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A Novel Analytical Approximate Solution for Strongly Nonlinear Third-Order Jerk Equations Using a Modified Iteration Method

Gamal M. Ismail[1,2,*], Mohammed I. Yamani[1]

[1] *Department of Mathematics, Faculty of Science, Islamic University of Madinah, Madinah, 42351, Saudi Arabia*
[2] *Department of Mathematics, Faculty of Science, Sohag University, Sohag 82524, Egypt*

Abstract. Nonlinear jerk equations, characterized by a third-order time derivative, play a crucial role in modeling various physical phenomena across disciplines like mechanics, circuits, and biology. Accurately solving these equations is essential for understanding and predicting the behavior of such systems. However, obtaining analytical solutions for nonlinear jerk equations can be challenging, necessitating the development of robust and accurate approximation methods. This work explores, for the first time, the application of the modified iteration approach to solve third-order jerk equations. By comparing the obtained approximate solutions with both exact and existing analytical solutions for established engineering problems, we demonstrate the superior accuracy and rapid convergence of the proposed method. The significantly reduced error percentages highlight the effectiveness of the modified iteration approach in providing precise solutions for nonlinear jerk equations, paving the way for its application in a wide range of oscillation problems within nonlinear sciences and engineering.

2020 Mathematics Subject Classifications: 34A34, 34C15, 34C25, 35L65, 37N30

Key Words and Phrases: Modified iteration method, Third-order jerk equations, Analytical solutions, Nonlinear oscillator, Exact solution

1. Introduction

Differential equations are widely employed to describe the evolution of complex systems in many fields, including physics and engineering. Nonlinear differential equations play a crucial role in modeling various scientific phenomena and have many applications in various branches of science. Researchers seek effective solutions to such problems using analytical or numerical approaches.

Recently, there has been an increasing interest in analytical solutions for nonlinear differential equations. The key issue in studying nonlinear differential equations is finding

*Corresponding author.
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Email addresses: gamalm2010@yahoo.com (G. M. Ismail), shingl6600@gmail.com (M. I. Yamani)

the exact solutions. On the other hand, computing the exact results can be challenging, especially in equations with a high degree of nonlinearity, mainly using traditional analytical techniques. To overcome this problem, many novel approaches are presented to deal with this shortage. For example, harmonic balance method [1–3], variational iteration method [4, 5], Hamiltonian approach [6, 7], modified algebraic method [8, 9], global residue harmonic balance method [10, 11], energy balance method [12–14]. linearizing method [15], multiple scales method [16], non-perturbative approach [17, 18], Adomian decomposition method [19], optimal variational iteration method [20] and Galerkin method [21]. These analytical approaches have been extensively utilized to examine the frequency and periodic solutions of nonlinear oscillators.

Nonlinear third-order Jerk equations are useful for analyzing structures with rotating and translating movements, such as machine tools or robots [22]. Nonlinear third-order jerk equations are differential equations that describe the evolution of a system's jerk, which is the rate of change of acceleration. In simpler terms, it represents the "snap" or "jolt" felt when acceleration changes abruptly. Jerk equations are powerful tools for modeling complex dynamical systems exhibiting rich and often unpredictable behavior. Understanding their properties and developing effective methods for their analysis is crucial for advancing our knowledge in various scientific and engineering disciplines.

Nonlinear third-order Jerk equations may explain a variety of physical issues, including third-order mechanical oscillators [23]. Nowadays, due to the need of knowing the analytical solutions of the nonlinear Jerk equations. Several diverse techniques have been proposed to find the analytical solutions of like these problems such as block method [24], harmonic balance method [25, 26], homotopy perturbation method [27], parameter perturbation method [28], Mickens iteration method [29], Linstedt-Poincare methods [30], residue harmonic balance method [31], multiple scales Lindstedt-Poincare method [32], differential transform method [33], modified harmonic balance method [34], variational iteration method [35] and homotopy asymptotic method [36] to solve the present problems. Recently, Ismail and Abu-Zinadah [37] used the global error minimization method for solving the current problems.

In this study, we apply the modified iteration technique, a powerful analytical method with high accuracy and efficiency, to obtain higher-order analytic approximations for nonlinear Jerk equations. The primary advantage of this method lies in its ability to provide both simplicity and accuracy when solving higher-order differential equations. The numerical solution is obtained using the fourth-order Runge-Kutta method. A comparison between the analytical and numerical solutions, presented through tables and corresponding figures, emphasizes the accuracy of the modified iteration technique.

2. The iteration procedure

Consider a non-linear equation

$$\dddot{x} + f(x, \dot{x}, \ddot{x}) = 0, \quad x(0) = A, \quad \dot{x}(0) = 0. \tag{1}$$

Rewrite Eq. (1) to be in the form:

$$\ddot{x} + \omega^2 x = \omega^2 x - f(x, \dot{x}, \ddot{x}) \approxeq F(x_k, \dot{x}_k, \ddot{x}_k), \tag{2}$$

where $\dot{x}$ and $\ddot{x}$ represent the first and second derivatives with respect to time, respectively, and ω is an unknown constant.

