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Wardrop Optimal Networks

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Abstract. Optimal flow allocation in directed networks is often analyzed through two types of equilibria. The first, called user equilibrium, occurs when the travel times on all utilized routes are the same and shorter than every unused route. The second, called system optimum, minimizes the average travel time across all routes. These two concepts represent the optimality for individual users and for the network as a whole, respectively, and generally do not coincide. Our main objective in this paper is to introduce and examine the properties of networks where the user equilibrium and the system optimum are identical, termed Wardrop optimal networks. We achieve this by providing a set of necessary and sufficient conditions for Wardrop optimal flows and using this characterization we investigate the main properties of Wardrop optimal networks. Moreover, we illustrate that these flows remain consistent under several important transformations as well as under uniform changes in their latency functions.

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1. Introduction

The efficient distribution of flow in directed networks is often examined through two different key concepts, known as Wardrop's Principles: *user equilibrium* and *system optimum* [30]. Wardrop's principles are fundamental concepts in transportation and network theory, providing a basis for understanding traffic flow and congestion management. According to Wardrop's first principle, user equilibrium is reached when travel times on all routes in use are the same, and no traveler can reduce their travel time by switching routes; this state is also referred to as a *Wardrop equilibrium* [7, 19]. The second principle asserts that the network flow configuration minimizes the total travel time for all users, representing an optimal state from the perspective of the network as a whole.

In practice, user equilibrium and system optimum typically differ, leading to inefficiencies in network usage. Achieving the system optimum typically requires a form of central

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control or coordination because individual users acting in their self-interest do not naturally lead to system optimal flows. To achieve the system optimum, traffic may need to be redistributed by encouraging or forcing users to follow less congested routes, even if their travel time increases. This redistribution reduces the overall congestion and minimizes the total travel time or cost for all users combined. At user equilibrium, the total travel time is generally higher than at system optimum because individuals are not incentivized to consider the overall system performance. In many networks, user equilibrium represents a "selfish" solution, whereas the system optimum represents a "cooperative" solution. The ratio of the total travel time at user equilibrium to that at system optimum is known as the *price of anarchy*. A flow that is both system optimum and user equilibrium is called *Wardrop optimal flow* and networks that admit such flows are termed *Wardrop optimal networks* [4, 13] (see also [2, 3, 5] where Wardrop optimal networks with dynamic flow assignment are examined and [14] presenting, without proofs, some of the main properties of Wardrop optimal networks we establish in this paper). By construction, the price of anarchy of such networks is optimal, i.e., equal to 1, a desirable property in network theory with applications in fields such as economics [26, 28], transportation [6, 23, 25, 29, 30], and communications [18, 20, 21, 24], and can be also applied to areas like multiprocessor task scheduling [8] and media flow with double diffusion [12]. For a comprehensive review of the literature in approaches bridging user equilibrium and system optimum see [22].

This paper focuses on analyzing these flows and their corresponding networks. A flow travelling from initial to target node is allocated across multiple links, causing congestion, which in turn increases travel times, captured by link-specific latency functions that are strictly increasing. We establish a connection between the user equilibrium and system optimum through the associated *Pigovian network* and derive conditions for the existence and uniqueness of optimal flows in convex networks.

The paper is organized in the following way. In Section 2 we outline the basic definitions we use throughout the paper. Section 3 examines in detail the user equilibrium and introduces the *discrete user equilibrium*. Section 4 explores the system optimum, presenting necessary and sufficient conditions for its characterization. In Section 5 we introduce Wardrop optimal networks and provide a characterization for differentiable convex networks which demonstrates that these flows are maintained under specific transformations.

2. Preliminaries

The networks we will examine have n links $\mathbf{I}_n = \{1, 2, \ldots, n\}$ connecting the origin with the destination node [1]. Let $\phi_i \geq 0$ denote the flow via link i, for every $i \in \mathbf{I}_n$. The total flow is distributed among the links such that

$$\sum_{i=1}^{n} \phi_i = 1.$$

The support of a flow ϕ is denoted by $supp(\phi) = \{k \in \mathbf{I}_n \mid \phi_k \neq 0\}$. Additionally, we set

$$\mathbf{S}^{n-1} = \{\phi \in \mathbb{R}_+^n : \sum_{i=1}^{n} \phi_i = 1\}$$

and

$$\text{Int}(\mathbf{S}^{n-1}) = \{\phi \in \mathbf{S}^{n-1} : supp(\phi) = \mathbf{I}_n\}.$$

On each link, the flow induces congestion, increasing the traversal delay [30]. This delay is captured by continuous and strictly increasing latency functions

$$l_i(x) : [0,1] \to \mathbb{R}, \quad i \in \mathbf{I}_n.$$

Such a network of can thus be characterized by the vector of latency functions $\mathbf{N_n} = (l_1(x), \dots, l_n(x))$. If all latency functions $l_i(x)$ are differentiable, the network is called a *differentiable network*; if they are convex, it is termed a *convex network*.

A *user equilibrium* or *Wardrop equilibrium* of a network $\mathbf{N_n} = (l_1(x), \dots, l_n(x))$ is a flow $\phi = (\phi_1, \dots, \phi_n) \in \mathbf{S}^{n-1}$ such that

$$l_k(\phi_k) = \min_{i \in \mathbf{I}_n}\{l_i(\phi_i)\}, \quad \forall k \in \mathbf{I}_n \text{ with } \phi_k > 0,$$

i.e., the delay is equal among all used links and less than that of any unused link [6].

The *overall delay* of a network $\mathbf{N_n} = (l_1(x), \dots, l_n(x))$ for $\phi = (\phi_1, \dots, \phi_n) \in \mathbf{S}^{n-1}$ is given by

$$\sum_{i=1}^{n} \phi_i l_i(\phi_i).$$

A *system optimum* of a network $\mathbf{N_n} = (l_1(x), \dots, l_n(x))$ is a flow $(\phi_1, \dots, \phi_n) \in \mathbf{S}^{n-1}$ that minimizes this average delay [6].

3. User Equilibrium

In what follows, we explore several properties that will be essential throughout. The next lemma demonstrates that user equilibrium remains unchanged under mappings which are strictly increasing. This is because the minimum value in a set is preserved when a strictly increasing function is applied.

