \quad i = 0, p-2 \\ p-1 & \text{if} \quad i = 1, p-1 \\ i & \quad otherwise \end{cases}$$

and

$$h_i(F) = \begin{cases} \alpha_2 i & \text{if} \quad i = 0, p-2 \\ \alpha_2 i - 1 & \text{if} \quad i = 1, p-1 \\ (\alpha_2 + 1)i & \quad otherwise \end{cases}$$

Therefore, $v_i(G) - h_i(F) = -\alpha_2 i$ for every $i \in \mathbb{Z}_p$. Consequently, $\{v_i(G) - h_i(F)\} = \mathbb{Z}_p$. By theorem 4, then the two bases $v(G)$ and $h(F)$ are orthogonal. Moreover, the edges set $E(F)$ of the bases $F$ can be represented by:

$$E(F) = \{((\alpha_2 i)_0\,, ((\alpha_2+1)i)_1)\,;\ i = 0, p-2\}$$
$$\cup\{((\alpha_2 i - 1)_0\,, (((\alpha_2+1)i)-1)_1); i = 1, p-1\}$$
$$\cup\{(((\alpha_2+1)i)_0\,, ((\alpha_2+2)i)_1)\,;\ i \in \mathbb{Z}_n \backslash \{0, 1, p-2, p-1\}\}.$$

and hence $F \cong C_3(1, p-4, 1) \cup (p-5)C_2(0, 0)$.

**Theorem 21.** *Let* $p > 7$ *to be a prime integer, then there is an ODC of* $K_{p,p}$ *by* $C_4(0, 0, 0, 0) \cup 2C_3(0, 0, 0) \cup (p-7)C_2(0, 0)$.

*Proof.* For any prime integer $p > 7$, the vectors $v(G)$ and $u(F)$ of the starters $G$ and $F$ are defined respectively as:

$$v_i(G) = \begin{cases} 0 & \text{if} \quad i = 0 \\ (\frac{p-1}{2})i - 1 & \text{if} \quad i = 1, p-1 \\ (\frac{p-3}{2})i - 1 & \quad otherwise \end{cases}$$

and

$$u_i(F) = \begin{cases} 0 & \text{if} \quad i = 0 \\ (\frac{p+1}{2})i - 1 & \text{if} \quad i = 1, p-1 \\ (\frac{p-1}{2})i - 1 & \quad otherwise \end{cases}$$

Thus, for any $i \in \mathbb{Z}_p$, $v_i(G) - u_i(F) = -i$. Consequently,
$\{v_i(G) - u_i(F);\ \ i \in \mathbb{Z}_p\} = \mathbb{Z}_p$. By theorem 4, then the two bases $v(G)$ and $u(F)$ are orthogonal. Moreover, the edges set $E(G)$ and $E(F)$ of the bases $G$ and $F$ respectively, can be represented by:

$$E(G) = \left\{\left(\left((\tfrac{p-1}{2})i - 1\right)_0, \left((\tfrac{p+1}{2})i - 1)\right)_1\right); i \in \{1, p-1\}\right\}$$
$$\cup\{(0_0, 0_1)\}$$
$$\cup\left\{\left(\left((\tfrac{p-3}{2})i - 1\right)_0, \left((\tfrac{p-1}{2})i - 1)\right)_1\right); i \in \mathbb{Z}_p \backslash \{0, 1, p-1\}\right\}$$

and

$$E(F) = \left\{\left(\left((\tfrac{p+1}{2})i - 1\right)_0, \left((\tfrac{p+3}{2})i - 1)\right)_1\right); i \in \{1, p-1\}\right\}$$
$$\cup\{(0_0, 0_1)\}$$
$$\cup\left\{\left(\left((\tfrac{p-1}{2})i - 1\right)_0, \left((\tfrac{p+1}{2})i - 1)\right)_1\right); i \in \mathbb{Z}_p \backslash \{0, 1, p-1\}\right\}.$$

Hence, $G \cong F \cong C_4(0, 0, 0, 0) \cup 2C_3(0, 0, 0) \cup (p-7)C_2(0, 0)$.