According to Ref [38], Eq. (2) can be rewritten as

$$\ddot{x}_{k+1} + \omega_k^2 x_{k+1} = F(x_k, \dot{x}_k, \ddot{x}_k), \quad k = 0, 1, 2, ..., \tag{3}$$

and the imputes of starting functions are

$$x_0(t) = A \cos \omega_0 t. \tag{4}$$

It is further required that for each k, the solution to Eq. (3), is to satisfy initial conditions

$$x_{k+1}(0) = A, \quad \dot{x}_{k+1}(0) = 0. \tag{5}$$

3. Applications

In this section, we illustrate the fundamental concept of the modified iteration approach by considering the following non-linear differential equation:

$$\dddot{x} + f(x, \dot{x}, \ddot{x}), \quad x(0) = 0, \quad \dot{x}(0) = A, \quad \ddot{x}(0) = 0. \tag{6}$$

Following [3], the general non-linear third order Jerk equation has the form

$$\dddot{x} = -\gamma\, \dot{x} - \alpha \dot{x}^3 - \beta x^2 \dot{x} + \delta x \dot{x} \ddot{x} - \varepsilon \dot{x}\, \ddot{x}^2, \tag{7}$$

with

$$x(0) = 0, \quad \dot{x}(0) = A, \quad \ddot{x}(0) = 0. \tag{8}$$

where the parameters α, β, γ, δ and ε are constants.

3.1. Jerk function containing time's velocity times acceleration and velocity

In this case at $\gamma = \delta = 1$, $\alpha = \beta = \varepsilon = 0$, in Eq. (7), the non-linear equation is in the following [1]:

$$\dddot{x} + \dot{x} - x\dot{x}\ddot{x} = 0, \quad x(0) = 0, \quad \dot{x}(0) = A, \quad \ddot{x}(0) = 0. \tag{9}$$

Eq. (9) can be rewritten in the form

$$\dot{x} = y, \quad \dot{y} = \ddot{x}, \tag{10}$$

then Eq. (10) becomes

$$\ddot{y} + y - \bar{y}y\dot{y} = 0, \quad y(0) = A, \quad \dot{y}(0) = 0. \tag{11}$$

where $\bar{y}$ is the integration of x

$$\ddot{y} + \omega^2 y = \omega^2 y - y + \bar{y}y\dot{y}. \tag{12}$$

The iteration technique according to Eq. (3) is

$$\ddot{y}_{k+1} + \omega_k^2 y_{k+1} = \omega_k^2 y_k - y_k + \bar{y}_k y_k \dot{y}_k. \tag{13}$$

3.1.1. First iteration solution

For the first iteration, at $k = 0$, we get

$$\ddot{y}_1 + \omega_0^2 y_1 = \omega_0^2 y_0 - y_0 + \bar{y}_0 y_0 \dot{y}_0. \tag{14}$$

According to Eq. (4), we have

$$y_0(t) = A\cos\omega_0 t. \tag{15}$$

Inserting Eq. (15) into Eq. (14), to obtain

$$\ddot{y}_1 + \omega_0^2 y_1 = \frac{1}{4}\left(-4A - A^3 + 4A\omega_0^2\right)\cos(\omega_0 t) - \frac{A^3}{4}\cos(3\omega_0 t). \tag{16}$$

A secular term is a term in the solution that grows linearly or polynomially with time. Secular terms often arise when using naive perturbation methods on systems with natural frequencies that are close to being resonant with the perturbation frequency. Secular terms are undesirable for several reasons, for example, breakdown of perturbation theory and loss of periodicity:

To avoid secular term from equation (16), we obtain

$$\omega_0 = \frac{1}{2}\sqrt{4 + A^2}. \tag{17}$$

Solving Eq. (16) with initial conditions (5), the first approximate solution y_1 of is obtained as

$$y_1 = \left(A + \frac{A^3}{32\omega_0^2}\right)\cos(\omega_1 t) - \frac{A^3}{32\omega_0^2}\cos(3\omega_1 t). \tag{18}$$

3.1.2. Second iteration solution

For the second level of iteration continuing to $k = 1$. Substituting Eq. (18) into Eq. (13), to obtain

$$\begin{aligned} \ddot{y}_2 + \omega_1^2 y_2 = &-\left(\tfrac{A(32+9A^2)(6144+4608A^2+1136A^4+93A^6-384(4+A^2)^2\omega_1^2}{3072(4+A^2)^3}\right)\cos(\omega_1 t) \\ &+\left(\tfrac{A^3(9216+7936A^2+2296A^4+223A^6-64(4+A^2)^2\omega_1^2)}{512(4+A^2)^3}\right)\cos(3\omega_1 t) \\ &+\left(\tfrac{A^5(3328+1928A^2+279A^4)}{1536(4+A^2)^3}\right)\cos(5\omega_1 t) + \left(\tfrac{13A^7(32+9A^2)}{6144(4+A^2)^3}\right)\cos(7\omega_1 t) \\ &-\left(\tfrac{A^9}{2048(4+A^2)^3}\right)\cos(9\omega_1 t). \end{aligned} \tag{19}$$

