Differential Sandwich Theorems of Analytic Functions Defined by Linear Operators

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Abstract. In this paper, we obtain some applications of first order differential subordination and superordination results involving a linear operator and other linear operators for certain normalized analytic functions. Some of our results generalize previously known results.

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1. Introduction

Let \( H(U) \) be the class of analytic functions in the open unit disk \( U = \{z \in \mathbb{C} : |z| < 1 \} \) and let \( H[a,k] \) be the subclass of \( H(U) \) consisting of functions of the form:
\[
f(z) = a + a_k z^k + a_{k+1} z^{k+1} + \ldots \quad (a \in \mathbb{C}). \tag{1}
\]

For simplicity \( H[a] = H[a,1] \). Also, let \( \mathcal{A} \) be the subclass of \( H(U) \) consisting of functions of the form:
\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{2}
\]

If \( f, g \in H(U) \), we say that \( f \) is subordinate to \( g \) or \( f \) is superordinate to \( g \), written \( f(z) \prec g(z) \) if there exists a Schwarz function \( \omega \), which (by definition) is analytic in \( U \) with \( \omega(0) = 0 \) and \( |\omega(z)| < 1 \) for all \( z \in U \), such that \( f(z) = g(\omega(z)), z \in U \). Furthermore, if the function \( g \) is univalent in \( U \), then we have the following equivalence, [cf., e.g., 6, 16, 17]:
\[
f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(U) \subset g(U).\]

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Let $\phi : \mathbb{C}^2 \times U \to \mathbb{C}$ and $h(z)$ be univalent in $U$. If $p(z)$ is analytic in $U$ and satisfies the first order differential subordination:

$$\phi \left( p(z), zp'(z) ; z \right) \prec h(z),$$

then $p(z)$ is a solution of the differential subordination (3). The univalent function $q(z)$ is called a dominant of the solutions of the differential subordination (3) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (3). A univalent dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants of (3) is called the best dominant. If $p(z)$ and $\phi \left( p(z), zp'(z) ; z \right)$ are univalent in $U$ and if $p(z)$ satisfies first order differential superordination:

$$h(z) \prec \phi \left( p(z), zp'(z) ; z \right),$$

then $p(z)$ is a solution of the differential superordination (4). An analytic function $q(z)$ is called a subordinant of the solutions of the differential superordination (4) if $q(z) \prec p(z)$ for all $p(z)$ satisfying (4). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants of (4) is called the best subordinant. Using the results of Miller and Mocanu [17], Bulboaca [5] considered certain classes of first order differential superordinations as well as superordination-preserving integral operators [6]. Ali et al. [1], have used the results of Bulboaca [5] to obtain sufficient conditions for normalized analytic functions to satisfy:

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where $q_1$ and $q_2$ are given univalent functions in $U$ with $q_1(0) = q_2(0) = 1$. Also, Tuneski [25] obtained a sufficient condition for starlikeness of $f$ in terms of the quantity $\frac{f''(z)f(z)}{(f'(z))^2}$.

Recently, Shanmugam et al. [24] obtained sufficient conditions for the normalized analytic function $f$ to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z)$$

and

$$q_1(z) \prec \frac{z^2f'(z)}{f(z)^2} \prec q_2(z).$$

They [24] also obtained results for functions defined by using Carlson-Shaffer operator [7], Ruscheweyh derivative [20] and Sǎlăgean operator [22].

For functions $f$ given by (1) and $g \in \mathcal{A}$ given by $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, the Hadamard product (or convolution) of $f$ and $g$ is defined by

$$(f \ast g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g \ast f)(z).$$  \hspace{1cm} (5)

For functions $f, g \in \mathcal{A}$, we define the linear operator $D^n_\lambda : \mathcal{A} \to \mathcal{A} \ (\lambda \geq 0, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, \ldots\})$ by:

$$D^n_\lambda (f \ast g)(z) = (f \ast g)(z).$$
The linear operator $D^1_\lambda (f \ast g)(z) = D_\lambda (f \ast g)(z) = (1 - \lambda)(f \ast g)(z) + \lambda z ((f \ast g)(z))'$, (6)

and (in general)

$$D^n_\lambda (f \ast g)(z) = D_\lambda (D^{n-1}_\lambda (f \ast g)(z))$$

$$= z + \sum_{k=2}^{\infty} [1 + \lambda (k - 1)]^n a_k b_k z^k \quad (\lambda \geq 0; n \in \mathbb{N}_0).$$ (7)