## 4. Tree- binary error detecting and correcting codes

Various combinatorial designs and related structures can be utilized to create codes using the incidence matrix. The interaction between designs and codes has led to many intriguing and valuable results [12] as a good survey. Lately, there has been a heightened focus on codes derived from graphs. The literature extensively delves into the relationship between codes and graphs from multiple perspectives. The primary aim of these studies is to select a specific class of graphs and form codes from the graph's adjacency matrix. The characteristics of the graph can give rise to diverse types of codes, including self-dual codes, self-orthogonal codes, authentication codes, etc. Further details on this subject can be found in [2, 4, 5, 8, 9, 13, 15, 17–19]. Binary codes can be produced from various graphs such as Paley graphs and Latin square graphs [3, 23]. Furthermore, non-isomorphic codes have been generated from non-isomorphic graphs. This section focuses on binary codes originating from the row span of the incidence matrices of particular graphs that appear as induced subgraphs of complete bipartite graphs. These codes are termed binary graph-codes. Additionally, if each codeword in a graph-code corresponds to a graph that is isomorphic to graph $G$, the code is known as a binary $G-$code. By utilizing an ODC of a complete bipartite graph, we develop binary codes that ensure the inner product of any two codewords is less than or equal to 1. Therefore, binary $G-$codes are viewed as a distinct subset of orthogonal codes. The unique properties of an ODC of a complete bipartite graph suggest that binary $G-$codes derived from ODCs may serve as effective error detection and correction codes.

**Theorem 22.** *Let there is an ODC of $K_{n,n}$ by a tree. Then there is a tree- binary code of length $n^2$.*

*Proof.* Given an ODC $\mathcal{G}=\{G_a^i : a \in \mathbb{Z}_n, i \in \{1,2\}\}$ of $K_{n,n}$ by a tree $G$. The incidence matrix $L = L(s,t)$ for such ODC $\mathcal{G}=\{G_a^i;\, a \in \mathbb{Z}_n, i \in \{1,2\}\}$ is a $2n \times n^2$ binary matrix showing the relation between the edges of $K_{n,n}$ and the the members of $\mathcal{G}$ such that every row in $L$ corresponds to a unique graph in $\mathcal{G}$ and every column in $L$ corresponds to a unique edge in $K_{n,n}$. Let the edge $(\alpha_0, \beta_1) \in E(K_{n,n})$. Then $L$ has a unique column $t$ corresponds to this edge $(\alpha_0, \beta_1)$. Such column $t$ is defined from an injective function $C : E(K_{n,n}) \to \{0, 1, \cdots, n^2-1\}$ where $C((\alpha_0, \beta_1)) = t = n\alpha + \beta;\ \alpha, \beta \in \mathbb{Z}_n$. For each a graph $G_a^i$ : $(a \in \mathbb{Z}_n, i \in \{1,2\})$, $a \in \mathbb{Z}_n$ from $\mathcal{G}$, there is exactly one row $s \in \{0, 1, \cdots, 2n-1\}$ in $L$ corresponds to $G_a^i$ according the injective function $R_G : \mathbb{Z}_n \times \{1,2\} \to \{0, 1, \cdots, 2n-1\}$ where

$$R_G(a,i) = s = \begin{cases} a & \text{if} \quad i = 1 \\ a+n & \text{if} \quad i = 2. \end{cases}$$

Then, The incidence matrix $L = L(s,t)$ of an ODC $\mathcal{G}$ is defined as

$$L(s,t) = \begin{cases} 1 & \text{if} \quad (\alpha_0, \beta_1) \in E\left(G_a^i\right) \\ 0 & \text{if} \quad (\alpha_0, \beta_1) \notin E\left(G_a^i\right) \end{cases}$$

The rows of $L$ can form a graph binary code. This binary code will be denoted as $C_r$. The encoding process in $C_r$ is manipulated in two consecutive steps. Suppose that $(a, i) \in \mathbb{Z}_n \times \{1, 2\}$ is a plain text. . In order to build the cipher text for $(a, i)$, firstly, we calculate $s = R_G(a, i)$. using the above definition of $R_G$. Secondly, the cipher text corresponds to the plain text $(a, i)$ is the concatenation of bits of the row $s$ in the matrix $L$so will be the codeword $l_{s0}l_{s1}l_{s2}...l_{s(n^2-1)}$ is the corresponding cipher text (of length $n^2$ ) to the a plain text $(a, i)$. The definition of the matrix $L = L(s, t)$ implies that every row in $L$ is isomorphic to the tree $G$. Whence every codeword in $C_r$ is also so. Thus, $C_r$ is a tree binary code induced from an ODC of $K_{n,n}$ by a tree $G$.