To avoid dominating terms, we obtain

$$\omega_1 = \sqrt{\frac{6144 + 4608A^2 + 1136A^4 + 93A^6}{6144 + 3072A^2 + 384A^4}}. \tag{20}$$

After solving Eq. (19) with the initial conditions, we have the second approximate solution.

$$\begin{aligned} y_2 = & \left(\tfrac{A^3(103680+85120A^2+23485A^4+2178A^6)+720A(4+A^2)^2(256+63A^2)\omega_1^2}{46080(4+A^2)^3 3\omega_1^2}\right)\cos(\omega_2 t) \\ & + \left(\tfrac{A^3(103680+85120A^2+23485A^4+2178A^6)+720A(4+A^2)^2(256+63A^2)\omega_1^2}{4096(4+A^2)^3\omega_1^2}\right)\cos(3\omega_2 t) \\ & + \left(\tfrac{A^5(3328+1928A^2+279A^4)}{36864(4+A^2)^3\omega_1^2}\right)\cos(5\omega_2 t) + \left(\tfrac{13A^7(32+9A^2)}{294912(4+A^2)^3\omega_1^2}\right)\cos(7\omega_2 t) \\ & + \left(\tfrac{A^9}{163840(4+A^2)^3\omega_1^2}\right)\cos(9\omega_2 t). \end{aligned} \tag{21}$$

3.1.3. Third iteration solution

For the third level of iteration, continuing to $k = 2$, we substituting y_2 from Eq. (21) into the right-hand side of Eq. (13), we have.

$$\ddot{y}_3 + \omega_2^2 y_3 = \omega_2^2 y_2 - y_2 + \bar{y}_2 y_2 \dot{y}_2. \tag{22}$$

Solving Equation (22), and avoiding secular terms, we can obtain ω_2 by using Mathematica command software program. The findings found for y_3, need too much space and cannot be shown here. However, the numerical values will be shown in the findings and discussion part.

$$\begin{aligned} \omega_2 = & (\surd(1/(4 + A^2)^9 A(147083960950218502963200 0+ 4826192468679044628480000A^2 \\ & +7296583778952951103488000A^4 + 6729655286686996758528000A^6 \\ & +4224062453812034745139200A^8 + 1905190794429295126118400A^{10} \\ & +635148579378340823040000A^{12} + 158430275820381536256000A^{14} \\ & +29559820082388423147520A^{16} + 4072969446127757557760A^{18} \\ & +402650199557972162560A^{20} + 27028701991773424000A^{22} \\ & +1103351520402981540A^{24} + 20664999056050911A^{26}))) \\ & /(3840\sqrt{7}\surd(\tfrac{1}{(4+A^2)^7} A(6144 + 4608A^2 + 1136A^4 + 93A^6)^2(23592960 \\ & +24330240A^2 + 9397760A^4 + 1618520A^6 + 105309A^8))). \end{aligned} \tag{23}$$

Integrating Eq. (21), we obtain the analytical solution of Eq. (9) in the form:

$$\begin{aligned} x = & \left(\left(\tfrac{4A\left(23592960+24330240A^2+9397760A^4+1618520A^6+105309A^8\right)}{\omega_2}\right)\sin(\omega_2 t) \right. \\ & - \left(\tfrac{20A^3\left(49152+43008A^2+12640A^4+1245A^6\right)}{\omega_2}\right)\sin(3\omega_2 t) \\ & + \left(\tfrac{26624A^5+15424A^7+2232A^9}{\omega_2}\right)\sin(5\omega_2 t) - \left(\tfrac{2080A^7+585A^9}{7\omega_2}\right)\sin(7\omega_2 t) \\ & \left. + \left(\tfrac{A^9}{\omega_2}\right)\sin(9\omega_2 t)\right) / \left(3840(4 + A^2)(6144 + 4608A^2 + 1136A^4 + 93A^6)\right) \end{aligned} \tag{24}$$

3.2. Jerk function containing velocity times acceleration-squared, and velocity

Another case of the Jerk equation is considered by putting $\gamma = \varepsilon = 1$, $\alpha = \beta = \delta = 0$, in Eq. (7) to obtain [1]:

$$\dddot{x} + \dot{x} + \dot{x}\ddot{x}^2 = 0, \quad x(0) = 0, \quad \dot{x}(0) = A, \quad \ddot{x}(0) = 0. \tag{25}$$