Lemma 1. *Consider a network* $\mathbf{N_n} = (l_1(x), \dots, l_n(x))$ *and a strictly increasing and continuous function* $f(x) : \mathbb{R} \to \mathbb{R}$*, if a flow* $(\phi_1, \dots, \phi_n) \in \mathbf{S}^{n-1}$ *is user equilibrium of* $\mathbf{N_n}$ *then it is also user equilibrium of* $f(\mathbf{N_n}) = (f(l_1(x)), \dots, f(l_n(x)))$.

On the other hand, the above property doesn't hold for system optimums as it is illustrated in the following example.

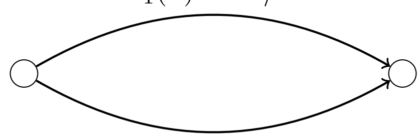


Figure 1: A network $\mathbf{N}$ with two parallel links

Example 1. *We examine the network* $\mathbf{N} = \{\frac{x^2}{3}, \frac{2x}{3}\}$ *presented in Fig. 1 which was utilized in [1] to highlight the inefficiency due to congestion externalities. The system optimum that minimizes the average delay of the network is* $\phi^{SO} = (\frac{2}{3}, \frac{1}{3})$ *and the user equilibrium that equates delay on the two links is* $\phi^{WE} \approx (0.73, 0.27)$.

The network obtained by taking the image of $\mathbf{N}$ *via the continuous and strictly increasing function* $f(x) = x^2$ *is*

$$f(\mathbf{N}) = \{\tfrac{x^4}{9}, \tfrac{4x^2}{9}\}.$$

It is straightforward to see that the user equilibrium of $f(\mathbf{N})$ *is the same but the new system optimum is* $\phi^{SO'} \approx \{0.69, 0.31\}$.

The example demonstrates that, the system optimum unlike the user equilibrium is not generally maintained by the type of latency function we consider i.e., continuous, convex and strictly increasing. This raises the following question: do Wardrop optimal flows remain consistent under such transformations? We will address this question in Section 5.

The below property will be very useful in our subsequent constructions. It can be easily proved by a simple geometric inspection of the curves of the strictly increasing latency functions on the same coordinate system. Below we provide a more detailed algebraic proof (similar constructions exist in the literature for the user equilibrium, see e.g. [27]).

Proposition 1. *There exists a unique user equilibrium for any network* $\mathbf{N_n}$.

Proof. Let $\mathbf{N_n} = (l_1(x), \cdots, l_n(x))$. For each $i \in \mathbf{I}_n$, we set $a_i = l_i(0)$, $b_i = l_i(1)$ and consider the inverse (again continuous, strictly increasing) function $l_i^{-1} : [a_i, b_i] \to [0, 1]$ of each latency function $l_i : [0, 1] \to [a_i, b_i]$.

We define a continuous and increasing function $\mathbf{l}_i^{-1} : \mathbb{R} \to [0, 1]$ by

$$\mathbf{l}_i^{-1}(y) := \begin{cases} 0, & \text{if } y < a_i \\ l_i^{-1}(y), & \text{if } a_i \le y \le b_i \\ 1, & \text{if } y > b_i \end{cases}$$

as well as a function $L^{-1} : \mathbb{R} \to [0, n]$ by

$$L^{-1}(y) := \sum_{i \in \mathbf{I}_n} \mathbf{l}_i^{-1}(y).$$

Then $L^{-1} : \mathbb{R} \to [0, n]$ is also a continuous and increasing function such that $L^{-1}(y) = 0$ for any $0 \leq y \leq \min_{i \in \mathbf{I}_n}\{a_i\}$ and $L^{-1}(y) = n$ for any $y \geq \max_{i \in \mathbf{I}_n}\{b_i\}$.

Since

$$l_j^{-1} : [a_j, b_j] \to [0, 1]$$

is the strictly increasing function on the interval $[a_j, b_j]$ where

$$a_j = \min_{i \in \mathbf{I}_n}\{a_i\},$$

therefore the function $L^{-1} : [a_j, b_j] \to [0, n]$ is also strictly increasing on the same interval $[a_j, b_j]$ where $a_j = \min_{i \in \mathbf{I}_n}\{a_i\}$. It can be easily seen that $L^{-1}(a_j) = 0$ and $L^{-1}(b_j) \geq l_j^{-1}(b_j) = 1$.

Let $L^{-1}(b_j) = 1$. Then the unique user equilibrium is

$$\phi = (0, 0 \cdots, 0, \underbrace{1}_{j}, 0, \cdots 0)$$

with $\mathbf{supp}(\phi) = \{j\}$. Indeed, since

$$L^{-1}(b_j) = \sum_{i \in \mathbf{I}_n} \mathbf{l}_i^{-1}(b_j) = 1,$$

it is easy to see that $\mathbf{l}_j^{-1}(b_j) = l_j^{-1}(b_j) = 1$ or equivalently $l_j(1) = b_j$ and meanwhile $\mathbf{l}_i^{-1}(b_j) = 0$ for every $i \in \mathbf{I}_n$ with $i \neq j$ or equivalently $b_j \leq a_i$ for every $i \in \mathbf{I}_n$ with $i \neq j$. The last inequality means that $l_i(0) = a_i \geq b_j = l_j(1)$ for every $i \in \mathbf{I}_n$ with $i \neq j$.

Let $L^{-1}(b_j) > 1$. Since $L^{-1}([0, a_j]) = 0$, the function L^{-1} is strictly increasing on the interval $[a_j, b_j]$, and $L^{-1}(b_j) > 1$. Then there always exists a unique $c \in (a_j, b_j)$ such that $L^{-1}(c) = 1$.

Let us define the following sets

$$\alpha := \{k \in \mathbf{I}_n : c \in (a_k, b_k)\} \text{ and } \beta := \{s \in \mathbf{I}_n : c \notin (a_s, b_s)\}.$$

Obviously, due to the construction, we have that $j \in \alpha \neq \emptyset$, $\alpha \cup \beta = \mathbf{I}_n$, and $\alpha \cap \beta = \emptyset$. Let us define $\phi_k := l_k^{-1}(c)$ for every $k \in \alpha$. It is obvious that

$$0 = l_k^{-1}(a_k) < \phi_k = l_k^{-1}(c) < l_k^{-1}(b_k) = 1, \qquad \forall\, k \in \alpha.$$

We then obtain that

$$\begin{aligned} 1 = L^{-1}(c) = \sum_{i \in \mathbf{I}_n} \mathbf{l}_i^{-1}(c) &= \sum_{k \in \alpha} \mathbf{l}_k^{-1}(c) + \sum_{s \in \beta} \mathbf{l}_s^{-1}(c) \\ &= \underbrace{\sum_{k \in \alpha} l_k^{-1}(c)}_{\text{positive term}} + \sum_{s \in \beta} \mathbf{l}_s^{-1}(c) > \sum_{s \in \beta} \mathbf{l}_s^{-1}(c). \end{aligned} \tag{1}$$