The significance of the Hamming distance in binary codes lies in its ability to determine error detection and correction capabilities within the code. Consider a binary code $C$, where each codeword has a length of $h$. Let $X = x_1x_2 \cdots x_h$ and $Y = y_1y_2 \cdots y_h$ be two codewords from $C$.The distance $d(X, Y)$ between $X$ and $Y$ is calculated as the count of differing bits between $X$ and $Y$. Hence, $d(X, Y) = \sum_{i=1}^{h} d(x_i, y_i)$ where, $d(x_i, y_i) = 0$ if $x_i = y_i$ and $d(x_i, y_i) = 1$ if $x_i \neq y_i$. The distance $d(X, Y)$ for any codewords $X$ and $Y$ in the binary code $C$ follows specific properties:

(i) $d(X, Y) \geq 0$.

(ii) $d(X, Y) = 0$ if and only if $X = Y$.

(iii) $d(Y, X) = d(X, Y)$.

(iv) $d(X, Z) \leq d(X, Y) + d(Y, Z)$.

The minimum distance $d(C)$ of a code $C$ is defined as

$$d(C) = \min\{d(X, Y);\ X, Y \in C \text{ and } X \neq Y\}.$$

**Lemma 1.** *Let $C_r$ be a tree binary codes constructed from the rows of the incidence matrix of an ODC of $K_{n,n}$ by a tree. Then the minimum distance $d(C_r)$ is $2n - 2$.*

*Proof.* From the relation between ODC and the incidence matrix $L$, there exists $n$ 1's in every row. Any two rows have at most 1 position of 1's common. Then the minimum distance of $C_r$ is $2(n-1) = 2n - 2$.

**Theorem 23.** *The binary code $C_r$ can detect up to $2n - 3$ errors or correct up to $\lfloor \frac{2n-3}{2} \rfloor$ errors.*

*Proof.* From Lemma 1, the minimum distance of $C_r$ is $2(n-1)$. Thus, in order to change any codeword to another codeword requires at least $2n - 2$ bit changes. Whence, $C_r$ can detect up to $2n - 2 - 1 = 2n - 3$ errors, since any $2n - 3$ transmission errors cannot change one codeword to another. Hence, in order to have a guarantee the detection of up to $k$ errors in all cases, the minimum distance of the code $C_r$, $d(C_r) = k + 1$. The Geometric concept for finding $d(C_r)$ in error detection is shown in Figure 4.

Suppose that the codeword $u$ was sent but for some reason $v$ was received and $d(u,v) \leq \frac{2n-3}{2}$, that is less than $\frac{2n-3}{2}$ errors occurred. Subsequently, the distance between $v$ and any codeword, excluding $u$, exceeds $\frac{2n-3}{2}$. Let $w \in C_r$ then $d(u,w) \geq 2n-2$. Therefore, if $u$ is the closest codeword to $v$, the amount of bit changes needed to switch from $u$ to $v$ (the number of errors in the transmission channel) is lower than the number of errors needed to switch from any other codeword to $v$. The substitution of $v$ with $u$ enables $C_r$ to rectify up to $2n$ errors. Therefore, to ensure the correction of $k$ errors in all instances, the minimum distance of code $C_r$ $d(C_r) = 2k+1$.The geometric principle for determining $d(C_r)$ in error correction is presented in Figure 5.

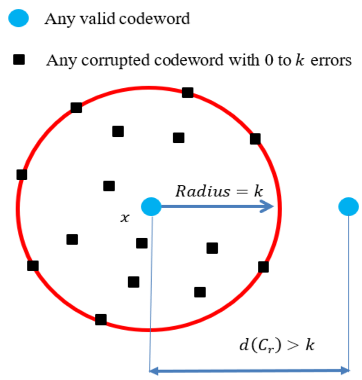


Figure 4: The Geometric concept for finding $d(C_r)$ in error detection.