Similar to application (1), Eq. (25) can be rewritten in the form

$$\ddot{y} + y + y\dot{y}^2 = 0 \quad y(0) = A, \quad \dot{y}(0) = 0. \tag{26}$$

Following the iteration scheme (3), we have

$$\ddot{y}_{k+1} + \omega_{k+1}^2 y_{k+1} = \omega_k^2 y_k - y_k - y_k \dot{y}_k^2. \tag{27}$$

3.2.1. First iteration solution

For first iteration at $k = 0$, we get

$$\ddot{y}_1 + \omega_0^2 y_1 = \frac{1}{4}\left(-4\dot{y} + 4\dot{y}\omega_0^2 - A^3\omega_0^2\right)\cos(\omega_0 t) + \frac{A^3\omega_0^2}{4}\cos(3\omega_0 t). \tag{28}$$

To avoid secular term from equation (28), we obtain

$$\omega_0 = \frac{2}{\sqrt{4 - A^2}}. \tag{29}$$

Solving Eq. (28) with initial conditions (5), the first approximate solution y_1 is obtained as

$$y_1 = \left(A + \frac{A^3}{32}\right)\cos(\omega_1 t) - \frac{A^3}{32}\cos(3\omega_1 t). \tag{30}$$

3.2.2. Second iteration solution

For the second level of iteration continuing to $k = 1$ gives,

$$\begin{aligned} \ddot{y}_2 + \omega_1^2 y_2 = &\left(-\tfrac{A(32+A^2)(2048+(-2048+512A^2-48A^4+7A^6)\omega_1^2)}{65536}\right)\cos(\omega_1 t) \\ &+\left(\tfrac{A^3(1024+(7168+1280A^2+56A^4+3A^6)\omega_1^2)}{32768}\right)\cos(3\omega_1 t) \\ &-.\left(\tfrac{A^5(1792+88A^2+A^4)\omega_1^2}{32768}\right)\cos(5\omega_1 t) \\ &+\left(\tfrac{15A^7(32+A^2)\omega_1^2}{131072}\right)\cos(7\omega_1 t) \\ &-\left(\tfrac{9A^9\omega_1^2)}{131072}\right)\cos(9\omega_1 t). \end{aligned} \quad . \tag{31}$$

To avoid dominating terms in Eq (31), we obtain

$$\omega_1 = \frac{32\sqrt{2}\sqrt{32 + A^2}}{\sqrt{65536 - 14336A^2 + 1024A^4 - 176A^6 - 7A^8}}, \tag{32}$$

After solving Eq. (31) with the initial conditions, we have the second approximate solution.

$$\begin{aligned} y_2 = &\left(A + \frac{7A^3}{256} + \frac{A^5}{384} + \frac{35A^7}{196608} + \frac{23A^9}{1966080} + \frac{A^3}{256\omega_1^2}\right)\cos(\omega_2 t) \\ &- \left(\frac{A^3(1024+(7168+1280A^2+56A^4+3A^6)\omega_1^2)}{262144\omega_1^2}\right)\cos(3\omega_2 t) \\ &+ \left(\frac{A^5(1792+88A^2+A^4)}{786432}\right)\cos(5\omega_2 t) - \left(\frac{5A^7(32+A^2)}{2097152}\right)\cos(7\omega_2 t) \\ &+ \left(\frac{9A^9}{10485760}\right)\cos(9\omega_2 t). \end{aligned} \tag{33}$$

3.2.3. Third iteration solution

For the third level of iteration continuing to $k = 2$. Substituting y_2 from Eq. (32) into the right hand side of Eq. (13), we have

$$\ddot{y}_3 + \omega_2^2 y_3 = \omega_2^2 y_2 - y_2 - y_2\dot{y}_2^2. \tag{34}$$

Solving Eq. (34), and avoiding secular terms, we can obtain ω_2 in the same manner as application (1). The numerical data will be presented in the findings and discussion section.

$$\begin{aligned} \omega_2 = &\left(10485760\sqrt{\frac{23592960A+737280A^3+38400A^5+6360A^7-39A^9}{2048-512A^2+48A^4-7A^6}}\right) / \\ &\Big(\sqrt{((-2594073385365405696000A + 567453553048682496000A^3} \\ &-44754521296994304000A^5 - 3232564185661440000A^7 \\ &+172513374398054400A^9 + 2482284938657792 00A^{11} \\ &+19274524734259200A^{13} + 2109308770713600A^{15} \\ &+185788373401600A^{17} + 8293138432000A^{19} + 200006092800A^{21} \\ &+2252995200A^{23} + 174151100A^{25} - 997773A^{27})/(-2048 + 512A^2 \\ &-48A^4 + 7A^6))\Big) \end{aligned}. \tag{35}$$