Consequently, we derive that $\mathbf{l}_s^{-1}(c) < 1$ for any $s \in \beta$. Since $c \notin (a_s, b_s)$ for any $s \in \beta$ and $\mathbf{l}_s^{-1}(c) < 1$, due to the definition of the function $\mathbf{l}_s^{-1}(y)$, we must have that $\mathbf{l}_s^{-1}(c) = 0$ for any $s \in \beta$. Equivalently, this means that

$$l_k(\phi_k) = c \leq a_s = l_s(0), \quad \forall\ k \in \alpha \ \text{ and } \ \forall\ s \in \beta.$$

Moreover, since

$$\sum_{s\in\beta} \mathbf{l}_s^{-1}(c) = 0,$$

it follows from (1) that

$$L^{-1}(c) = \sum_{i\in\mathbf{I}_n} \mathbf{l}_i^{-1}(c) = \sum_{k\in\alpha} \mathbf{l}_k^{-1}(c) + \underbrace{\sum_{s\in\beta} \mathbf{l}_s^{-1}(c)}_{\text{zero terms}} = \sum_{k\in\alpha} l_k^{-1}(c) = \sum_{k\in\alpha} \phi_k = 1.$$

Hence, the flow $\phi = (\phi_1, \phi_2, \cdots, \phi_n)$ which is defined as follows $\phi_k := l_k^{-1}(c)$ for every $k \in \alpha$ and $\phi_k := 0$ for every $k \in \beta = \mathbf{I}_n \setminus \alpha$ is user equilibrium and it is unique due to the uniqueness of the point c. This completes the proof.

3.1. Discrete User Equilibrium

The definition of user equilibrium, as it was introduced [6, 7, 30] and studied in the literature since then, assumes that the traffic flow can be represented as a positive real number. The advantage of this representation is that it can be infinitely divided into smaller flows and thus we can always obtain the exact value of flow distribution that creates the user optimum.

On the other hand this is just an approximation of the situation in real life scenarios, where for transportation networks individual vehicles can not be split and more over they can have varying sizes. The same can be said for information networks and parallel processing in relation respectively to the packages and the processes that run in parallel.

To address these situations, in what follows, we will introduce a notion of user equilibrium for Discrete Multi-type flow. An additional advantage of this representation is that the resulting discrete structures could be algebraically recognized in a manner similar to [10] for crisp graphs and [17], [16] for fuzzy structures. We will consider again a network with n parallel links, $\mathbf{I}_n = \{1, 2, \ldots, n\}$, where constants $c_1, \ldots, c_n$, represent the cost of using the link i, for $i = 1, \ldots, n$, for a transportation network this can be for example, the transit time of a unit traffic through the link i.

The flow is given by a flow-vector $\phi = (\phi_1, \phi_2, \ldots, \phi_N)$, $\phi_i > 0$, $N \geq n$, where ϕ_i is the size of the packet or vehicle i. We assume that all flow is distributed among all n links and denote the total flow size by

$$\mathcal{X} = \sum_{i=1}^{N} \phi_i.$$

The elementary flow of each type is unsplittable in the sense that a vehicle or a packet ϕ_k is routed in whole along one and only one link $i = 1, 2, \ldots, n$. Hence it holds

$$\sum_{k=1}^{n}\sum_{i=1}^{N} \delta_{ik}\phi_i^{(k)} = \mathcal{X},$$

where

$$\delta_{ik} = \begin{cases} 1, & \text{if } \phi_i \text{ is routed along the link } k \\ 0, & \text{otherwise.} \end{cases}$$

In order to introduce the user equilibrium from the discrete case we will consider the particular case of two links, $\mathbf{I}_2 = \{1, 2\}$. Let $\mathbf{X} = \{\phi_1, \phi_2, \ldots, \phi_N\}$ be a flow-set, $\phi_i \in \mathbb{R}^+$. Let $\mathcal{P}_2^X$ be a partition of $\mathbf{X}$ into two subsets X' and X'', where $X' \neq \emptyset$, $X'' \neq \emptyset$, $X' \cup X'' = \mathbf{X}$, $X' \cap X'' = \emptyset$, and $X'' = \mathbf{X} \setminus X'$. To further simplify, we assume that the latency functions

$$l_k(x) : [0, T] \to \mathbb{R},\ k \in \mathbf{I}_n,\ T = \textstyle\sum_{i=1}^{N} \phi_i$$

of the two links, are given by

$$l_k(x) = c_i x,\ \ k = 1, 2.$$

Hence, according to this setup, the user equilibrium can be described as the partition $\mathcal{P}_2^X$ that minimizes the difference between the time needed for each of the two flow subsets X' and X'' to travel across the two links. In other words, to find the user equilibrium one must solve the optimization problem

$$\min_{\mathcal{P}_2^X} \left\{ \left| c_k \sum_{\phi_i \in X'} \delta_{ik}\phi_i - c_k \sum_{\phi_i \in X''} \delta_{ik}\phi_i \right| \right\}, \quad k = 1, 2$$

with $\phi_i > 0$.

Consequently, the discrete user equilibrium does not equalize the link delays as in the continuous case, rather than minimize the absolute difference between them. This can be further generalized for more than two parallel links by minimizing the sum of absolute differences between all pairs of links delays.

Moreover, the link latency function can be further enhanced by adding a maximum capacity and a constant toll price. Hence, if we denote by R_i, $i \in \mathbf{I}_n$, the capacity of the link i, it can be assumed that the flow on the link i cannot exceed R_i, that is

$$\sum_{i=1}^{N} \delta_{ik}\phi_i^{(k)} \leq R_i, \quad k = 1, 2, \ldots, n.$$

If we additionally assume that the price of sending the flow through a link $i = 1, 2, \ldots, n$ is increased by a constant toll price p_i, $i = 1, 2, \ldots, n$. Then the delay on each link is given by

$$c_k \sum_{i=1}^{N} \delta_{ik}\phi_i + p_k, \quad k = 1, 2, \ldots, n.$$

4. System Optimum

Our next step is to investigate the properties of the system optimum. The networks we will examine from now on are assumed to be differentiable. We start with the following important result.