From (7), we can easily deduce that

$$\lambda z \left( D^n_\lambda (f \ast g)(z) \right)' = D^{n+1}_\lambda (f \ast g)(z) - (1 - \lambda) D^n_\lambda (f \ast g)(z) (\lambda > 0).$$ (8)

The linear operator $D^n_\lambda (f \ast g)(z)$ was introduced by Aouf and Seoudy [3] and we observe that $D^n_\lambda (f \ast g)(z)$ reduces to several interesting many other linear operators considered earlier for different choices of $n$, $\lambda$ and the function $g(z)$:

(i) For $b_k = 1$ (or $g(z) = \frac{z}{1 - g}$), we have $D^n_\lambda (f \ast g)(z) = D^n_\lambda f(z)$, where $D^n_\lambda$ is the general-

ized Sălăgean operator (or Al-Oboudi operator [2]) which yield Sălăgean operator $D^n_\lambda$

for $\lambda = 1$ introduced and studied by Sălăgean [21];

(ii) For $n = 0$ and

$$g(z) = z + \sum_{k=2}^{\infty} \frac{(a_1)_k - 1 \ldots (a_l)_k}{(b_1)_k - 1 \ldots (b_m)_k} z^k$$ (9)

$$\left( a_i \in \mathbb{C}; i = 1, \ldots, l; b_j \in \mathbb{C} \setminus \mathbb{Z}_0^- = \{0, -1, -2, \ldots\}; j = 1, \ldots, m; l \leq m + 1; l, m \in \mathbb{N}_0; z \in U \right),$$

where

$$ (x)_k = \begin{cases} 1 & (k = 0; x \in \mathbb{C}^* \subseteq \mathbb{C} \setminus \{0\}) \\ x(x+1) \ldots (x+k-1) & (k \in \mathbb{N}; x \in \mathbb{C}) \end{cases},$$

we have $D^0_\lambda (f \ast g)(z) = (f \ast g)(z) = H_{i,m} (a_1; b_1) f(z)$, where the operator $H_{i,m} (a_1; b_1)$

is the Dziok-Srivastava operator introduced and studied by Dziok and Srivastava [10]

([see also 11, 12]). The operator $H_{i,m} (a_1; b_1)$, contains in turn many interesting opera-

tors such as, Hohlov linear operator (see [13]), the Carlson-Shaffer linear operator (see

[7, 21]), the Ruscheweyh derivative operator (see [20]), the Bernardi-Libera-Livingston

operator (see [4, 14, 15]) and Owa-Srivastava fractional derivative operator (see [19]);

(iii) For $n = 0$ and

$$g(z) = z + \sum_{k=2}^{\infty} \left[ \frac{1 + l + \lambda (k - 1)}{1 + l} \right]^s z^k \quad (\lambda \geq 0; l, s \in \mathbb{N}_0),$$ (10)

we see that $D^0_\lambda (f \ast g)(z) = (f \ast g)(z) = I(s, \lambda, l)f(z)$, where $I(s, \lambda, l)$ is the general-

ized multiplier transformations which was introduced and studied by Cătășă et al. [8]. The

operator $I(s, \lambda, l)$, contains as special cases, the multiplier transformation $I(s, l)$ (see

[9]) for $\lambda = 1$, the generalized Sălăgean operator $D^0_\lambda$ introduced and studied by Al-

Oboudi [2] which in turn contains as special case the Sălăgean operator $D^n_\lambda$ (see [21]);
(iv) For \( g(z) \) of the form (9), the operator \( D_\lambda^n(f \ast g)(z) = D_\lambda^n(a_1, b_1)f(z) \), introduced and studied by Selvaraj and Karthikeyan [23].

In this paper, we will derive several subordination results, superordination results and sandwich results involving the operator \( D_\lambda^n(f \ast g)(z) \) and some of its special operators by some choices of \( n, \lambda \) and the function \( g(z) \).

2. Preliminaries

In order to prove our subordinations and superordinations, we need the following definition and lemmas.

**Definition 1.** [17] Denote by \( Q \), the set of all functions \( f \) that are analytic and injective on \( U \setminus \mathbb{E}(f) \), where

\[
\mathbb{E}(f) = \left\{ \zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty \right\},
\]

and are such that \( f'(\zeta) \neq 0 \) for \( \zeta \in \partial U \setminus \mathbb{E}(f) \).

