EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 17, No. 4, 2024, 3242-3253
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

c-quasi (τ_1, τ_2)-continuous multifunctions

Prapart Pue-on[1], Areeyuth Sama-Ae[2], Chawalit Boonpok[1,*]

[1] *Mathematics and Applied Mathematics Research Unit, Department of Mathematics, Faculty of Science, Mahasarakham University, Maha Sarakham, 44150, Thailand*
[2] *Department of Mathematics and Computer Science, Faculty of Science and Technology, Prince of Songkla University, Pattani Campus, Pattani, 94000, Thailand*

Abstract. Our main purpose is to introduce the concepts of upper and lower c-quasi (τ_1, τ_2)-continuous multifunctions. Moerover, several characterizations of upper and lower c-quasi (τ_1, τ_2)-continuous multifunctions are established.

2020 Mathematics Subject Classifications: 54C08, 54C60, 54E55

Key Words and Phrases: $\tau_1\tau_2$-open set, upper c-quasi (τ_1, τ_2)-continuous multifunction, lower c-quasi (τ_1, τ_2)-continuous multifunction

---

1. Introduction

It is well-known that the branch of mathematics called topology is concerned with all questions directly or indirectly related to continuity. Continuity is an important concept for the study and investigation in the theory of classical point set topology. Generalization of this concept by using stronger and weaker forms of open sets. Many authors have researched and investigated several stronger and weaker forms of continuous functions and multifunctions. Viriyapong and Boonpok [58] investigated some characterizations of (Λ, sp)-continuous functions by utilizing the notions of (Λ, sp)-open sets and (Λ, sp)-closed sets due to Boonpok and Khampakdee [13]. Dungthaisong et al. [33] introduced and studied the concept of $g_{(m,n)}$-continuous functions. Duangphui et al. [32] introduced and investigated the notion of $(\mu, \mu')^{(m,n)}$-continuous functions. Moreover, several characterizations of almost (Λ, p)-continuous functions, strongly $\theta(\Lambda, p)$-continuous functions, almost strongly $\theta(\Lambda, p)$-continuous functions, $\theta(\Lambda, p)$-continuous functions, weakly (Λ, b)-continuous functions, $\theta(\star)$-precontinuous functions, $(\Lambda, p(\star))$-continuous functions, $\star$-continuous functions, θ-$\mathscr{I}$-continuous functions, almost (g, m)-continuous functions, pairwise M-continuous functions, (τ_1, τ_2)-continuous functions, almost (τ_1, τ_2)-continuous functions, weakly (τ_1, τ_2)-continuous functions, almost quasi (τ_1, τ_2)-continuous functions

---

*Corresponding author.

DOI: https://doi.org/10.29020/nybg.ejpam.v17i4.5393

Email addresses: `prapart.p@msu.ac.th` (P. Pue-on), `areeyuth.s@psu.ac.th` (A. Sama-Ae), `chawalit.b@msu.ac.th` (C. Boonpok)

and weakly quasi (τ_1, τ_2)-continuous functions were presented in [53], [55], [17], [49], [26], [12], [9], [11], [5], [2], [3], [27], [24], [19], [39] and [31], respectively. Gentry and Hoyle III [34] introduced and investigated the concept of c-continuous functions. In particular, some characterizations of c-continuous functions were studied in [41], [42] and [46], respectively.