Proposition 2. *Consider a network* $\mathbf{N_n} = (l_1(x), \ldots, l_n(x))$*, then for every* $i, j \in \mathbf{I}_n$ *we have*

$$l_i(\phi_i) + \phi_i l_i'(\phi_i) = l_j(\phi_j) + \phi_j l_j'(\phi_j).$$

provided that $\phi = (\phi_1, \ldots, \phi_n) \in \mathrm{Int}(\mathbf{S}^{n-1})$ *is system optimum of the network.*

Proof. Let $(\phi_1, \cdots, \phi_i, \cdots, \phi_j, \cdots \phi_n) \in \mathrm{int}\mathbf{S}^{n-1}$. The total delay of the network at this flow is

$$T = \phi_1 l_1(\phi_1) + \cdots + \phi_i l_i(\phi_i) + \cdots + \phi_j l_j(\phi_j) + \phi_n l_n(\phi_n).$$

For $0 < \epsilon < 1$, the total delay of the network at the flow

$$x^{+\epsilon,-\epsilon}_{\{i,j\}} = (\phi_1, \cdots, \phi_i + \epsilon, \cdots, \phi_j - \epsilon, \cdots \phi_n) \in \mathrm{int}\mathbf{S}^{n-1}$$

is

$$T^{+\epsilon} = \phi_1 l_1(\phi_1) + \cdots + (\phi_i + \epsilon) l_i(\phi_i + \epsilon) + \cdots + (\phi_j - \epsilon) l_j(\phi_j - \epsilon) + \cdots + \phi_n l_n(\phi_n).$$

The delay $T^{+\epsilon}$ is less than the delay T if the difference

$$\begin{aligned}
T^{+\epsilon} - T &= (\phi_i + \epsilon) l_i(\phi_i + \epsilon) - \phi_i l_i(\phi_i) + (\phi_j - \epsilon) l_j(\phi_j - \epsilon) - \phi_j l_j(\phi_j) \\
&= (\phi_i + \epsilon) l_i(\phi_i + \epsilon) - (\phi_i + \epsilon) l_i(\phi_i) + \epsilon l_i(\phi_i) \\
&\quad + (\phi_j - \epsilon) l_j(\phi_j - \epsilon) - (\phi_j - \epsilon) l_j(\phi_j) - \epsilon l_j(\phi_j) \\
&= (\phi_i + \epsilon)[l_i(\phi_i + \epsilon) - l_i(\phi_i)] + \epsilon l_i(\phi_i) \\
&\quad - (\phi_j - \epsilon)\left[l_j(\phi_j) - l_j(\phi_j - \epsilon)\right] - \epsilon l_j(\phi_j)
\end{aligned}$$

is negative. By dividing with ϵ and taking the limit with $\epsilon \to 0$, the above quantity becomes

$$l_i(\phi_i) + \phi_i l_i'(\phi_i) - l_j(\phi_j) - \phi_j l_j'(\phi_j).$$

If this is negative, the marginal difference $T^{+\epsilon} - T$ is negative, and therefore we will obtain a better total latency by removing flow from ϕ_j and adding it to ϕ_i. Hence, at any given flow as above, if we reduce the flow ϕ_j by ϵ while at the same time increase the flow ϕ_i by the same amount we will obtain a better total delay if

$$l_i(\phi_i) + \phi_i l_i'(\phi_i) < l_j(\phi_j) + \phi_j l_j'(\phi_j). \tag{2}$$

Similarly we can consider the total delay of the flow

$$x^{-\epsilon} = (\phi_i - \epsilon, \phi_j + \epsilon) \in \mathbf{S}^1$$

as follows

$$T^{-\epsilon} = (\phi_i - \epsilon) l_i(\phi_i - \epsilon) + (\phi_j + \epsilon) l_j(\phi_j + \epsilon).$$

In this case for the new delay $T^{-\epsilon}$ to be less than T, the difference $T^{-\epsilon} - T$ must be negative and using the same syllogism it turns out that the new total delay $T^{-\epsilon} - T$ is lower than T if the quantity

$$-l_i(\phi_i) - \phi_i l_i'(\phi_i) + l_j(\phi_j) + \phi_j l_j'(\phi_j)$$

is negative. Therefore, if we reduce the flow ϕ_i by ϵ while at the same time increase the flow ϕ_j by the same amount we will obtain a better total delay if

$$l_i(\phi_i) + \phi_i l_i'(\phi_i) > l_j(\phi_j) + \phi_j l_j'(\phi_j). \tag{3}$$

By consider both Equations 2 and 3 we deduce that an optimal flow $(\phi_i, \phi_j) \in \mathrm{Int}\mathbf{S}^1$ implies

$$l_i(\phi_i) + \phi_i l_i'(\phi_i) = l_j(\phi_j) + \phi_j l_j'(\phi_j), \tag{4}$$

which completes the proof.

We can generalize the above as follows.

Theorem 1. *Consider a network* $\mathbf{N_n} = (l_1(x), \ldots, l_n(x))$, *then it holds*

$$l_i(\phi_i) + \phi_i l_i'(\phi_i) = l_j(\phi_j) + \phi_j l_j'(\phi_j), \quad \text{for every } i, j \in \mathbf{I}_n \text{ with } \phi_i, \phi_j > 0$$

and

$$l_i(0) \geq l_j(\phi_j) + \phi_j l_j'(\phi_j), \quad \text{for every } i, j \in \mathbf{I}_n \text{ with } \phi_i = 0,\ \phi_j > 0,$$

provided that $\phi = (\phi_1, \ldots, \phi_n) \in \mathbf{S}^{n-1}$ *is system optimum of the network.*

Proof. We only have to consider the case of a system optimum

$$\phi = (\phi_1, \ldots, \phi_n) \in \mathbf{S}^{n-1}$$

on $\mathbf{N_n}$ such that there exists an $i \in \mathbf{I}_n$ with $\phi_i = 0$. Using a similar argument as in the previous proposition we get that if there exists any $j \in \mathbf{I}_n$ such that

$$l_i(\phi_i) + \phi_i l_i'(\phi_i) < l_j(\phi_j) + \phi_j l_j'(\phi_j), \tag{5}$$

then we will achieve lower total delay by ϵ-reducing ϕ_j while at the same time ϵ-increasing ϕ_i. Since $x \in \mathbf{S}^{n-1}$ is system optimum this is not possible. Thus, taking also into account that $\phi_i = 0$, it must hold

$$l_i(0) \geq l_j(\phi_j) + \phi_j l_j'(\phi_j), \quad \text{for every } i, j \in \mathbf{I}_n \text{ with } \phi_i = 0 \text{ and } \phi_j > 0.$$

Now, for differentiable latency functions $l_i(x)$, we construct the following functions

$$p_i(x) = l_i(x) + x l_i'(x).$$

In this way, for every network $\mathbf{N_n} = (l_1(x), \ldots, l_n(x))$, we construct a corresponding (*Pigovian*) network $\mathbf{PN}_n = (p_1(x), \ldots, p_n(x))$. Theorem 1 is then reformulated to describe the relationship between the two networks.