**Lemma 1.** [17] Let \( q(z) \) be univalent in the unit disk \( U \) and \( \theta \) and \( \varphi \) be analytic in a domain \( D \) containing \( q(U) \) with \( \varphi(w) \neq 0 \) when \( w \in q(U) \). Set

\[
\psi(z) = zq'(z)\varphi(q(z)) \quad \text{and} \quad h(z) = \theta(q(z)) + \psi(z).
\]

(11)

Suppose that

(i) \( \psi(z) \) is starlike univalent in \( U \),

(ii) \( \Re \left\{ \frac{zh'(z)}{\psi(z)} \right\} > 0 \) for \( z \in U \).

If \( p(z) \) is analytic with \( p(0) = q(0) \), \( p(U) \subset D \) and

\[
\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)),
\]

(12)

then \( p(z) \prec q(z) \) and \( q(z) \) is the best dominant.

Taking \( \theta(w) = \alpha w \) and \( \varphi(w) = \gamma \) in Lemma 1, Shanmugam et al. [24] obtained the following lemma.

**Lemma 2.** [24] Let \( q(z) \) be univalent in \( U \) with \( q(0) = 1 \). Let \( \alpha \in \mathbb{C}; \gamma \in \mathbb{C}^* \), further assume that

\[
\Re \left\{ 1 + \frac{zq''(z)}{q'(z)} \right\} > \max \left\{ 0, -\Re \left( \frac{\alpha}{\gamma} \right) \right\}.
\]

(13)

If \( p(z) \) is analytic in \( U \), and

\[
\alpha p(z) + \gamma zp'(z) \prec \alpha q(z) + \gamma zq'(z),
\]

then \( p(z) \prec q(z) \) and \( q(z) \) is the best dominant.
Lemma 4. [5] Let \( q(z) \) be convex univalent in \( U \) and \( \vartheta \) and \( \phi \) be analytic in a domain \( D \) containing \( q(U) \). Suppose that

\[
\begin{align*}
(1) & \quad \Re \left\{ \frac{\vartheta'(q(z))}{\varphi(q(z))} \right\} > 0 \text{ for } z \in U, \\
(2) & \quad \Psi(z) = zq'(z) \phi(q(z)) \text{ is starlike univalent in } U.
\end{align*}
\]

If \( p(z) \in H[q(0), 1] \cap Q \), with \( p(U) \subseteq D \), and \( \vartheta(p(z)) +zp'(z)\phi(p(z)) \) is univalent in \( U \) and

\[
\vartheta(q(z)) + zq'(z)\phi(q(z)) \prec \vartheta(p(z)) +zp'(z)\phi(p(z)),
\]

then \( q(z) \prec p(z) \) and \( q(z) \) is the best subordinant.

Taking \( \vartheta(w) = \alpha w \) and \( \phi(w) = \gamma \) in Lemma 3, Shanmugam et al. [24] obtained the following lemma.

Lemma 4. [24] Let \( q(z) \) be convex univalent in \( U \), \( q(0) = 1 \). Let \( \alpha \in \mathbb{C}; \gamma \in \mathbb{C}^+ \) and \( \Re \left( \frac{\alpha}{\gamma} \right) > 0 \). If \( p(z) \in H[q(0), 1] \cap Q \), \( \alpha p(z) + \gamma z p'(z) \) is univalent in \( U \) and

\[
\alpha q(z) + \gamma z q'(z) \prec \alpha p(z) + \gamma z p'(z),
\]

then \( q(z) \prec p(z) \) and \( q(z) \) is the best subordinant.

3. Sandwich Results

Unless otherwise mentioned, we assume throughout this paper that \( \lambda > 0 \) and \( n \in \mathbb{N}_0 \).

Theorem 1. Let \( q(z) \) be univalent in \( U \) with \( q(0) = 1 \), and \( \gamma \in \mathbb{C}^+ \). Further, assume that

\[
\Re \left\{ 1 + \frac{zq''(z)}{q(z)} \right\} > \max \left\{ 0, -\Re \left( \frac{1}{\gamma} \right) \right\}
\]

(15)

If \( f, g \in \mathcal{A} \) satisfy the following subordination condition:

\[
\left( 1 + \frac{\gamma}{\lambda} \right) \frac{zD_n^{n+1}(f \ast g)(z)}{[D_n^1(f \ast g)(z)]^2} + \frac{\gamma}{\lambda} \left\{ \frac{zD_n^{n+2}(f \ast g)(z)}{[D_n^1(f \ast g)(z)]^2} - 2 \frac{z [D_n^{n+1}(f \ast g)(z)]^2}{[D_n^1(f \ast g)(z)]^3} \right\} \prec q(z) + \gamma z q'(z),
\]