Integrating Eq. (33), we obtain the analytical solution of Eq. (25) in the form:

$$\begin{aligned} x = \frac{1}{220200960\ \omega_2}\Big(&-28A\left(-7864320 - 245760A^2 - 12800A^4 - 2120A^6 + 13A^8\right)\sin(\omega_2 t) \\ &+140A^3\left(-16384 - 2048A^2 - 160A^4 + A^6\right)\sin(3\omega_2 t) \\ &+(100352A^5 + 4928A^7 + 56A^9)\sin(5\omega_2 t) \\ &- (2400A^7 + 75A^9)\sin(7\omega_2 t) + 21A^9\sin(9\omega_2 t)\Big). \end{aligned} \tag{36}$$

4. Discussions

In this section, the approximate analytical solutions of Eqs. (9) and (25), obtained using the modified iteration approach as shown in Eqs. (24) and (36), are compared with those obtained from fourth-order Runge-Kutta numerical solutions and other known analytical methods from the literature. This comparison is presented in Figures 1 – 4 and Tables $1-6$. The graphical representations clearly demonstrate that the results obtained from the present approach are in excellent agreement with those obtained using the fourth-order Runge-Kutta method. Moreover, the present approach accurately predicts the periodic behavior of the equations over a wide range.

To demonstrate the exceptional accuracy of the modified iteration technique, we examine two cases of non-linear third-order jerk equations. We compare the approximate results obtained using this technique with existing analytical solutions from the literature and with numerical integration results to validate the accuracy of the solutions derived.

For the same cases discussed using the block method [24], harmonic balance method [25], homotopy perturbation method [27], Linstedt-Poincare methods [30], residue harmonic balance method [31], multiple scales Lindstedt-Poincare [32], modified harmonic balance method [34], differential transform method [33], and global error minimization method [37], the present results were compared with those obtained by the modified iteration technique, as shown in Tables $1-6$. The numerical results show excellent agreement with the third-order approximate analytical solutions obtained in this study using the modified iteration technique.

A comparison between the higher-order approximate solution, the differential transform method [33], and the modified global error minimization method [37] with the corresponding numerical solution is presented in Figures $1-4$ for $A=0.3$ and $A=1$. It is clear that the approximation of the solution using the modified iteration technique agrees with the differential transform method, the modified global error minimization method, and the numerical solution. Furthermore, the approximate frequencies agree well with the corresponding exact solutions, implying that using the modified iteration technique with higher orders produces realistic results.

We looked at the percentage error (%) by the definition to confirm the accuracy.

$$Error = \left|\frac{T_e - T_{App}}{T_e}\right| \times 100\%.$$

where the various approximate periods obtained by T_{App} and T_e represents the corresponding exact period of the oscillator.

Table 1: Comparison of the approximate and exact solutions for Eq. (9).

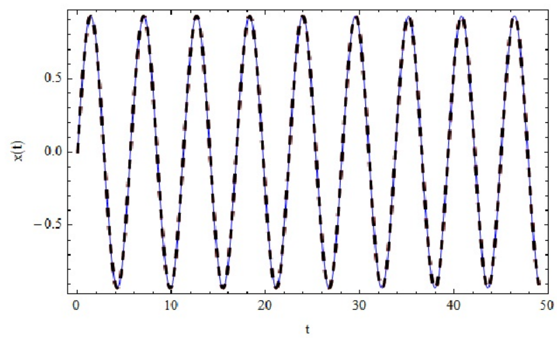
A	T_e	$T_{3\text{current}}$	T_3 [37]	T_3 [32]	T_3 [31]	T_3 [27]	T_3 [30]	T_3 [25]	T_3 [34]
0.1	6.275347	6.275347	6.275347	6.2753468	6.275346837	6.27534684	6.275348	6.275346	6.275346837
		0	0	$3.19e^{-6}$	$2.59e^{-6}$	$2.55e^{-6}$	$1.59e^{-5}$	$1.59e^{-5}$	$2.59e^{-6}$
0.2	6.252016	6.252016	6.252016	6.2520158	6.25201599	6.25201599	6.252028	6.252003	6.25201599
		0	0	$3.19e^{-6}$	$1.59e^{-7}$	$1.59e^{-7}$	$1.92e^{-4}$	$2.08e^{-4}$	$1.59e^{-7}$
0.5	6.096061	6.096061	6.096061	6.0960246	6.09606050	6.09605904	6.096491	6.095585	6.096060516
		0	0	$5.97e^{-4}$	$8.20e^{-6}$	$3.12e^{-5}$	0.00706	0.00781	$7.93e^{-7}$
1	5.626007	5.62587	5.62602	5.6245487	5.62599289	5.62579479	5.630343	5.619852	5.62599937
		$2.43e^{-5}$	$2.31e^{-4}$	0.02592	$2.51e^{-4}$	0.00377	0.07707	0.10940	$1.36e^{-4}$
2	4.491214	4.48492	4.47661	4.4664554	4.49012538	4.48208113	4.509311	4.442883	4.49112308
		0.14014	0.47234	0.55127	0.02424	0.20335	0.40294	1.07612	0.00202