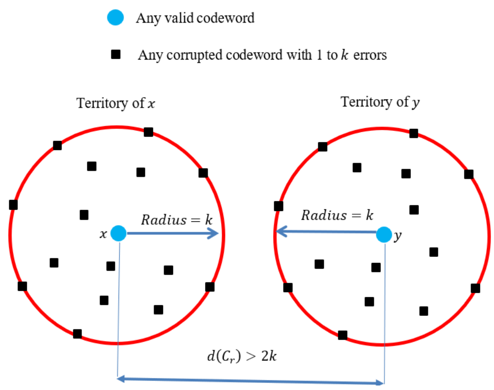


Figure 5: The Geometric concept for finding $d(C_r)$ in error correction.

**Example 2.** *Let the vector* $(0, 1, 2, 1)$ *be a symmetric base for an ODC of* $K_{4,4}$ *by* $\tau^{0_1}(2, 1)$*. The two orthogonal decompositions generated from the vector* $(0, 1, 2, 1)$ *are shown in Figure 6.The incidence matrix* $L = L(s, t)$ *for such ODC is a* $8 \times 16$ *binary matrix where the rows correspond to elements of ODC* $\mathcal{G} = \{G^i_{0+x} : x \in \mathbb{Z}_4, i \in \{1, 2\}\}$ *of* $K_{4,4}$*. The following matrix is the incidence matrix* $L(s, t)$ *for the ODC (described in Figure 6) of* $K_{4,4}$ *by* $\tau^{0_1}(2, 1)$*. From the rows of the matrix* $L(s, t)$ *we construct the code* $C_r$ *such that* $C_r$ *has 8 codewords each of length 16.*

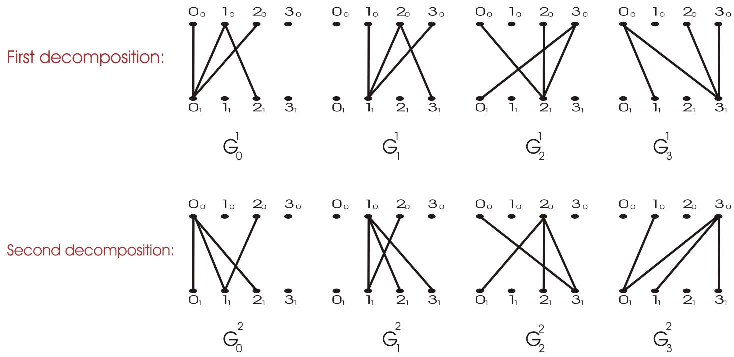


Figure 6: An ODC of $K_{4,4}$ by $\tau^{0_1}(2, 1)$ where the vector $(0, 1, 2, 1)$ is the symmetric base for this ODC.

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1
\end{bmatrix}
$$

The incidence matrix $L(s,t)$ for the ODC described in Figure 6

$$\downarrow$$

$$
C_r = \left\{ \begin{array}{c} 1000101010000000, 0000010001010100, 001000000010100, \\ 01010010000000, 1110000001000000, 0000011100000010, \\ 0001000010110000, 0000100000001101 \end{array} \right\}
$$

The code $C_r$ deduced from the rows of the above incidence matrix $L(s,t)$

## 5. Recursive construction of an ODC by disjoint trees

Hereafter, we will show how to use an ODC of small complete bipartite graphs to construct an ODC of larger complete bipartite graphs. In the following, for simplicity, if $(x, y) \in \mathbb{Z}_m \times \mathbb{Z}_n,$ we use $xy$ for $(x, y)$.

**Theorem 24.** *Let $m \equiv 1 \bmod 6$ or $m \equiv 5 \bmod 6$ and let $n$ be a positive integer. If $v(G) = (v_0, v_1, \cdots, v_{n-1})$ be a symmetric base for an ODC of $K_{n,n}$ by a certain tree $G$, then there is an ODC of $K_{mn,mn}$ by $mG$ with respect to $\mathbb{Z}_n \times \mathbb{Z}_m,$ (the Cartesian product of the groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$).*