Theorem 2. *Consider a network* $\mathbf{N_n} = (l_1(x), \dots, l_n(x))$ *and a flow* $\phi = (\phi_1, \dots, \phi_n) \in \mathbf{S}^{n-1}$*, then if* ϕ *is system optimum of* $\mathbf{N_n}$ *it is also user equilibrium of*

$$\mathbf{PN}_n = (p_1(x), \dots, p_n(x)).$$

Proposition 1 allows us now the achieve the following uniqueness result.

Proposition 3. *A convex network* $\mathbf{N_n} = (l_1(x), \dots, l_n(x))$ *has a unique system optimum.*

Proof. All the latency functions are differentiable in $[0, 1]$, hence the total delay

$$\sum_{i=1}^{n} \phi_i l_i(\phi_i)$$

has at least one minimum.

Therefore, there is at least one system optimum of $\mathbf{N_n}$. Moreover, we know that the latency functions are differentiable and convex and so $p_i(x)$ will be strictly increasing and continuous. Thus, according to Proposition 1 there is a unique user equilibrium of $\mathbf{PN}_n = (p_1(x), \dots, p_n(x))$. Ther result follows by applying Theorem 2.

The following characterization of system optimums is achieved by combining the previous results.

Theorem 3. *Given a convex network* $\mathbf{N_n} = (l_1(x), \dots, l_n(x))$ *and a flow* $\phi = (\phi_1, \dots, \phi_n) \in \mathbf{S}^{n-1}$ *the following conditions are equivalent.*

i) The flow ϕ *is the system optimum of* $\mathbf{N_n}$.

ii) The flow ϕ *is the user equilibrium of*

$$\mathbf{PN}_n = (p_1(x), \dots, p_n(x)).$$

iii) It holds

$$l_i(\phi_i) + \phi_i l_i'(\phi_i) = l_j(\phi_j) + \phi_j l_j'(\phi_j), \text{ for every } i, j \in \mathbf{I}_n \text{ with } \phi_i, \phi_j > 0$$

and

$$l_i(0) \geq l_j(\phi_j) + \phi_j l_j'(\phi_j), \text{ for every } i, j \in \mathbf{I}_n, \text{ with } \phi_i = 0, \text{ and } \phi_j > 0.$$

iv) It holds

$$l_k(\phi_k) + \phi_k l'(\phi_k) = \min_{i \in \mathbf{I}_n} \{l_i(\phi_i) + \phi_i l_i'(\phi_i)\}, \text{ for every } \phi_k > 0.$$

5. Wardrop Optimal Networks

Flows that are at the same time system optimum and user equilibrium are called Wardrop optimal flows (WOFs). In this section we will investigate networks admitting such flows i.e., Wardrop optimal networks (WONs). First we need to identify some necessary conditions for system optimums. By applying Theorem 3 we get the following two propositions.

Proposition 4. *If the flow* $\phi = (\phi_1, \dots, \phi_n) \in \mathbf{S}^{n-1}$ *is WOF of the network* $\mathbf{N_n} = (l_1(x), \dots, l_n(x))$ *then it holds*

i) $\phi_i l_i'(\phi_i) = \phi_j l_j'(\phi_j)$, *for every* $i, j \in \mathbf{I}_n$, *with* $\phi_i, \phi_j > 0$,

ii) $l_i(0) \geq l_j(\phi_j) + \phi_j l_j'(\phi_j)$, *for every* $i, j \in \mathbf{I}_n$, *with* $\phi_i = 0$ *and* $\phi_j > 0$.

Proposition 5. *If the flow* $\phi = (\phi_1, \dots, \phi_n) \in \mathbf{S}^{n-1}$ *is user equilibrium of the convex network* $\mathbf{N_n} = (l_1(x), \dots, l_n(x))$ *and the following conditions hold*

i) $\phi_i l_i'(\phi_i) = \phi_j l_j'(\phi_j)$, *for every* $i, j \in \mathbf{I}_n$, *with* $\phi_i, \phi_j > 0$,

ii) $l_i(0) \geq l_j(\phi_j) + \phi_j l_j'(\phi_j)$, *for every* $i, j \in \mathbf{I}_n$, *with* $\phi_i = 0$ *and* $\phi_j > 0$,

then the flow ϕ *is also system optimum.*

Combining the above propositions we arrive at the following result.

Theorem 4. *If the flow* $(\phi_1, \dots, \phi_n) \in \mathbf{S}^{n-1}$ *is a user equilibrium of a convex network*

$$\mathbf{N_n} = (l_1(x), \dots, l_n(x)),$$

then the following conditions are equivalent

i) It holds

$$\phi_i l_i'(\phi_i) = \phi_j l_j'(\phi_j), \text{ for every } i, j \in \mathbf{I}_n \text{ with } \phi_i, \phi_j > 0,$$

and

$$l_i(0) \geq l_j(\phi_j) + \phi_j l_j'(\phi_j), \text{ for every } i, j \in \mathbf{I}_n \text{ with } \phi_i = 0, \text{ and } \phi_j > 0.$$

ii) The flow ϕ *is a system optimum.*

On the other hand, we get the following requirements for a flow that is a system optimum.

Proposition 6. *If the flow* $\phi = (\phi_1, \dots, \phi_n) \in \mathbf{S}^{n-1}$ *is system optimum of a network* $\mathbf{N_n} = (l_1(x), \dots, l_n(x))$ *and it holds*

$$l_i(\phi_i) = l_j(\phi_j), \text{ for every } i, j \in \mathbf{I}_n \text{ with } \phi_i, \phi_j > 0,$$

then it is also a user equilibrium.

Hence we can arrive at the next Theorem.