In 1961, Marcus [43] introduced the notion of quasicontinuous functions. Popa [47] introduced and studied the notion of quasi-continuous multifunctions. Viriyapong and Boonpok [59] introduced and studied the concept of weakly quasi (Λ, sp)-continuous multifunctions. Furthermore, several characterizations of $(\tau_1, \tau_2)\delta$-semicontinuous multifunctions, almost weakly (τ_1, τ_2)-continuous multifunctions, $\star$-continuous multifunctions, $\beta(\star)$-continuous multifunctions, α-$\star$-continuous multifunctions, almost α-$\star$-continuous multifunctions, almost quasi $\star$-continuous multifunctions, weakly α-$\star$-continuous multifunctions, $s\beta(\star)$-continuous multifunctions, weakly $s\beta(\star)$-continuous multifunctions, $\theta(\star)$-quasi continuous multifunctions, almost $\imath^\star$-continuous multifunctions, weakly (Λ, sp)-continuous multifunctions, $\alpha(\Lambda, sp)$-continuous multifunctions, almost $\alpha(\Lambda, sp)$-continuous multifunctions, weakly $\alpha(\Lambda, sp)$-continuous multifunctions, almost $\beta(\Lambda, sp)$-continuous multifunctions, slightly (Λ, sp)-continuous multifunctions, (τ_1, τ_2)-continuous multifunctions, almost (τ_1, τ_2)-continuous multifunctions, weakly (τ_1, τ_2)-continuous multifunctions and weakly quasi (τ_1, τ_2)-continuous multifunctions were investigated in [6], [29], [4], [8], [18], [25], [7], [22], [21], [16], [10], [20], [23], [36], [14], [28], [54], [15], [51], [38], [56] and [52], respectively. Neubrunn [44] and Holá et al. [35] extended the concept of c-continuous functions to the setting of multifunctions. Lipski [40] introduced the notion of c-quasicontinuous multifunctions as a generalization of c-continuous multifunctions [44] and quasi-continuous multifunctions [47]. Noiri and Popa [45] introduced and investigated the notion of C-m-continuous multifunctions. Popa and Noiri [48] investigated some characterizations of C-quasicontinuous multifunctions. Khampakdee et al. [37] introduced and studied the notion of c-(τ_1, τ_2)-continuous multifunctions. In this paper, we introduce the concepts of upper and lower c-qausi (τ_1, τ_2)-continuous multifunctions. We also investigate several characterizations of upper and lower c-qausi (τ_1, τ_2)-continuous multifunctions.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [30] if $A = \tau_1\text{-Cl}(\tau_2\text{-Cl}(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. Let A be a subset of a bitopological space (X, τ_1, τ_2). The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [30] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [30] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 1. [30] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2\text{-}Cl(A)$ *and* $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Cl(A)) = \tau_1\tau_2\text{-}Cl(A)$.

(2) If $A \subseteq B$, *then* $\tau_1\tau_2\text{-}Cl(A) \subseteq \tau_1\tau_2\text{-}Cl(B)$.

(3) $\tau_1\tau_2\text{-}Cl(A)$ *is* $\tau_1\tau_2$*-closed.*

(4) A *is* $\tau_1\tau_2$*-closed if and only if* $A = \tau_1\tau_2\text{-}Cl(A)$.

(5) $\tau_1\tau_2\text{-}Cl(X - A) = X - \tau_1\tau_2\text{-}Int(A)$.

A bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$*-compact* [30] if every cover of X by $\tau_1\tau_2$-open sets of X has a finite subcover. A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)r$*-open* [57] (resp. $(\tau_1, \tau_2)s$*-open* [6], $(\tau_1, \tau_2)p$*-open* [6], $(\tau_1, \tau_2)\beta$*-open* [6]) if $A = \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$ (resp. $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A))$, $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$, $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A)))$). The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open) set is said to be $(\tau_1, \tau_2)r$*-closed* (resp. $(\tau_1, \tau_2)s$*-closed*, $(\tau_1, \tau_2)p$*-closed*, $(\tau_1, \tau_2)\beta$*-closed*). A subset A of a bitopological space (X, τ_1, τ_2) is called $\alpha(\tau_1, \tau_2)$*-open* [60] if $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A)))$. The complement of an $\alpha(\tau_1, \tau_2)$-open set is called $\alpha(\tau_1, \tau_2)$*-closed*. The intersection of all $(\tau_1, \tau_2)s$-closed sets of X containing A is called the $(\tau_1, \tau_2)s$*-closure* [6] of A and is denoted by $(\tau_1, \tau_2)\text{-sCl}(A)$. The union of all $(\tau_1, \tau_2)s$-open sets of X contained in A is called the $(\tau_1, \tau_2)s$*-interior* [6] of A and is denoted by $(\tau_1, \tau_2)\text{-sInt}(A)$.

Lemma 2. *For a subset* A *of a bitopological space* (X, τ_1, τ_2)*, the following properties hold:*

(1) $(\tau_1, \tau_2)\text{-}sCl(A) = \tau_1\tau_2\text{-}Int(\tau_1\tau_2\text{-}Cl(A)) \cup A$ [6];

(2) $(\tau_1, \tau_2)\text{-}sInt(A) = \tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A)) \cap A$ [50].

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, following [1] we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and

$$F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}.$$

In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$.

3. Upper and lower c-quasi (τ_1, τ_2)-continuous multifunctions

In this section, we introduce the notions of upper and lower c-quasi (τ_1, τ_2)-continuous multifunctions. Moreover, we investigate some characterizations of upper and lower c-quasi (τ_1, τ_2)-continuous multifunctions.