Table 2: Comparison between the numerical solution and analytical solutions at $A = 0.2$ for Eq. (9)

t	Block Method [24]	MDTM [4/4] and [5/5] [33]	Present Solution	Numerical Solution
0	0	0	0	0
0.125	0.024934	0.024951214	0.024934943	0.024935034
0.25	0.049480	0.049511005	0.049480664	0.049480891
0.375	0.073253	0.073293861	0.073253553	0.073253560
0.5	0.095881	0.095926024	0.095881202	0.095881465
0.625	0.117008	0.117051228	0.117007962	0.117008021
0.75	0.1363	0.136336241	0.136300404	0.136300551
0.875	0.153453	0.153476143	0.153452636	0.153452785
1	0.168191	0.168199251	0.168191377	0.168191348
1.125	0.180281	0.180271603	0.180280677	0.180280833
1.25	0.180281	0.180271603	0.180280677	0.180280833
1.375	0.195778	0.195739784	0.195778569	0.195778587
1.5	0.198937	0.198888450	0.198936744	0.198936684
1.625	0.198949	0.198896377	0.198949462	0.198949302
1.75	0.195816	0.195763266	0.195816515	0.195816396
1.875	0.189589	0.189539059	0.189588706	0.189588464
2	0.180367	0.180323081	0.180366838	0.180366565

Table 3: Comparison between the numerical solution and analytical solutions at $A = 0.4$ for Eq. (9)

t	Block Method [24]	MDTM [4/4] and [5/5] [33]	Present Solution	Numerical Solution
0	0	0	0	0
0.125	0.049869	0.049869861	0.049869861	0.049870029
0.25	0.098960	0.098960562	0.098960558	0.098961002
0.375	0.146501	0.146501385	0.146501379	0.146501402
0.5	0.191739	0.191738916	0.191738937	0.191739420
0.625	0.233947	0.233946661	0.233946816	0.233946758
0.75	0.272436	0.272435646	0.272436179	0.272435964
0.875	0.306566	0.306566039	0.306567314	0.306566352
1	0.33576	0.335759596	0.335761911	0.335759593
1.125	0.359513	0.359512477	0.359515583	0.359513002
1.25	0.377408	0.377407791	0.377410047	0.377407810
1.375	0.389127	0.389126999	0.389124206	0.389127373
1.5	0.39446	0.394459308	0.394443406	0.394459515
1.625	0.393308	0.393308207	0.393266152	0.393308138
1.75	0.385695	0.385694562	0.385607769	0.385694560
1.875	0.371756	0.371755941	0.371755564	0.371600658
2	0.351742	0.351742248	0.351491092	0.351741644



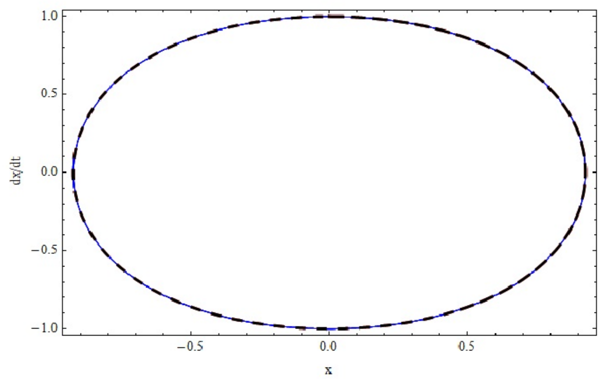
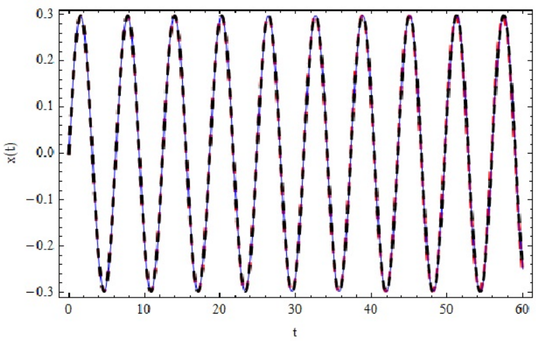


Figure 1: Comparison between higher order analytical solution (black line), MGEMM [37] (red line), and the numerical solution (blue line) at $A = 1$, for Eq. (9).