(16)

then

\[
\frac{zD_n^{n+1}(f \ast g)(z)}{[D_n^1(f \ast g)(z)]^2} \prec q(z)
\]

and \( q(z) \) is the best dominant.
Proof. Define a function \( p(z) \) by

\[
p(z) = \frac{zD_{\lambda}^{n+1}(f \ast g)(z)}{[D_{\lambda}^{n}(f \ast g)(z)]^2} \quad (z \in U).
\]

Then the function \( p(z) \) is analytic in \( U \) and \( p(0) = 1 \). Therefore, differentiating (17) logarithmically with respect to \( z \) and using the identity (8) in the resulting equation, we have

\[
\left( 1 + \frac{\gamma}{\lambda} \right) \frac{zD_{\lambda}^{n+1}(f \ast g)(z)}{[D_{\lambda}^{n}(f \ast g)(z)]^2} + \frac{\gamma}{\lambda} \left\{ \frac{zD_{\lambda}^{n+2}(f \ast g)(z)}{[D_{\lambda}^{n}(f \ast g)(z)]^2} - 2 \frac{z[D_{\lambda}^{n+1}(f \ast g)(z)]^2}{[D_{\lambda}^{n}(f \ast g)(z)]^3} \right\} = p(z) + \gamma z p'(z),
\]

that is,

\[
p(z) + \gamma z p'(z) \prec q(z) + \gamma z q'(z).
\]

Therefore, Theorem 1 now follows by applying Lemma 2.

Putting \( q(z) = \frac{1 + Az}{1 + Bz} \) \((-1 \leq B < A \leq 1) \) in Theorem 1, we obtain the following corollary.

**Corollary 1.** Let \( \gamma \in \mathbb{C}^* \) and

\[
\Re \left\{ \frac{1 - Bz}{1 + Bz} \right\} > \max \left\{ 0, -\Re \left( \frac{1}{\gamma} \right) \right\}.
\]

If \( f, g \in \mathcal{A} \) satisfy the following subordination condition:

\[
\left( 1 + \frac{\gamma}{\lambda} \right) \frac{zD_{\lambda}^{n+1}(f \ast g)(z)}{[D_{\lambda}^{n}(f \ast g)(z)]^2} + \frac{\gamma}{\lambda} \left\{ \frac{zD_{\lambda}^{n+2}(f \ast g)(z)}{[D_{\lambda}^{n}(f \ast g)(z)]^2} - 2 \frac{z[D_{\lambda}^{n+1}(f \ast g)(z)]^2}{[D_{\lambda}^{n}(f \ast g)(z)]^3} \right\} \prec \frac{1 + Az}{1 + Bz} + \gamma \frac{(A - B)z}{(1 + Bz)^2},
\]

then

\[
\frac{zD_{\lambda}^{n+1}(f \ast g)(z)}{[D_{\lambda}^{n}(f \ast g)(z)]^2} \prec \frac{1 + Az}{1 + Bz}
\]

and the function \( \frac{1 + Az}{1 + Bz} \) is the best dominant.

**Remark 1.** Taking \( g(z) = \frac{z}{1 - z} \) in Theorem 1, we obtain the subordination result of Nechita [18, Theorem 14].

**Remark 2.** Taking \( \lambda = 1 \) and \( g(z) = \frac{z}{1 - z} \) in Theorem 1, we obtain the subordination result for Sălăgean operator which was obtained by Shanmugam et al. [24, Theorem 5.4] and also obtained by Nechita [18, Corollary 16].
Corollary 4. Let \( q \in A \) holds. If \( f \in \mathcal{A} \) is the best dominant.

Corollary 2. Let \( q(z) \) be univalent in \( U \) with \( q(0) = 1 \), and \( \gamma \in \mathbb{C}^* \). Further assume that (15) holds. If \( f \in \mathcal{A} \) satisfies the following subordination condition:

\[
\frac{z^2 \left( H_{i,m} (a_1; b_1) f(z) \right)'}{\left[ H_{i,m} (a_1; b_1) f(z) \right]^2} - \frac{z}{\left[ H_{i,m} (a_1; b_1) f(z) \right]^2} \prec q(z) + \gamma z q'(z),
\]

then

\[
\frac{z^2 \left( H_{i,m} (a_1; b_1) f(z) \right)'}{\left[ H_{i,m} (a_1; b_1) f(z) \right]^2} \prec q(z)
\]

and \( q(z) \) is the best dominant.