Definition 1. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called upper c-quasi (τ_1, τ_2)-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $F(x)$ and having $\sigma_1\sigma_2$-compact complement and for each $\tau_1\tau_2$-open set U of X containing x, there exists a nonempty $\tau_1\tau_2$-open set G such that $G \subseteq U$ and $F(G) \subseteq V$. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called upper c-quasi (τ_1, τ_2)-continuous if F has this property at every point of X.*

Theorem 1. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is upper c-quasi (τ_1, τ_2)-continuous at $x \in X$ if and only if for every $\sigma_1\sigma_2$-open set V of Y containing $F(x)$ and having $\sigma_1\sigma_2$-compact complement, there exists a $(\tau_1, \tau_2)s$-open set U of X containing x such that $F(U) \subseteq V$.*

Proof. Suppose that F is upper c-quasi (τ_1, τ_2)-continuous at $x \in X$. Let V be any $\sigma_1\sigma_2$-open set of Y having $\sigma_1\sigma_2$-compact complement such that $F(x) \subseteq V$. For each $\tau_1\tau_2$-open set U of X containing x, there exists a nonempty $\tau_1\tau_2$-open set G_U of X such that $G_U \subseteq U$ and $F(G_U) \subseteq V$. Put $W = \cup\{G_U \mid U \text{ is } \tau_1\tau_2\text{-open}, x \in U\}$. Then, W is a $\tau_1\tau_2$-open set and $x \in \tau_1\tau_2$-Cl(W). Let $H = W \cup \{x\}$, then $W \subseteq H \subseteq \tau_1\tau_2$-Cl$(W)$; hence H is a $(\tau_1, \tau_2)s$-open set of X containing x and $F(H) \subseteq V$.

Conversely, let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y having $\sigma_1\sigma_2$-compact complement such that $F(x) \subseteq V$. Let U be a $\tau_1\tau_2$-open set U of X containing x. For each $x_0 \in F^+(V)$, there exists a $(\tau_1, \tau_2)s$-open set U_{x_0} of X containing x_0 such that $F(U_{x_0}) \subseteq V$. Therefore, we have $U_{x_0} \subseteq F^+(V)$ and $F^+(V) = \cup_{x_0 \in F^+(V)} U_{x_0}$. Thus, $F^+(V)$ is $(\tau_1, \tau_2)s$-open in X and hence $F^+(V) \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(F^+(V)))$. Put $G = \tau_1\tau_2$-Int$(F^+(V)) \cap U$. Then, G is $\tau_1\tau_2$-open, $G \subseteq U$ and $G \neq \emptyset$ because $x \in \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(F^+(V)))$ implies $\tau_1\tau_2$-Int$(F^+(V)) \cap U \neq \emptyset$. On the other hand, we have

$$F(G) \subseteq F(\tau_1\tau_2\text{-Int}(F^+(V))) \subseteq F(F^+(V)) \subseteq V.$$

This shows that F is upper c-quasi (τ_1, τ_2)-continuous at x.

Definition 2. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be lower c-quasi (τ_1, τ_2)-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y having $\sigma_1\sigma_2$-compact complement such that $F(x) \cap V \neq \emptyset$ and for each $\tau_1\tau_2$-open set U of X containing x, there exists a nonempty $\tau_1\tau_2$-open set G such that $G \subseteq U$ and $F(z) \cap V \neq \emptyset$ for each $z \in G$. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be lower c-quasi (τ_1, τ_2)-continuous if F has this property at every point of X.*

Theorem 2. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is lower c-quasi (τ_1, τ_2)-continuous at $x \in X$ if and only if for every $\sigma_1\sigma_2$-open set V of Y having $\sigma_1\sigma_2$-compact complement with $F(x) \cap V \neq \emptyset$, there exists a $(\tau_1, \tau_2)s$-open set U of X containing x such that $F(z) \cap V \neq \emptyset$ for every $z \in U$.*

Proof. The proof is similar to that of Theorem 1.

Definition 3. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be c-quasi (τ_1, τ_2)-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$ and having $\sigma_1\sigma_2$-compact complement and for each $\tau_1\tau_2$-open set U of X containing x, there exists a nonempty $\tau_1\tau_2$-open set G such that $G \subseteq U$ and $f(G) \subseteq V$. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be c-quasi (τ_1, τ_2)-continuous if f has this property at each point of X.*

Corollary 1. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is c-quasi (τ_1, τ_2)-continuous at a point $x \in X$ if and only if for every $\sigma_1\sigma_2$-open set V of Y containing $f(x)$ and having $\sigma_1\sigma_2$-compact complement, there exists a $(\tau_1, \tau_2)s$-open set U of X containing x such that $f(U) \subseteq V$.*

Theorem 3. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper c-quasi (τ_1, τ_2)-continuous;

(2) $F^+(V)$ is $(\tau_1, \tau_2)s$-open in X for every $\sigma_1\sigma_2$-open set V of Y having $\sigma_1\sigma_2$-compact complement;

(3) $F^-(K)$ is $(\tau_1, \tau_2)s$-closed in X for every $\sigma_1\sigma_2$-compact $\sigma_1\sigma_2$-closed set K of Y;