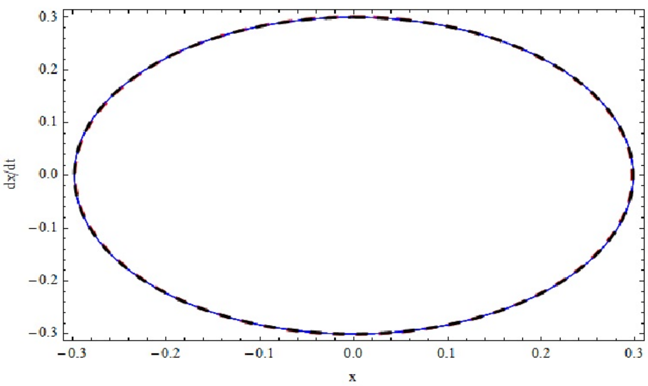


Figure 2: Comparison between higher order analytical solution (black line), MDTM [33] (red line), and the numerical solution (blue line) at $A = 0.3$, for Eq. (9).

Table 4: Comparison of the approximate and exact solutions for Eq. (25).

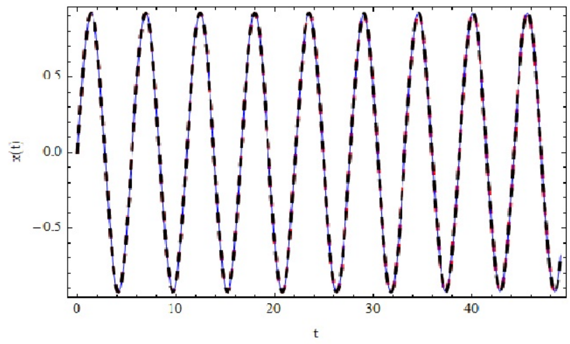
A	T_e	$T_{3\text{current}}$	T_3 [37]	T_3 [32]	T_3 [31]	T_3 [27]	T_3 [30]	T_3 [25]	T_3 [34]
0.1	6.27533378	6.27533378	6.27533378	6.27533378	6.2753338	6.27533378	6.275329	6.2753264	6.27533378
		0	0	0	0	0	$7.61e^{-5}$	$1.18e^{-4}$	0
0.2	6.251809	6.25180898	6.25180898	6.25180884	6.25180911	6.25182078	6.251740	6.251690	6.2518089
		$3.19e^{-7}$	$3.19e^{-7}$	$2.56e^{-6}$	$1.76e^{-6}$	$1.88e^{-4}$	0.00110	0.00190	$1.59e^{-6}$
0.5	6.088449	6.08845097	6.08845017	6.08841902	6.08848374	6.08815979	6.085649	6.083668	6.088450
		$3.24e^{-5}$	$1.92e^{-5}$	$4.92e^{-4}$	$5.71e^{-4}$	0.00475	0.04599	0.07853	$1.64e^{-5}$
1	5.527200	5.527510790	5.527656919	5.52576588	5.52994105	5.50818960	5.477174	5.441398	5.527497
		0.00562	0.00827	$2.59e^{-4}$	0.04959	0.343943	0.90509	1.55236	0.00537
2	4.690247	4.6831871	4.7771790	4.68572454	4.72603111	4.44735707	4.412733	4.155936	4.683269
		0.15052	1.85346	0.09642	0.76295	5.17847	5.91683	11.39196	0.14878

Table 5: Comparison between the numerical solution and analytical solutions at $A = 0.2$ for Eq. (25)

t	Block Method [24]	MDTM [4/4] and [5/5] [33]	Present Solution	Numerical Solution
0	0	0	0	0
0.125	0.024934	0.024934942	0.024934943	0.024935034
0.25	0.049480	0.049480663	0.049480664	0.049480891
0.375	0.073253	0.073253551	0.073253552	0.073253560
0.5	0.095881	0.095881197	0.095881199	0.095881461
0.625	0.117008	0.117007942	0.117007949	0.117008005
0.75	0.1363	0.136300334	0.136300356	0.136300499
0.875	0.153452	0.153452432	0.153452503	0.153452647
1	0.168191	0.168190868	0.168191071	0.168191036
1.125	0.18028	0.180279542	0.180280059	0.180280209
1.25	0.189525	0.189523851	0.189525026	0.189524937
1.375	0.195777	0.195774284	0.195776702	0.195776711
1.5	0.198934	0.198929290	0.198933855	0.198933786
1.625	0.198945	0.198937278	0.198945268	0.198945100
1.75	0.195811	0.195797675	0.195810754	0.195810626
1.875	0.189581	0.189560997	0.189581165	0.189580915
2	0.180357	0.180327919	0.180357378	0.180357098