Taking \( g(z) \) of the form (9) in Theorem 1, we obtain the following subordination result for the Dziok-Srivastava operator.

Corollary 3. Let \( q(z) \) be univalent in \( U \) with \( q(0) = 1 \), and \( \gamma \in \mathbb{C}^* \). Further assume that [15] holds. If \( f \in \mathcal{A} \) satisfies the following subordination condition:

\[
\left( 1 + \frac{\gamma}{\lambda} \right) \frac{z D_{\lambda}^{n+1}(a_1; b_1) f(z)}{D_{\lambda}^n(a_1; b_1) f(z)} + \frac{\gamma}{\lambda} \left\{ \frac{z D_{\lambda}^{n+2}(a_1; b_1) f(z)}{D_{\lambda}^n(a_1; b_1) f(z)} - \frac{z}{D_{\lambda}^n(a_1; b_1) f(z)} \right\} \prec q(z) + \gamma z q'(z),
\]

then

\[
\frac{z D_{\lambda}^{n+1}(a_1; b_1) f(z)}{D_{\lambda}^n(a_1; b_1) f(z)} \prec q(z)
\]

and \( q(z) \) is the best dominant.

Taking \( n = 0, \lambda = 1 \) and

\[
g(z) = z + \sum_{k=2}^{\infty} \left( \frac{l+k}{1+} \right)^k \lambda^k (l, s \in \mathbb{N}_0),
\]

in Theorem 1, we obtain the following subordination result for the multiplier transformations \( I(s, l) \).

Corollary 4. Let \( q(z) \) be univalent in \( U \) with \( q(0) = 1 \), and \( \gamma \in \mathbb{C}^* \). Further assume that (15) holds. If \( f \in \mathcal{A} \) satisfies the following subordination condition:

\[
\frac{z^2 (I(s,l) f(z))'}{[I(s,l) f(z)]^2} - \frac{z}{[I(s,l) f(z)]^2} \prec q(z) + \gamma z q'(z),
\]
then
\[ \frac{z^2 (I(s, l)f(z))'}{[I(s, l)f(z)]^2} < q(z) \]
and \( q(z) \) is the best dominant.

**Remark 3.** Taking \( n = 0, \lambda = 1 \) and \( g(z) = \frac{z}{1 - z} \) in Theorem 1, we obtain the subordination result of Shanmugam et al. [24, Theorem 3.4] and also obtained by Nechita [18, Corollary 17].

Now, by appealing to Lemma 4 it can be easily prove the following theorem.

**Theorem 2.** Let \( q(z) \) be convex univalent in \( U \) with \( q(0) = 1 \). Let \( \gamma \in \mathbb{C} \) with \( \Re(\gamma) > 0 \). If \( f, g \in \mathcal{A}, \frac{zD_{\lambda}^{n+1}(f * g)(z)}{[D_{\lambda}^n(f * g)(z)]^2} \in H[1, 1] \cap Q, \)

\[
\left(1 + \frac{\gamma}{\lambda}\right) \frac{zD_{\lambda}^{n+1}(f * g)(z)}{[D_{\lambda}^n(f * g)(z)]^2} + \frac{\gamma}{\lambda} \left\{ \frac{zD_{\lambda}^{n+2}(f * g)(z)}{[D_{\lambda}^n(f * g)(z)]^2} - 2z \left[ \frac{D_{\lambda}^{n+1}(f * g)(z)}{[D_{\lambda}^n(f * g)(z)]^3} \right] \right\}
\]

is univalent in \( U \), and the following superordination condition

\[ q(z) + \gamma q'(z) \prec \left(1 + \frac{\gamma}{\lambda}\right) \frac{zD_{\lambda}^{n+1}(f * g)(z)}{[D_{\lambda}^n(f * g)(z)]^2} + \frac{\gamma}{\lambda} \left\{ \frac{zD_{\lambda}^{n+2}(f * g)(z)}{[D_{\lambda}^n(f * g)(z)]^2} - 2z \left[ \frac{D_{\lambda}^{n+1}(f * g)(z)}{[D_{\lambda}^n(f * g)(z)]^3} \right] \right\}
\]
holds, then

\[ q(z) \prec \frac{zD_{\lambda}^{n+1}(f * g)(z)}{[D_{\lambda}^n(f * g)(z)]^2} \]
and \( q(z) \) is the best subordinant.