(4) $\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(F^-(B))) \subseteq F^-(\sigma_1\sigma_2$-Cl$(B))$ for every subset B of Y having the $\sigma_1\sigma_2$-compact $\sigma_1\sigma_2$-closure;

(5) (τ_1, τ_2)-sCl$(F^-(B)) \subseteq F^-(\sigma_1\sigma_2$-Cl$(B))$ for every subset B of Y having the $\sigma_1\sigma_2$-compact $\sigma_1\sigma_2$-closure;

(6) $F^+(\sigma_1\sigma_2$-Int$(B)) \subseteq (\tau_1, \tau_2)$-sInt$(F^+(B))$ for every subset B of Y such that

$$Y - \sigma_1\sigma_2\text{-}Int(B)$$

is $\sigma_1\sigma_2$-compact.

Proof. (1) ⇒ (2): Let V be any $\sigma_1\sigma_2$-open set of Y having $\sigma_1\sigma_2$-compact complement and $x \in F^+(V)$. By Theorem 1, there exists a $(\tau_1, \tau_2)s$-open set U of X containing x such that $F(U) \subseteq V$. Therefore, we have $x \in U \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(F^+(V)))$. Thus, $F^+(V) \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(F^+(V)))$ and hence $F^+(V)$ is $(\tau_1, \tau_2)s$-open in X.

(2) ⇒ (3): The proof follows immediately from the fact that $F^+(Y - B) = Y - F^-(B)$ for every subset B of Y.

(3) ⇒ (4): Let B be any subset of Y having the $\sigma_1\sigma_2$-compact $\sigma_1\sigma_2$-closure. Then, $F^-(\sigma_1\sigma_2$-Cl$(B))$ is $(\tau_1, \tau_2)s$-closed in X. By Lemma 2, we have

$$\begin{aligned}\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(F^-(B))) &\subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(F^-(\sigma_1\sigma_2\text{-Cl}(B))))\\ &\subseteq (\tau_1, \tau_2)\text{-sCl}(F^-(\sigma_1\sigma_2\text{-Cl}(B)))\\ &= F^-(\sigma_1\sigma_2\text{-Cl}(B)).\end{aligned}$$

Thus, $\tau_1\tau_2$-Int($\tau_1\tau_2$-Cl($F^-(B)$)) $\subseteq F^-(\sigma_1\sigma_2$-Cl$(B))$.

(4) ⇒ (5): Let B be any subset of Y having the $\sigma_1\sigma_2$-compact $\sigma_1\sigma_2$-closure. It follows from Lemma 2 that

$$\begin{aligned}(\tau_1,\tau_2)\text{-sCl}(F^-(B)) &= F^-(B) \cup \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(F^-(B))) \\ &\subseteq F^-(B) \cup F^-(\sigma_1\sigma_2\text{-Cl}(B)) \\ &= F^-(\sigma_1\sigma_2\text{-Cl}(B)).\end{aligned}$$

(5) ⇒ (6): Let B be any subset of Y such that $Y - \sigma_1\sigma_2$-Int(B) is $\sigma_1\sigma_2$-compact. Then by Lemma 2, we have

$$\begin{aligned}X - (\tau_1,\tau_2)\text{-sInt}(F^+(B)) &= (\tau_1,\tau_2)\text{-sCl}(X - F^+(B)) \\ &= (\tau_1,\tau_2)\text{-sCl}(F^-(Y - B)) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(Y - B)) \\ &= F^-(Y - \sigma_1\sigma_2\text{-Int}(B)) \\ &= X - F^+(\sigma_1\sigma_2\text{-Int}(B))\end{aligned}$$

and hence $F^+(\sigma_1\sigma_2$-Int$(B)) \subseteq (\tau_1,\tau_2)$-sInt$(F^+(B))$.

(6) ⇒ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y such that $F(x) \subseteq V$ and having $\sigma_1\sigma_2$-compact complement. Then, $F^+(V) = F^+(\sigma_1\sigma_2$-Int$(V)) \subseteq (\tau_1,\tau_2)$-sInt$(F^+(V))$. Put $U = (\tau_1,\tau_2)$-sInt$(F^+(V))$. Then, U is a $(\tau_1,\tau_2)s$-open set U of X containing x and $F(U) \subseteq V$. Thus, F is upper c-quasi (τ_1,τ_2)-continuous at x. This shows that F is upper c-quasi (τ_1,τ_2)-continuous.