Theorem 5. *Given a flow* $\phi = (\phi_1, \dots, \phi_n) \in \mathbf{S}^{n-1}$ *of a convex network*

$$\mathbf{N_n} = (l_1(x), \dots, l_n(x)),$$

the following conditions are equivalent:

i) The flow ϕ *Wardrop optimal.*

ii) It holds:

$$l_i(\phi_i) = l_j(\phi_j), \text{ for every } i, j \in \mathbf{I}_n \text{ with } \phi_i, \phi_j > 0,$$
$$\phi_i l_i'(\phi_i) = \phi_j l_j'(\phi_j), \text{ for every } i, j \in \mathbf{I}_n \text{ with } \phi_i, \phi_j > 0,$$
$$l_i(0) \geq l_j(\phi_j) + \phi_j l_j'(\phi_j), \text{ for every } i, j \in \mathbf{I}_n \text{ with } \phi_i = 0, \text{ and } \phi_j > 0.$$

If we only consider internal flows, then we get the following corollary as a particular case of the above result.

Corollary 1. *Given a flow* $\phi = (\phi_1, \dots, \phi_n) \in \mathrm{Int}(\mathbf{S}^{n-1})$ *of a convex network*

$$\mathbf{N_n} = (l_1(x), \dots, l_n(x)),$$

the following conditions are equivalent:

i) The flow ϕ *Wardrop optimal.*

ii) It holds:

$$l_i(\phi_i) = l_j(\phi_j), \text{ for every } i, j \in \mathbf{I}_n \text{ with } \phi_i, \phi_j > 0,$$
$$\phi_i l_i'(\phi_i) = \phi_j l_j'(\phi_j), \text{ for every } i, j \in \mathbf{I}_n \text{ with } \phi_i, \phi_j > 0.$$

The next result provides some insight on the effect of appropriate transformations on the latency functions of our networks.

Theorem 6. *If the flow* $\phi = (\phi_1, \dots, \phi_n) \in \mathbf{S}^{n-1}$ *is the WOF of a convex network*

$$\mathbf{N_n} = (l_1(x), \dots, l_n(x)),$$

then it is also WOF of the network

$$f(\mathbf{N_n}) = (f(l_1(x)), \dots, f(l_n(x))),$$

where $f(x) : \mathbb{R} \to \mathbb{R}$ *is a a continuous, strictly increasing and convex function.*

Proof. Since ϕ is WOF of $\mathbf{N_n}$ we get

$$l_i(\phi_i) = l_j(\phi_j), \text{ for every } i, j \in \mathbf{I}_n \text{ with } \phi_i, \phi_j > 0, \tag{6}$$

from where we can derive the first item of Theorem 5 for the set $f(\mathbf{N_n})$:

$$f(l_i(\phi_i) = f(l_j(\phi_j)), \text{ for every } i, j \in \mathbf{I}_n \text{ with } \phi_i, \phi_j > 0.$$

Since ϕ is also optimal flow of $\mathbf{L}_n$, we have

$$\phi_i l_i'(\phi_i) = \phi_j l_j'(\phi_j), \text{ for every } i, j \in \mathbf{I}_n \text{ with } \phi_i, \phi_j > 0, \tag{7}$$

and

$$l_i(0) \geq l_j(\phi_j) + \phi_j l_j'(\phi_j), \text{ for every } i, j \in \mathbf{I}_n \text{ with } \phi_i = 0, \text{ and } \phi_j > 0. \tag{8}$$

For $\phi_i, \phi_j > 0$, it holds

$$\phi_i (f(l_i(\phi_i)))' = \phi_i f'(l_i(\phi_i)) l_i'(\phi_i) \overset{(6),(7)}{=} \phi_j f'(l_j(\phi_j)) l_j'(\phi_j) = \phi_j (f(l_j(\phi_j)))'. \tag{9}$$

Hence the second condition of Theorem 5 is satisfied by ϕ.

Now let $\phi_i = 0$ and $\phi_j > 0$. Since f is increasing, from Equation 8 we obtain

$$f(l_i(0)) \geq f(l_j(\phi_j) + \phi_j l_j'(\phi_j)).$$

We need to prove the third condition of Theorem 5 which is

$$f(l_i(0)) \geq f(l_j(\phi_j)) + \phi_j \left(f(l_j(\phi_j))\right)', \text{ for every } i, j \in \mathbf{I}_n \text{ with } \phi_i = 0, \text{ and } \phi_j > 0.$$

Therefore, it suffices to show that

$$f(l_j(\phi_j) + \phi_j l_j'(\phi_j)) \geq f(l_j(\phi_j)) + \phi_j \left(f(l_j(\phi_j))\right)' \Rightarrow$$
$$f(l_j(\phi_j) + \phi_j l_j'(\phi_j)) - f(l_j(\phi_j)) \geq \phi_j f'(l_j(\phi_j))\, l_j'(\phi_j) \Rightarrow$$
$$\frac{f(l_j(\phi_j) + \phi_j l_j'(\phi_j)) - f(l_j(\phi_j))}{\phi_j l_j'(\phi_j)} \geq f'(l_j(\phi_j)).$$

If we set $a = l_j(\phi_j)$ and $b = \phi_j l_j'(\phi_j)$, the last inequality can be rewritten as

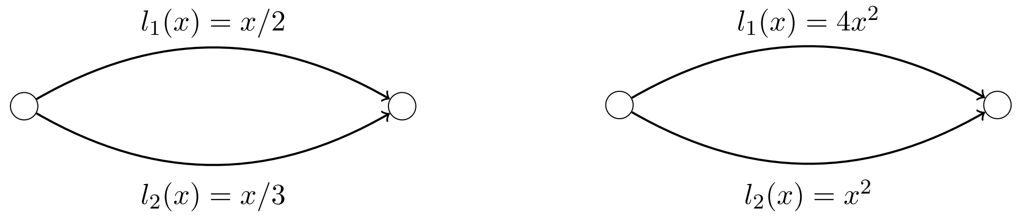
$$\frac{f(a+b) - f(a)}{b} \geq f'(a).$$

Since $b > 0$, the left hand side of the above inequality is the average increase of $f(x)$ at the interval $[a, a+b]$ and the right hand side is the rate of change of $f(x)$ at a. Therefore, from the convexity of $f(x)$, we deduce that the inequality holds true. The proof is completed.

The above Theorem shows that WOFs are preserved by continuous, strictly increasing and convex functions. Nevertheless, this does not hold for a system optimum which is not user equilibrium. Indeed, as it shown in Example 1 the system optimum of the network of Figure 1 does not remain the same after taking the image of $\mathbf{N}$ via the continuous, strictly increasing and convex function $f(x) = x^2$.