*Proof.* Let the vector $u(F) = (u_0, u_1, \cdots, u_{m-1}) = (0, 1, \cdots, m-1)$ represent a base in $K_{m,m}$ with respect to $\mathbb{Z}_m$. From the definition of $u(F)$, We can conclude that $\{u_i - u_{-i} + i; i \in \mathbb{Z}_m\} = \mathbb{Z}_m$. Therefore, $u(F)$ is a symmetric base an ODC of $K_{m,m}$. Whence, $m \equiv 1 \bmod 6$ or $m \equiv 5 \bmod 6$ the base $F$ is isomorphic to $mK_2$. Since the vector $v(G) = (v_0, v_1, \cdots, v_{n-1})$ is a symmetric base for an ODC of $K_{n,n}$. Taking the Cartesian product of the two vectors $v(G)$ and $u(F)$ implies that $v(G) \times u(F) = ((v_0, u_0), (v_0, u_1), \cdots, (v_i, u_j), \cdots, (v_{n-1}, u_{m-1}); i \in \mathbb{Z}_n, j \in \mathbb{Z}_m)$ is also a symmetric base of an ODC of $K_{mn,mn}$ by $G \times F$ with respect to $\mathbb{Z}_n \times \mathbb{Z}_m,$ as

$$\{(v_i, u_j) - (v_{-i}, u_{-j}) + (i, j); i \in \mathbb{Z}_n,\ j \in \mathbb{Z}_m\} = \mathbb{Z}_n \times \mathbb{Z}_{m,}.$$

Moreover,the edge set of the the base $G \times F$ is

$$E(G \times F) = \left\{(v_i, u_j)_0, +((v_i, u_j) + (i, j))_1; i \in \mathbb{Z}_n,\ j \in \mathbb{Z}_m\right\}.$$

From the vectors $u(F)$ and $v(G)$ the base $G \times F$ is isomorphic to $mG$ whenever $m \equiv 1 \bmod 6$ or $m \equiv 5 \bmod 6$ and $n$ be a positive integer. Note that the the vertices of $K_{mn,mn}$ are labelled by the elements of $\mathbb{Z}_n \times \mathbb{Z}_m \times \{0, 1\}$.For the vertex $(x, y, i)$ we write $(xy)_0$ where $x \in \mathbb{Z}_n,\ y \in \mathbb{Z}_n,\ i \in \{0, 1\}$, and $\mathbb{Z}_n = \{0, 1, 2, \cdots, n-1\}$ is the group of all residual classes modulo $n$, $\mathbb{Z}_m = \{0, 1, 2, \cdots, m-1\}$ is the group of all residual classes modulo $m$.

**Example 3.** *Given the vector $v(F) = (0, 1, 2, 3, 4)$ as a symmetric base for an ODC of $K_{5,5}$ by $5K_2$. If there is an ODC of $K_{3,3}$ by $\tau^{0_1}(1, 1)$ where the vector $v(G) = (0, 1, 1)$ is the symmetric base for this ODC then we have a guarantee that there is an ODC of $K_{15,15}$ by $5\tau^{0_1}(1, 1)$ and the*

$$v(G) \times u(F) = (00, 01, 02, 03, 04, 10, 11, 12, 13, 14, 10, 11, 12, 13, 14)$$

*the base $G \times F$ is illustrated in Figure 7*

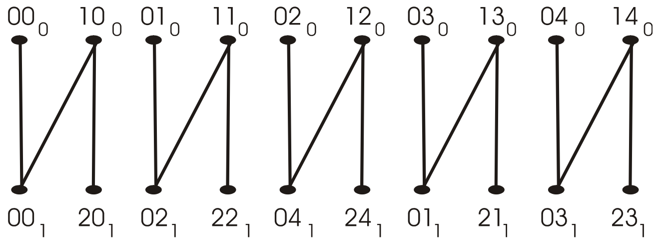


Figure 7: Symmetric base for an ODC of $K_{15,15}$ by $5\tau^{0_1}(1,1)$.

## 6. Concluding Remarks

The paper delves into the concept of graph decomposition, particularly focusing on orthogonal decompositions. In this context, a graph $H$ is divided into subgraphs in such a way that any two subgraphs share at most one edge. These decompositions are referred to as $G-$orthogonal decompositions if each subgraph is isomorphic to the graph $G$. The applications of such decompositions are widespread, encompassing fields like statistics, information theory, and experimental design theory. The document also introduces an approach for constructing orthogonal decompositions of regular graphs and discusses its utilization in creating tree-orthogonal decompositions of complete bipartite graphs. Additionally, the use of orthogonal decompositions in designing hamming tree-codes is explored, supported by examples showcasing their effectiveness in error detection and correction during data transmission. In future wok, further investigations will be planed to manipulate our approach to work on irregular graphs.

## Declaration of Conflicting Interests

The authors declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

## Acknowledgements

We are grateful to the reviewers for a number of valuable remarks and suggestions.

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