Theorem 4. *For a multifunction $F : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$, the following properties are equivalent:*

(1) F is lower c-quasi (τ_1,τ_2)-continuous;

(2) $F^-(V)$ is $(\tau_1,\tau_2)s$-open in X for every $\sigma_1\sigma_2$-open set V of Y having $\sigma_1\sigma_2$-compact complement;

(3) $F^+(K)$ is $(\tau_1,\tau_2)s$-closed in X for every $\sigma_1\sigma_2$-compact $\sigma_1\sigma_2$-closed set K of Y;

(4) $\tau_1\tau_2$-Int($\tau_1\tau_2$-Cl($F^+(B)$)) $\subseteq F^+(\sigma_1\sigma_2$-Cl$(B))$ for every subset B of Y having the $\sigma_1\sigma_2$-compact $\sigma_1\sigma_2$-closure;

(5) (τ_1,τ_2)-sCl$(F^+(B)) \subseteq F^+(\sigma_1\sigma_2$-Cl$(B))$ for every subset B of Y having the $\sigma_1\sigma_2$-compact $\sigma_1\sigma_2$-closure;

(6) $F^-(\sigma_1\sigma_2$-Int$(B)) \subseteq (\tau_1,\tau_2)$-sInt$(F^-(B))$ for every subset B of Y such that

$$Y - \sigma_1\sigma_2\text{-}Int(B)$$

is $\sigma_1\sigma_2$-compact.

Proof. The proof is similar to that of Theorem 3.

Corollary 2. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is c-quasi (τ_1, τ_2)-continuous;

(2) $f^{-1}(V)$ is $(\tau_1, \tau_2)s$-open in X for every $\sigma_1\sigma_2$-open set V of Y having $\sigma_1\sigma_2$-compact complement;

(3) $f^{-1}(K)$ is $(\tau_1, \tau_2)s$-closed in X for every $\sigma_1\sigma_2$-compact $\sigma_1\sigma_2$-closed set K of Y;

(4) $\tau_1\tau_2$-Int($\tau_1\tau_2$-Cl($f^{-1}(B)$)) $\subseteq f^{-1}$($\sigma_1\sigma_2$-Cl(B)) for every subset B of Y having the $\sigma_1\sigma_2$-compact $\sigma_1\sigma_2$-closure;

(5) (τ_1, τ_2)-sCl($f^{-1}(B)$) $\subseteq f^{-1}$($\sigma_1\sigma_2$-Cl(B)) for every subset B of Y having the $\sigma_1\sigma_2$-compact $\sigma_1\sigma_2$-closure;

(6) f^{-1}($\sigma_1\sigma_2$-Int(B)) $\subseteq (\tau_1, \tau_2)$-sInt($f^{-1}(B)$) for every subset B of Y such that

$$Y - \sigma_1\sigma_2\text{-}Int(B)$$

is $\sigma_1\sigma_2$-compact.

Corollary 3. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is upper c-quasi (τ_1, τ_2)-continuous if $F^-(K)$ is $(\tau_1, \tau_2)s$-closed in X for every $\sigma_1\sigma_2$-compact set K of Y.*

Proof. Let V be any $\sigma_1\sigma_2$-open set of Y having $\sigma_1\sigma_2$-compact complement. Then, $Y - V$ is a $\sigma_1\sigma_2$-compact $\sigma_1\sigma_2$-closed set. By the hypothesis, $F^-(Y - V)$ is $(\tau_1, \tau_2)s$-closed in X. Thus, $F^+(V)$ is $(\tau_1, \tau_2)s$-open in X and by Theorem 3, F is upper c-quasi (τ_1, τ_2)-continuous.

Corollary 4. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is lower c-quasi (τ_1, τ_2)-continuous if $F^+(K)$ is $(\tau_1, \tau_2)s$-closed in X for every $\sigma_1\sigma_2$-compact set K of Y.*

Proof. The proof is similar to that of Corollary 3.

For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, by $\mathrm{Cl}F_{\circledast} : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ [30] we denote a multifunction defined as follows: $\mathrm{Cl}F_{\circledast}(x) = \sigma_1\sigma_2\text{-Cl}(F(x))$ for each $x \in X$.

Definition 4. [30] *A subset A of a bitopological space (X, τ_1, τ_2) is said to be:*

(1) $\tau_1\tau_2$-paracompact if every cover of A by $\tau_1\tau_2$-open sets of X is refined by a cover of A which consists of $\tau_1\tau_2$-open sets of X and is $\tau_1\tau_2$-locally finite in X;

(2) $\tau_1\tau_2$-regular if for each $x \in A$ and each $\tau_1\tau_2$-open set U of X containing x, there exists a $\tau_1\tau_2$-open set V of X such that $x \in V \subseteq \tau_1\tau_2$-Cl($V$) $\subseteq U$.