Example 2. *Let us examine the linear network* $\mathbf{W_1} = \{\frac{x}{2}, \frac{x}{3}\}$ *shown in Fig. 2a. The system optimum that minimizes the average delay of the network and the user equilibrium that equates delay on the two links are identical and equal with* $\phi = (0.4, 0.6)$.

Similarly for the quadratic network $\mathbf{W_2} = \{4x^2, x^2\}$ *of Fig. 2b, it can easily be verified that the system optimum and the user equilibrium are identical and equal with* $\phi = (1/3, 2/3)$.



(a) $\mathbf{W_1}$: Linear Wardrop optimal network (b) $\mathbf{W_2}$: Quadratic Wardrop optimal network

Figure 2: Two Wardrop optimal networks

The next corollaries illustrate the effect of some fundamental transformations on WOFs.

Corollary 2. *If a flow* $\phi \in \mathbf{S}^{n-1}$ *is WOF of a convex network* $\mathbf{N_n} = (l_1(x), \ldots, l_n(x))$ *then it is also a WOF of the following networks.*

i) $\mathbf{N_n} + \mathbf{b} = (l_1(x) + b, \ldots, l_n(x) + b)$, *for every* $b > 0$.

ii) $\mathbf{aN_n} = (al_1(x), \ldots, al_n(x))$, *for every* $a > 0$.

For the particular case of internal flows, we can also state the following.

Corollary 3. *If the flow* $\phi = (\phi_1, \ldots, \phi_n) \in int\mathbf{S}^{n-1}$ *is the WOF of a convex network* $\mathbf{N_n} = (l_1(x), \ldots, l_n(x))$, *then it is also WOF of*

$$f(\mathbf{N_n}) = (f(l_1(x)), \ldots, f(l_n(x))),$$

where $f(x) : \mathbb{R} \to \mathbb{R}$ *is a continuous, strictly increasing function.*

Next we examine networks with identical latency functions.

Proposition 7. *If the latency functions across all links of a network are identical then the WOF is uniformly distributed.*

Proof. It suffices to observe that $\phi = (\frac{1}{n}, \ldots, \frac{1}{n})$ is the WOF of $\mathbf{N_n}$.

Similarly we have

Proposition 8. *Any internal flow* $\mathbf{p} = (p_1, \cdots, p_n) \in \mathrm{int}\mathbf{S}^{n-1}$ *is the WOF of* $\mathbf{N_n} = (\frac{\phi_1}{p_1}, \ldots, \frac{\phi_n}{p_n})$.

This result together with Corollary 3, gives the following proposition.

Proposition 9. *An internal flow* $\mathbf{p} = (p_1, \cdots, p_n) \in \mathrm{int}\mathbf{S}^{n-1}$ *is the WOF of* $\mathbf{N_n} = (f(\frac{\phi_1}{p_1}), \ldots, f(\frac{\phi_n}{p_n}))$*, where* $f(x)$ *is any continuous, strictly increasing function.*

By using Theorem 5 we obtain the next proposition.

Proposition 10. *If a flow* $\phi = (\phi_1, \ldots, \phi_n) \in \mathbf{S}^{n-1}$ *is at the same time, WOF of the convex networks* $\mathbf{N_n} = (l_1(x), \ldots, l_n(x))$ *and* $\overline{\mathbf{N}}_\mathbf{n} = (\bar{l}_1(x), \ldots, \bar{l}_n(x))$ *then it is also WOF of* $\mathbf{N_n}\overline{\mathbf{N}}_\mathbf{n} = (l_1(x)\bar{l}_1(x), \ldots, l_n(x)\bar{l}_n(x))$.

Proof. Since ϕ is WOF of the networks $\mathbf{N_n}$ and $\overline{\mathbf{N}}_\mathbf{n}$, from the first condition of Theorem 5 for the two networks we have

$$l_i(\phi_i) = l_j(\phi_j), \text{ for every } i, j \in \mathbf{I}_n \text{ with } \phi_i, \phi_j > 0, \tag{10}$$

and

$$\bar{l}_i(\phi_i) = \bar{l}_j(\phi_j), \text{ for every } i, j \in \mathbf{I}_n \text{ with } \phi_i, \phi_j > 0. \tag{11}$$

By multiplication of the two we get the first condition for the network $\mathbf{N_n}\overline{\mathbf{N}}_\mathbf{n}$:

$$l_i(\phi_i)\bar{l}_i(x) = l_j(\phi_j)\bar{l}_j(x), \text{ for every } i, j \in \mathbf{I}_n \text{ with } \phi_i, \phi_j > 0, \tag{12}$$

which establishes the first condition of Theorem 5 for $\mathbf{N_n}\overline{\mathbf{N}}_\mathbf{n}$. From the second condition of Theorem 5 for the networks $\mathbf{N_n}$ and $\overline{\mathbf{N}}_\mathbf{n}$ we have

$$\phi_i l_i'(\phi_i) = \phi_j l_j'(\phi_j), \text{ for every } i, j \in \mathbf{I}_n \text{ with } \phi_i, \phi_j > 0, \tag{13}$$

and

$$\phi_i \bar{l}_i'(\phi_i) = \phi_j \bar{l}_j'(\phi_j), \text{ for every } i, j \in \mathbf{I}_n \text{ with } \phi_i, \phi_j > 0. \tag{14}$$

To prove the second condition for $\mathbf{N_n}\overline{\mathbf{N}}_\mathbf{n}$ we need to prove that for every $\phi_i, \phi_j > 0$ it holds that

$$\phi_i(l_i(\phi_i)\bar{l}_i(\phi_i))' = \phi_j(l_j(\phi_j)\bar{l}_j(\phi_j))', \tag{15}$$

or equivalently

$$\phi_i l_i'(\phi_i)\bar{l}_i(\phi_i) + \phi_i l_i(\phi_i)\bar{l}_i'(\phi_i) = \phi_j l_j'(\phi_j)\bar{l}_j(\phi_j) + \phi_j l_j(\phi_j)\bar{l}_j'(\phi_j). \tag{16}$$

By multiplying Equations 11 and 13 we get

$$\phi_i l_i'(\phi_i)\bar{l}_i(\phi_i) = \phi_j l_j'(\phi_j)\bar{l}_j(\phi_j), \text{ for every } i, j \in \mathbf{I}_n \text{ with } \phi_i, \phi_j > 0. \tag{17}$$

By multiplying Equations 10 and 14 we get

$$\phi_i l_i(\phi_i)\bar{l}_i'(\phi_i) = \phi_j l_j(\phi_j)\bar{l}_j'(\phi_j), \text{ for every } i, j \in \mathbf{I}_n \text{ with } \phi_i, \phi_j > 0. \tag{18}$$

Equation 16 is now obtained by adding Equations 17 and 18.