Taking \( q(z) = \frac{1 + Az}{1 + Bz} (-1 \leq B < A \leq 1) \) in Theorem 2, we have the following corollary.

**Corollary 5.** Let \( \gamma \in \mathbb{C} \) with \( \Re(\gamma) > 0 \). If \( f, g \in \mathcal{A}, \frac{zD_{\lambda}^{n+1}(f * g)(z)}{[D_{\lambda}^n(f * g)(z)]^2} \in H[1, 1] \cap Q, \)

\[
\left(1 + \frac{\gamma}{\lambda}\right) \frac{zD_{\lambda}^{n+1}(f * g)(z)}{[D_{\lambda}^n(f * g)(z)]^2} + \frac{\gamma}{\lambda} \left\{ \frac{zD_{\lambda}^{n+2}(f * g)(z)}{[D_{\lambda}^n(f * g)(z)]^2} - 2z \left[ \frac{D_{\lambda}^{n+1}(f * g)(z)}{[D_{\lambda}^n(f * g)(z)]^3} \right] \right\}
\]
is univalent in \( U \), and the following superordination condition

\[
\frac{1 + Az}{1 + Bz} + \frac{(A - B)z}{(1 + Bz)^2} \prec \left(1 + \frac{\gamma}{\lambda}\right) \frac{zD_{\lambda}^{n+1}(f * g)(z)}{[D_{\lambda}^n(f * g)(z)]^2} + \frac{\gamma}{\lambda} \left\{ \frac{zD_{\lambda}^{n+2}(f * g)(z)}{[D_{\lambda}^n(f * g)(z)]^2} - 2z \left[ \frac{D_{\lambda}^{n+1}(f * g)(z)}{[D_{\lambda}^n(f * g)(z)]^3} \right] \right\}
\]
Corollary 7. Let \( q(z) \) be convex univalent in \( U \) with \( q(0) = 1 \). Let \( \gamma \in \mathbb{C} \) with \( \Re (\gamma) > 0 \). If \( f, g \in \mathcal{A} \),

\[
\frac{1 + Az}{1 + Bz} \frac{zD^{n+1}_\lambda(f * g)(z)}{D^{n}_\lambda(f * g)(z)}
\]

and \( q(z) \) is the best subordinant.

Remark 4. Taking \( g(z) = \frac{z}{1 - z} \) in Theorem 2, we obtain the superordination result of Nechita (18, Theorem 19).

Remark 5. Taking \( \lambda = 1 \) and \( g(z) = \frac{z}{1 - z} \) in Theorem 2, we obtain the following superordination result for Sălăgean operator which is obtained Shanmugam et al. [24, Theorem 5.5].

Taking \( n = 0, \lambda = 1 \) and \( g(z) \) of the form (9) in Theorem 2, we obtain the following superordination result for Dziok-Srivastava operator.

Corollary 6. Let \( q(z) \) be convex univalent in \( U \) with \( q(0) = 1 \). Let \( \gamma \in \mathbb{C} \) with \( \Re (\gamma) > 0 \). If \( f \in \mathcal{A} \),

\[
\frac{z^2 \left( H_{l,m} \left( a_1; b_1 \right) f(z) \right)'}{\left[ H_{l,m} \left( a_1; b_1 \right) f(z) \right]^2} \in H \left[ 1, 1 \right] \cap Q,
\]

is univalent in \( U \), and the following superordination condition

\[
q(z) + \gamma z q'(z) < \frac{z^2 \left( H_{l,m} \left( a_1; b_1 \right) f(z) \right)'}{\left[ H_{l,m} \left( a_1; b_1 \right) f(z) \right]^2} - \gamma z^2 \left( \frac{z}{H_{l,m} \left( a_1; b_1 \right) f(z)} \right)''
\]

holds, then

\[
q(z) < \frac{z^2 \left( H_{l,m} \left( a_1; b_1 \right) f(z) \right)'}{\left[ H_{l,m} \left( a_1; b_1 \right) f(z) \right]^2}
\]

and \( q(z) \) is the best subordinant.

Taking \( g(z) \) of the form (9) in Theorem 2, we obtain the following superordination result for the operator \( D^n_\lambda(a_1; b_1) \).