Lemma 3. [30] *If A is a $\tau_1\tau_2$-regular $\tau_1\tau_2$-paracompact set of a bitopological space (X, τ_1, τ_2) and U is a $\tau_1\tau_2$-open neighbourhood of A, then there exists a $\tau_1\tau_2$-open set V of X such that $A \subseteq V \subseteq \tau_1\tau_2$-$Cl(V) \subseteq U$.*

Lemma 4. [30] *If $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a multifunction such that $F(x)$ is $\tau_1\tau_2$-regular and $\tau_1\tau_2$-paracompact for each $x \in X$, then $ClF^+_{\circledast}(V) = F^+(V)$ for each $\sigma_1\sigma_2$-open set V of Y.*

Theorem 5. *Let $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a multifunction such that $F(x)$ is $\sigma_1\sigma_2$-paracompact and $\sigma_1\sigma_2$-regular for each $x \in X$. Then, F is upper c-quasi (τ_1, τ_2)-continuous if and only if $ClF_{\circledast} : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is upper c-quasi (τ_1, τ_2)-continuous.*

Proof. We put $G = \mathrm{Cl}F_{\circledast}$. Suppose that F is upper c-quasi (τ_1, τ_2)-continuous. Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $G(x)$ and having $\sigma_1\sigma_2$-connected complement. By Lemma 4, we have $x \in G^+(V) = F^+(V)$ and by Theorem 1, there exists a $(\tau_1, \tau_2)s$-open set U of X containing x such that $F(U) \subseteq V$. Since $F(z)$ is $\sigma_1\sigma_2$-paracompact and $\sigma_1\sigma_2$-regular for each $z \in U$, by Lemma 3 there exists a $\tau_1\tau_2$-open set W of X such that $F(z) \subseteq W \subseteq \sigma_1\sigma_2$-$\mathrm{Cl}(W) \subseteq V$; hence $G(z) \subseteq \sigma_1\sigma_2$-$\mathrm{Cl}(W) \subseteq V$ for each $z \in U$. Thus, $G(U) \subseteq V$ and hence G is upper c-quasi (τ_1, τ_2)-continuous.

Conversely, suppose that G is upper c-quasi (τ_1, τ_2)-continuous. Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$ and having $\sigma_1\sigma_2$-connected complement. By Lemma 4, we have $x \in F^+(V) = G^+(V)$ and hence $G(x) \subseteq V$. By Theorem 1, there exists a $(\tau_1, \tau_2)s$-open set U of X containing x such that $G(U) \subseteq V$. Thus, $U \subseteq G^+(V) = F^+(V)$ and so $F(U) \subseteq V$. This shows that F is upper c-quasi (τ_1, τ_2)-continuous.

Lemma 5. [30] *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, $ClF^-_{\circledast}(V) = F^-(V)$ for each $\sigma_1\sigma_2$-open set V of Y.*

Theorem 6. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is lower c-quasi (τ_1, τ_2)-continuous if and only if $ClF_{\circledast} : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is lower c-quasi (τ_1, τ_2)-continuous.*

Proof. By using Lemma 5 this is shown similarly as in Theorem 5.

Acknowledgements

This research project was financially supported by Mahasarakham University.

References

[1] C. Berge. *Espaces topologiques fonctions multivoques.* Dunod, Paris, 1959.

[2] C. Boonpok. Almost (g, m)-continuous functions. *International Journal of Mathematical Analysis*, 4(40):1957–1964, 2010.

[3] C. Boonpok. M-continuous functions in biminimal structure spaces. *Far East Journal of Mathematical Sciences*, 43(1):41–58, 2010.

[4] C. Boonpok. On continuous multifunctions in ideal topological spaces. *Lobachevskii Journal of Mathematics*, 40(1):24–35, 2019.

[5] C. Boonpok. On characterizations of $\star$-hyperconnected ideal topological spaces. *Journal of Mathematics*, 2020:9387601, 2020.

[6] C. Boonpok. $(\tau_1, \tau_2)\delta$-semicontinuous multifunctions. *Heliyon*, 6:e05367, 2020.

[7] C. Boonpok. Weak quasi continuity for multifunctions in ideal topological spaces. *Advances in Mathematics: Scientific Journal*, 9(1):339–355, 2020.

[8] C. Boonpok. Upper and lower $\beta(\star)$-continuity. *Heliyon*, 7:e05986, 2021.

[9] C. Boonpok. On some closed sets and low separation axioms via topological ideals. *European Journal of Pure and Applied Mathematics*, 15(3):1023–1046, 2022.

[10] C. Boonpok. $\theta(\star)$-quasi continuity for multifunctions. *WSEAS Transactions on Mathematics*, 21:245–251, 2022.

[11] C. Boonpok. On some spaces via topological ideals. *Open Mathematics*, 21:20230118, 2023.