Table 6: Comparison between the numerical solution and analytical solutions at $A = 0.4$ for Eq. (25)

t	Block Method [24]	MDTM [4/4] and [5/5] [33]	Present Solution	Numerical Solution
0	0	0	0	0
0.125	0.049869	0.049869861	0.049869863	0.049870029
0.25	0.098960	0.098960556	0.098960576	0.098961001
0.375	0.146501	0.146501354	0.146501423	0.146501386
0.5	0.191739	0.191738758	0.191738937	0.191739302
0.625	0.233946	0.233945991	0.233946416	0.233946249
0.75	0.272434	0.272433351	0.272434399	0.272434276
0.875	0.306562	0.306559422	0.306562164	0.306561832
1	0.335749	0.335743004	0.33575013	0.335749291
1.125	0.359492	0.359475362	0.359492741	0.359492282
1.25	0.37737	0.377332285	0.377371297	0.377370224
1.375	0.389065	0.388985315	0.389065644	0.389064877
1.5	0.394363	0.394211444	0.394363972	0.394363009
1.625	0.393169	0.392900613	0.393169669	0.393168420
1.75	0.385503	0.385060429	0.385504547	0.385503391
1.875	0.371507	0.370817684	0.371508112	0.371506638
2	0.351432	0.350416451	0.351432939	0.351431339



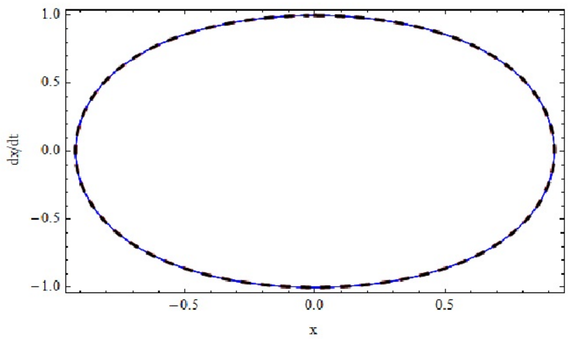


Figure 3: Comparison between higher order analytical solution (black line), MGEMM [37] (red line), and the numerical solution (blue line) at $A = 1$, for Eq. (25).

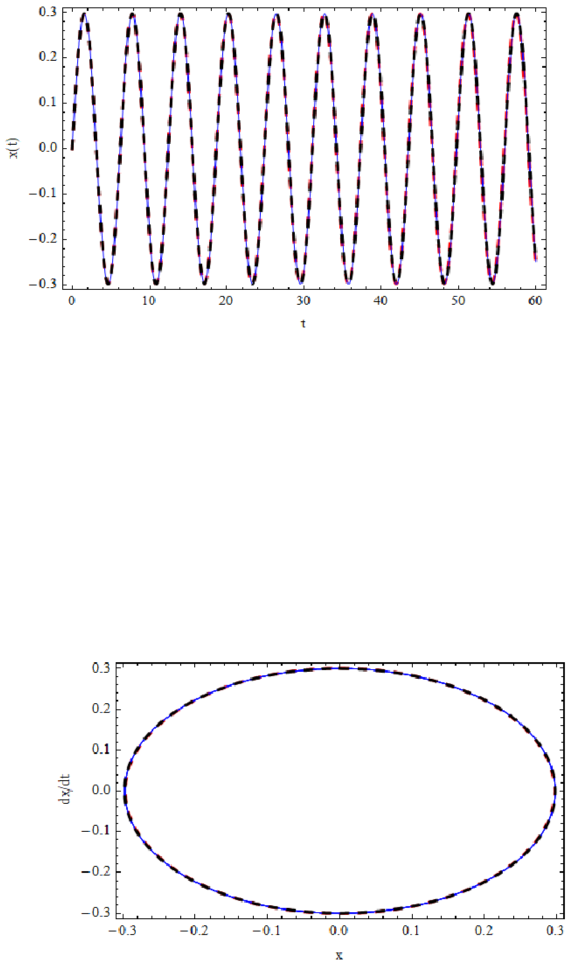


Figure 4: Comparison between higher order analytical solution (black line), MDTM [33] (red line), and the numerical solution (blue line) at $A = 0.3$, for Eq. (25).

5. Conclusion

In this study, a modified iteration approach was developed and employed to solve two nonlinear third-order jerk equations with broad engineering applications. The modified approach successfully determined approximate analytical solutions for these equations. A comparison of the obtained results with numerical solutions demonstrated the excellent accuracy of the proposed method. The present approach offers solutions in a readily usable analytical form, exhibiting superior accuracy and a wider range of applicability compared to other established analytical methods found in the literature. This technique proves to be a powerful and effective mathematical tool for solving highly nonlinear third-order differential equations arising in mathematical physics, applied mathematics, and engineering. Furthermore, the iterative nature of the method allows for the computation of higher-order approximations to achieve even greater accuracy if desired. The modified iteration technique facilitates the straightforward calculation of these higher-order terms, leading to solutions that closely approximate the exact solutions. Consequently, the present method demonstrates consistent and reliable performance, offering a simple yet effective approach for obtaining novel solutions to a variety of nonlinear problems.

Future research can focus on a deeper qualitative analysis of third-order systems, exploring the existence and uniqueness of solutions, stability analysis, and bifurcation behavior [39, 40].

Because of the great importance of applications of nonlinear third-order Jerk equations, more analytical and numerical methods will be developed to solve such problems in future work.

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