Corollary 7. Let \( q(z) \) be convex univalent in \( U \) with \( q(0) = 1 \). Let \( \gamma \in \mathbb{C} \) with \( \Re (\gamma) > 0 \). If \( f, g \in \mathcal{A} \),

\[
\frac{zD^{n+1}_\lambda(a_1; b_1)f(z)}{D^n_\lambda(a_1; b_1)f(z)} \in H \left[ 1, 1 \right] \cap Q,
\]

\[
\left( 1 + \frac{\gamma}{\lambda} \right) \frac{zD^{n+1}_\lambda(a_1; b_1)f(z)}{D^n_\lambda(a_1; b_1)f(z)} + \frac{\gamma}{\lambda} \left( \frac{zD^{n+2}_\lambda(a_1; b_1)f(z)}{D^n_\lambda(a_1; b_1)f(z)} - 2 \frac{D^{n+1}_\lambda(a_1; b_1)f(z)}{D^n_\lambda(a_1; b_1)f(z)} \right)
\]

holds, then

\[
q(z) < \frac{zD^{n+1}_\lambda(a_1; b_1)f(z)}{D^n_\lambda(a_1; b_1)f(z)}
\]
is univalent in $U$, and the following superordination condition

$$q(z) + \gamma q'(z) \prec \left(1 + \frac{\gamma}{\lambda}\right) \frac{z D_{\lambda}^{n+1}(a_1; b_1)f(z)}{[D_{\lambda}^n(a_1; b_1)f(z)]^2} + \frac{\gamma}{\lambda} \left(\frac{z D_{\lambda}^{n+2}(a_1; b_1)f(z)}{[D_{\lambda}^n(a_1; b_1)f(z)]^2} - 2 \frac{z D_{\lambda}^{n+1}(a_1; b_1)f(z)}{[D_{\lambda}^n(a_1; b_1)f(z)]^3}\right)$$

holds, then

$$q(z) \prec \frac{z D_{\lambda}^{n+1}(a_1; b_1)f(z)}{[D_{\lambda}^n(a_1; b_1)f(z)]^2}$$

and $q(z)$ is the best subordinant.

Taking $n = 0, \lambda = 1$ and $g(z)$ of the form (18) in Theorem 2, we obtain the following supordination result for the multiplier transformations $I(s, l)$.

**Corollary 8.** Let $q(z)$ be convex univalent in $U$ with $q(0) = 1$. Let $\gamma \in \mathbb{C}$ with $\Re(\gamma) > 0$. If $f \in \mathcal{A}$, $\frac{z^2 (I(s, l)f(z))'}{[I(s, l)f(z)]^2} \in H[1, 1] \cap Q$,

$$\frac{z^2 (I(s, l)f(z))'}{[I(s, l)f(z)]^2} - \gamma z^2 \left(\frac{z}{I(s, l)f(z)}\right)''$$

is univalent in $U$, and the following superordination condition

$$q(z) + \gamma q'(z) \prec \frac{z^2 (I(s, l)f(z))'}{[I(s, l)f(z)]^2} - \gamma z^2 \left(\frac{z}{I(s, l)f(z)}\right)''$$

holds, then

$$q(z) \prec \frac{z^2 (I(s, l)f(z))'}{[I(s, l)f(z)]^2}$$

and $q(z)$ is the best subordinant.

**Remark 6.** Taking $n = 0, \lambda = 1$ and $g(z) = \frac{z}{1 - z}$ in Theorem 2, we obtain the superordination result of Shanmugam et al. [24, Theorem 3.5].

Combining Theorem 1 and Theorem 2, we get the following sandwich theorem for the linear operator $D_{\lambda}^n(f \ast g)$.

**Theorem 3.** Let $q_1(z)$ be convex univalent in $U$ with $q_1(0) = 1, \gamma \in \mathbb{C}$ with $\Re(\gamma) > 0, q_2(z)$ be univalent in $U$ with $q_2(0) = 1$, and satisfies (15). If $f, g \in \mathcal{A}$, $\frac{z D_{\lambda}^{n+1}(f \ast g)(z)}{[D_{\lambda}^n(f \ast g)(z)]^2} \in H[1, 1] \cap Q$,

$$\left(1 + \frac{\gamma}{\lambda}\right) \frac{z D_{\lambda}^{n+1}(f \ast g)(z)}{[D_{\lambda}^n(f \ast g)(z)]^2} + \frac{\gamma}{\lambda} \left(\frac{z D_{\lambda}^{n+2}(f \ast g)(z)}{[D_{\lambda}^n(f \ast g)(z)]^2} - 2 \frac{z D_{\lambda}^{n+1}(f \ast g)(z)}{[D_{\lambda}^n(f \ast g)(z)]^3}\right)$$