[12] C. Boonpok. $\theta(\star)$-precontinuity. *Mathematica*, 65(1):31–42, 2023.

[13] C. Boonpok and J. Khampakdee. (Λ, sp)-open sets in topological spaces. *European Journal of Pure and Applied Mathematics*, 15(2):572–588, 2022.

[14] C. Boonpok and J. Khampakdee. On almost $\alpha(\Lambda, sp)$-continuous multifunctions. *European Journal of Pure and Applied Mathematics*, 15(2):626–634, 2022.

[15] C. Boonpok and J. Khampakdee. Slight (Λ, sp)-continuity and Λ_{sp}-extremally disconnectedness. *European Journal of Pure and Applied Mathematics*, 15(3):1180–1188, 2022.

[16] C. Boonpok and J. Khampakdee. Upper and lower weak $s\beta(\star)$-continuity. *European Journal of Pure and Applied Mathematics*, 16(4):2544–2556, 2023.

[17] C. Boonpok and J. Khampakdee. Almost strong $\theta(\Lambda, p)$-continuity for functions. *European Journal of Pure and Applied Mathematics*, 17(1):300–309, 2024.

[18] C. Boonpok and J. Khampakdee. Upper and lower α-$\star$-continuity. *European Journal of Pure and Applied Mathematics*, 17(1):201–211, 2024.

[19] C. Boonpok and C. Klanarong. On weakly (τ_1, τ_2)-continuous functions. *European Journal of Pure and Applied Mathematics*, 17(1):416–425, 2024.

[20] C. Boonpok and P. Pue-on. Continuity for multifunctions in ideal topological spaces. *WSEAS Transactions on Mathematics*, 19:624–631, 2020.

[21] C. Boonpok and P. Pue-on. Upper and lower $s\beta(\star)$-continuous multifunctions. *European Journal of Pure and Applied Mathematics*, 16(3):1634–1646, 2023.

[22] C. Boonpok and P. Pue-on. Upper and lower weakly α-$\star$-continuous multifunctions. *International Journal of Analysis and Applications*, 21:90, 2023.

[23] C. Boonpok and P. Pue-on. Upper and lower weakly (Λ, sp)-continuous multifunctions. *European Journal of Pure and Applied Mathematics*, 16(2):1047–1058, 2023.

[24] C. Boonpok and P. Pue-on. Characterizations of almost (τ_1, τ_2)-continuous functions. *International Journal of Analysis and Applications*, 22:33, 2024.

[25] C. Boonpok and N. Srisarakham. Almost α-$\star$-continuity for multifunctions. *International Journal of Analysis and Applications*, 21:107, 2023.

[26] C. Boonpok and N. Srisarakham. Weak forms of (Λ, b)-open sets and weak (Λ, b)-continuity. *European Journal of Pure and Applied Mathematics*, 16(1):29–43, 2023.

[27] C. Boonpok and N. Srisarakham. (τ_1, τ_2)-continuity for functions. *Asia Pacific Journal of Mathematics*, 11:21, 2024.

[28] C. Boonpok and M. Thongmoon. Weak $\alpha(\Lambda, sp)$-continuity for multifunctions. *European Journal of Pure and Applied Mathematics*, 16(1):465–478, 2023.

[29] C. Boonpok and C. Viriyapong. Upper and lower almost weak (τ_1, τ_2)-continuity. *European Journal of Pure and Applied Mathematics*, 14(4):1212–1225, 2021.

[30] C. Boonpok, C. Viriyapong, and M. Thongmoon. On upper and lower (τ_1, τ_2)-precontinuous multifunctions. *Journal of Mathematics and Computer Science*, 18:282–293, 2018.

[31] M. Chiangpradit, S. Sompong, and C. Boonpok. Weakly quasi (τ_1, τ_2)-continuous functions. *International Journal of Analysis and Applications*, 22:125, 2024.

[32] T. Duangphui, C. Boonpok, and C. Viriyapong. Continuous functions on bigeneralized topological spaces. *International Journal of Mathematical Analysis*, 5(24):1165–1174, 2011.

[33] T. Dungthaisong, C. Boonpok, and C. Viriyapong. Generalized closed sets in bigeneralized topological spaces. *International Journal of Mathematical Analysis*, 5(24):1175–1184, 2011.

[34] K. R. Gentry and H. B. Hoyle III. C-continuous functions. *Yokohama Mathematical Journal*, 18:71–76, 1970.

[35] L. Holá, V. Baláz, and T. Neubrunn. Remarks on C-continuous multifunctions. *Acta Mathematica Universitatis Comenianae*, 50/51:51–59, 1987.