Now, for the remaining part of the proof, from the last condition of Theorem 5 for $\mathbf{N_n}$ and $\overline{\mathbf{N}}_\mathbf{n}$, it holds that for every $i, j \in \mathbf{I}_n$, with $\phi_i = 0$ and $\phi_j > 0$

$$l_i(0) \geq l_j(\phi_j) + \phi_j\, l_j'(\phi_j) \quad \text{and} \quad \bar{l}_i(0) \geq \bar{l}_j(\phi_j) + \phi_j\, \bar{l}_j'(\phi_j).$$

By multiplying the above inequalities we get

$$l_i(0)\bar{l}_i(0) \;\geq\; l_j(\phi_j)\bar{l}_j(\phi_j) \;+\; l_j(\phi_j)\phi_j\,\bar{l}_j'(\phi_j) \;+\; \bar{l}_j(\phi_j)\phi_j\, l_j'(\phi_j) \;+\; \phi_j\, l_j'(\phi_j)\phi_j\,\bar{l}_j'(\phi_j). \quad (19)$$

From this, we get the third condition of Theorem 5 for the network $\mathbf{N_n}\overline{\mathbf{N}}_\mathbf{n}$, which is:

$$l_i(0)\bar{l}_i(0) \geq l_j(\phi_j)\bar{l}_j(\phi_j) + \phi_j\left(l_j(\phi_j)\bar{l}_j(\phi_j)\right)', \quad (20)$$

or equivalently

$$l_i(0)\bar{l}_i(0) \geq l_j(\phi_j)\bar{l}_j(\phi_j) + \phi_j l_j'(\phi_j)\bar{l}_j(\phi_j) + \phi_j\, l_j(\phi_j)\phi_j\bar{l}_j'(\phi_j). \quad (21)$$

In a similar way, we will establish the next proposition.

Proposition 11. *If the flow $\phi = (\phi_1, \ldots, \phi_n) \in \mathbf{S}^{n-1}$ is WOF of the convex networks*

$$\mathbf{N_n} = (l_1(x), \ldots, l_n(x)) \text{ and } \overline{\mathbf{N}}_\mathbf{n} = (\bar{l}_1(x), \ldots, \bar{l}_n(x)),$$

then it is also WOF of $\mathbf{N_n} + \overline{\mathbf{N}}_\mathbf{n} = (l_1(x) + \bar{l}_1(x), \ldots, l_n(x) + \bar{l}_n(x))$.

Proof. Since ϕ is WOF of the networks $\mathbf{N_n}$ and $\overline{\mathbf{N}}_\mathbf{n}$, from the first condition of Theorem 5, we have that for the two networks it holds

$$l_i(\phi_i) = l_j(\phi_j), \text{ for every } i, j \in \mathbf{I}_n \text{ with } \phi_i, \phi_j > 0, \quad (22)$$

and

$$\bar{l}_i(\phi_i) = \bar{l}_j(\phi_j), \text{ for every } i, j \in \mathbf{I}_n \text{ with } \phi_i, \phi_j > 0. \quad (23)$$

By adding the two we get the first condition for the network $\mathbf{N_n} + \overline{\mathbf{N}}_\mathbf{n}$

$$l_i(\phi_i) + \bar{l}_i(x) = l_j(\phi_j) + \bar{l}_j(x), \text{ for every } i, j \in \mathbf{I}_n \text{ with } \phi_i, \phi_j > 0, \quad (24)$$

which settles the first condition of Theorem 5 for $\mathbf{N_n} + \overline{\mathbf{N}}_\mathbf{n}$. For the next condition of Theorem 5, assume that for the networks $\mathbf{N_n}$ and $\overline{\mathbf{N}}_\mathbf{n}$ we have

$$\phi_i l_i'(\phi_i) = \phi_j l_j'(\phi_j), \text{ for every } i, j \in \mathbf{I}_n \text{ with } \phi_i, \phi_j > 0. \quad (25)$$

and

$$\phi_i \bar{l}_i'(\phi_i) = \phi_j \bar{l}_j'(\phi_j), \text{ for every } i, j \in \mathbf{I}_n \text{ with } \phi_i, \phi_j > 0. \quad (26)$$

By adding Equations 25 and 26 we get

$$\phi_i l_i'(\phi_i) + \phi_i \bar{l}_i'(\phi_i) = \phi_j l_j'(\phi_j) + \phi_j \bar{l}_j'(\phi_j), \quad (27)$$

or equivalently

$$\phi_i(l_i(\phi_i) + \bar{l}_i(\phi_i))' = \phi_j(l_j(\phi_j) + \bar{l}_j(\phi_j))', \tag{28}$$

as desired.

Now for the remaining part of the proof, if for $\mathbf{N_n}$ and $\overline{\mathbf{N}}_\mathbf{n}$ it holds

$$l_i(0) \geq l_j(\phi_j) + \phi_j\, l_j'(\phi_j) \quad \text{ and } \quad \bar{l}_i(0) \geq \bar{l}_j(\phi_j) + \phi_j\, \bar{l}_j'(\phi_j),$$

for every $i, j \in \mathbf{I}_n$ with $\phi_i = 0$ and $\phi_j > 0$, then by adding the above inequalities we get

$$l_i(0) + \bar{l}_i(0) \geq l_j(\phi_j) + \bar{l}_j(\phi_j) + \phi_j(l_j'(\phi_j) + \bar{l}_j'(\phi_j)). \tag{29}$$

From the above, the last condition of Theorem 5 for the network $\mathbf{N_n} + \overline{\mathbf{N}}_\mathbf{n}$ follows.

6. Conclusion and Future Work

We have investigated directed, parallel networks with congestion externalities and their equilibriums. We identified necessary and sufficient conditions for achieving the system optimum and we introduced Wardrop optimal networks admitting the same user and system equilibrium. In addition we establish a characterization of Wardrop optimal networks which allows us to identify important closure properties of the related class of networks.

Future research directions deriving from the results presented in this paper include the investigation of Wardrop optimal flows in networks with more general underlying graph structures by analyzing all potential paths from the origin to the destination, similarly to the approach introduced in [15]. Additionally, this framework could be utilized for the algebraic recognition of such networks, thereby enabling the use of graph recognizability to identify graph properties, such as determining if a graph is Eulerian [9] and k-colorable [11].

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