is univalent in $U$, and

$$q_1(z) + \gamma z q_1'(z)$$

$$< \left(1 + \frac{\gamma}{\lambda}\right) \frac{z D^{n+1}_\lambda(f \ast g)(z)}{[D^n_\lambda(f \ast g)(z)]^2} + \frac{\gamma}{\lambda} \left\{ \frac{z D^{n+2}_\lambda(f \ast g)(z)}{[D^n_\lambda(f \ast g)(z)]^2} - 2 \frac{z [D^{n+1}_\lambda(f \ast g)(z)]^2}{[D^n_\lambda(f \ast g)(z)]^3} \right\}$$

$$< q_2(z) + \gamma z q_2'(z)$$

holds, then

$$q_1(z) < \frac{z D^{n+1}_\lambda(f \ast g)(z)}{[D^n_\lambda(f \ast g)(z)]^2} < q_2(z)$$

and $q_1(z)$ and $q_2(z)$ are, respectively, the best subordinant and the best dominant.

Taking $q_i(z) = \frac{1 + A_i z}{1 + B_i z}$ ($i = 1, 2; -1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$) in Theorem 3, we have the following corollary.

**Corollary 9.** Let $\gamma \in \mathbb{C}$ with $\Re(\gamma) > 0$. If $f, g \in \mathcal{A}$, $\frac{z D^{n+1}_\lambda(f \ast g)(z)}{[D^n_\lambda(f \ast g)(z)]^2} \in H[1, 1] \cap Q$,

$$\left(1 + \frac{\gamma}{\lambda}\right) \frac{z D^{n+1}_\lambda(f \ast g)(z)}{[D^n_\lambda(f \ast g)(z)]^2} + \frac{\gamma}{\lambda} \left\{ \frac{z D^{n+2}_\lambda(f \ast g)(z)}{[D^n_\lambda(f \ast g)(z)]^2} - 2 \frac{z [D^{n+1}_\lambda(f \ast g)(z)]^2}{[D^n_\lambda(f \ast g)(z)]^3} \right\}$$

is univalent in $U$, and

$$\frac{1 + A_1 z}{1 + B_1 z} + \gamma \frac{(A_1 - B_1) z}{(1 + B_1 z)^2}$$

$$< \left(1 + \frac{\gamma}{\lambda}\right) \frac{z D^{n+1}_\lambda(f \ast g)(z)}{[D^n_\lambda(f \ast g)(z)]^2} + \frac{\gamma}{\lambda} \left\{ \frac{z D^{n+2}_\lambda(f \ast g)(z)}{[D^n_\lambda(f \ast g)(z)]^2} - 2 \frac{z [D^{n+1}_\lambda(f \ast g)(z)]^2}{[D^n_\lambda(f \ast g)(z)]^3} \right\}$$

$$< \frac{1 + A_2 z}{1 + B_2 z} + \gamma \frac{(A_2 - B_2) z}{(1 + B_2 z)^2}$$

holds, then

$$\frac{1 + A_1 z}{1 + B_1 z} < \frac{z D^{n+1}_\lambda(f \ast g)(z)}{[D^n_\lambda(f \ast g)(z)]^2} < \frac{1 + A_2 z}{1 + B_2 z}$$

and $\frac{1 + A_1 z}{1 + B_1 z}$ and $\frac{1 + A_2 z}{1 + B_2 z}$ are, respectively, the best subordinant and the best dominant.
Remark 7. Taking \( g(z) = \frac{z}{1 - z} \) in Theorem 3, we obtain sandwich result of Nechita [18, Theorem 19].

Remark 8. Taking \( \lambda = 1 \) and \( g(z) = \frac{z}{1 - z} \) in Theorem 3, we obtain sandwich result of Shanmugam et al. [24, Theorem 5.6].

Remark 9. Combining (i) Corollary 2 and Corollary 6; (ii) Corollary 3 and Corollary 7; (iii) Corollary 4 and Corollary 8, we obtain similar sandwich theorems for the corresponding linear operators.

Remark 10. Taking \( n = 0, \lambda = 1 \) and \( g(z) = \frac{z}{1 - z} \) in Theorem 3, we obtain the sandwich result of Shanmugam et al. [24, Corollary 3.6].

References