[36] J. Khampakdee and C. Boonpok. Upper and lower $\alpha(\Lambda, sp)$-continuous multifunctions. *WSEAS Transactions on Mathematics*, 21:684–690, 2022.

[37] J. Khampakdee, S. Sompong, and C. Boonpok. c-(τ_1, τ_2)-continuity for multifunctions. *European Journal of Pure and Applied Mathematics*, 17(3):2289–2299, 2024.

[38] C. Klanarong, S. Sompong, and C. Boonpok. Upper and lower almost (τ_1, τ_2)-continuous multifunctions. *European Journal of Pure and Applied Mathematics*, 17(2):1244–1253, 2024.

[39] B. Kong-ied, S. Sompong, and C. Boonpok. Almost quasi (τ_1, τ_2)-continuous functions. *Asia Pacific Journal of Mathematics*, 11:64, 2024.

[40] T. Lipski. Remarks on limits of sequences of C-quasicontinuous multivalued maps. *Radovi Matematički*, 7:17–27, 1991.

[41] P. E. Long and Michael D. Hendrix. Properties of c-continuous functions. *Yokohama Mathematical Journal*, 22:117–123, 1974.

[42] P. E. Long and L. L. Herrington. Properties of c-continuous functions and c^*-continuous functions. *Kyungpook Mathematical Journal*, 15:213–221, 1975.

[43] S. Marcus. Sur les fonctions quasicontinues au sens de S. Kempisty. *Colloquium Mathematicum*, 8:47–53, 1961.

[44] T. Neubrunn. C-continuity and closed graphs. *Časopis pro Pěstování Matematiky*, 110:172–178, 1985.

[45] T. Noiri and V. Popa. Some forms of C-continuity for multifunctions. *European Journal of Pure and Applied Mathematics*, 1(1):82–98, 2008.

[46] Ö. Orhan. Properties of C-continuous functions. *Hacettepe Bulletin Natural Sciences Engineering*, 7-8:77–83, 1978/79.

[47] V. Popa. On some decomposition of quasi-continuity of multifunctions (Romanian). *Studii şi Cercetări de Matematică*, 27:322–328, 1975.

[48] V. Popa and T. Noiri. Characterizations of C-quasicontinuous multifunctions. *Mathematica Balkanica*, 20:265–274, 2006.

[49] P. Pue-on and C. Boonpok. $\theta(\Lambda, p)$-continuity for functions. *International Journal of Mathematics and Computer Science*, 19(2):491–495, 2024.

[50] P. Pue-on, S. Sompong, and C. Boonpok. Almost quasi (τ_1, τ_2)-continuity for multifunctions. *International Journal of Analysis and Applications*, 22:97, 2024.

[51] P. Pue-on, S. Sompong, and C. Boonpok. Upper and lower (τ_1, τ_2)-continuous mulfunctions. *International Journal of Mathematics and Computer Science*, 19(4):1305–1310, 2024.

[52] P. Pue-on, S. Sompong, and C. Boonpok. Weakly quasi (τ_1, τ_2)-continuous multifunctions. *European Journal of Pure and Applied Mathematics*, 17(3):1553–1564, 2024.

[53] N. Srisarakham and C. Boonpok. Almost (Λ, p)-continuous functions. *International Journal of Mathematics and Computer Science*, 18(2):255–259, 2023.

[54] M. Thongmoon and C. Boonpok. Upper and lower almost $\beta(\Lambda, sp)$-continuous multifunctions. *WSEAS Transactions on Mathematics*, 21:844–853, 2022.

[55] M. Thongmoon and C. Boonpok. Strongly $\theta(\Lambda, p)$-continuous functions. *International Journal of Mathematics and Computer Science*, 19(2):475–479, 2024.

[56] M. Thongmoon, S. Sompong, and C. Boonpok. Upper and lower weak (τ_1, τ_2)-continuity. *European Journal of Pure and Applied Mathematics*, 17(3):1705–1716, 2024.

[57] C. Viriyapong and C. Boonpok. $(\tau_1, \tau_2)\alpha$-continuity for multifunctions. *Journal of Mathematics*, 2020:6285763, 2020.

[58] C. Viriyapong and C. Boonpok. (Λ, sp)-continuous functions. *WSEAS Transactions on Mathematics*, 21:380–385, 2022.

[59] C. Viriyapong and C. Boonpok. Weak quasi (Λ, sp)-continuity for multifunctions. *International Journal of Mathematics and Computer Science*, 17(3):1201–1209, 2022.

[60] N. Viriyapong, S. Sompong, and C. Boonpok. (τ_1, τ_2)-extremal disconnectedness in bitopological spaces. *International Journal of Mathematics and Computer Science*, 19(3):855–860, 2024.