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On the Family of Theorems on Metric Completeness

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Abstract. There have appeared a large family of theorems related the metric completeness. They are mainly concerned with generalizations of the Banach contraction on quasi-metric spaces and their extended artificial spaces. In this survey article, we classify the family according to our 2023 Metatheorem. Many known metric fixed point theorems belong to the family including the Rus-Hicks-Rhoades (RHR) theorem. Such results on metric spaces are consequences of our generalized forms of the Banach contraction principle for weak contractions or the RHR maps on quasi-metric spaces. We list a large number of examples of metric fixed point theorems which follow from our principles. Moreover, we add some comments on related papers in order to improve them.

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1. Prologue

It is well-known that complete metric spaces have a large number of properties and, conversely, many of them characterize the completeness.

In our study in the ordered fixed point theory, we derived the 2023 Metatheorem which is a set of equivalent logical statements. From 2022, we applied it to almost one hundred theorems and obtained nearly one thousand new facts in mathematics.

Let (X, q) be a quasi-metric space (without assuming the symmetry of a metric). A selfmap $f : X \to X$ is called a Banach contraction with a constant $\alpha \in (0, 1)$ if

$$q(f(x), f(y)) \leq \alpha\, q(x, y) \quad \forall x, y \in X.$$

A selfmap $f : X \to X$ is called a weak contraction or a Rus-Hicks-Rhoades contraction (or simply an RHR map) with $\alpha \in (0, 1)$ whenever

$$q(f(x), f^2(x)) \leq \alpha\, q(x, f(x)) \quad \forall x \in X.$$

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Since 2023, we studied RHR maps and related topics in [16]–[21]. We obtained the generalized Banach Contraction Principle and the RHR Contraction Principle (Theorem P) with their applications to scores of examples in the literature.

Even for metric spaces our previous Theorem H in [17], [19] gave equivalent formulations of extensions of fixed point theorems due to Banach, Rus-Hicks-Rhoades, Nadler, Covitz-Nadler, Ekeland, Takahashi, Caristi-Kirk, Oettli-Théra and others. Consequently, we give the common unified new proofs of them.

The present survey is to classify many known fixed point theorems for quasi-metric spaces which characterize their completeness. In fact, we obtain a family of old or new theorems and can give them a unified proof. Consequently, this will enhance the reader's understanding of metric fixed point theory.

This survey is organized as follows: Section 2 is for preliminaries on quasi-metric spaces. In Section 3, we introduce Theorem H in [17], [19], which is a consequence of the 2023 Metatheorem implying equivalent formulations of quasi-metric completeness. Section 4 devotes some remarks on the family (0) of theorems related metric completeness.

In Sections 5-10, we introduce major results in the subfamilies $(\alpha)-(\eta)$ of the family (0) corresponding to each equivalent formulations to the completeness in Theorem H. Finally, Section 11 is for the epilogue.

There are a large number of articles concerning metric completeness. However this survey would be a precious supplement of the history of metric fixed pint theory.

2. Preliminaries

We recall the following:

Definition 2.1. A *quasi-metric* on a non-empty set X is a function $q : X \times X \to \mathbb{R}^+ = [0, \infty)$ verifying the following conditions for all $x, y, z \in X$:

(a) (self-distance) $q(x, y) = q(y, x) = 0 \iff x = y$;
(b) (triangle inequality) $q(x, z) \leq q(x, y) + d(y, z)$.

A *metric* in a set X is a quasi-metric satisfying that for all $x, y \in X$,

(c) (symmetry) $q(x, y) = q(y, x)$.

For quasi-metric spaces, the convergence of a sequence, Cauchy sequences, completeness, orbits, and orbital continuity are routinely defined as follows:

Definition 2.2. ([2], [9])

(1) A sequence (x_n) in X *converges* to $x \in X$ if

$$\lim_{n\to\infty} q(x_n, x) = \lim_{n\to\infty} q(x, x_n) = 0.$$

(2) A sequence (x_n) is *left-Cauchy* if for every $\varepsilon > 0$, there is a positive integer $N = N(\varepsilon)$ such that $q(x_n, x_m) < \varepsilon$ for all $n > m > N$.

(3) A sequence (x_n) is *right-Cauchy* if for every $\varepsilon > 0$, there is a positive integer $N = N(\varepsilon)$ such that $q(x_n, x_m) < \varepsilon$ for all $m > n > N$.

(4) A sequence (x_n) is *Cauchy* if for every $\varepsilon > 0$ there is positive integer $N = N(\varepsilon)$ such that $q(x_n, x_m) < \varepsilon$ for all $m, n > N$; that is (x_n) is a *Cauchy sequence* if it is left and right Cauchy.

Definition 2.3. ([2], [9])
(1) (X, q) is *left-complete* if every left-Cauchy sequence in X is convergent;
(2) (X, q) is *right-complete* if every right-Cauchy sequence in X is convergent;
(3) (X, q) is *complete* if every Cauchy sequence in X is convergent.

Definition 2.4. Let (X, q) be a quasi-metric space and $T : X \to X$ a selfmap. The *orbit* of T at $x \in X$ is the set

$$O_T(x) = \{x, T(x), \cdots, T^n(x), \cdots\}.$$

The space X is said to be T-*orbitally complete* if every right-Cauchy sequence in $O_T(x)$ is convergent in X. A selfmap T of X is said to be *orbitally continuous* at $x_0 \in X$ if

$$\lim_{n\to\infty} T^n(x) = x_0 \Longrightarrow \lim_{n\to\infty} T^{n+1}(x) = T(x_0)$$

for any $x \in X$.

Note that every complete metric space is T-orbitally complete for all maps $T : X \to X$. There exists a T-orbitally complete metric space but it is not complete. Moreover, there exists an orbitally continuous map but it is not continuous.

For other terminology related quasi-metric spaces, see [2], [9],

3. The basic principle and subfamilies $(\alpha) - (\eta)$

Let (X, q) be a quasi-metric space and $\mathrm{Cl}(X)$ denote the family of all nonempty closed subsets of X (not necessarily bounded). For $A,\ B \in \mathrm{Cl}(X)$, set

$$H(A, B) = \max\{\sup\{q(a, B) : a \in A\},\ \sup\{q(b, A) : b \in B\}\},$$

where $q(a, B) = \inf\{q(a, b) : b \in B\}$. Then H is called a generalized Hausdorff distance and it may have infinite values.

Based on our 2023 Metatheorem [14] and the RHR theorem, we obtained the following in [17], [19]:

Theorem H. *Let (X, q) be a quasi-metric space and $0 < \alpha < 1$. Then the following statements are equivalent:*

(0) (X, q) *is complete.*

(α) *For a multimap $T : X \to \mathrm{Cl}(X)$, there exists an element $v \in X$ such that $H(T(v), T(w)) > \alpha\, q(v, w)$ for any $w \in X\backslash\{v\}$.*

(β) *If $\mathfrak{F}$ is a family of maps $f : X \to X$ such that, for any $x \in X\backslash\{f(x)\}$, there exists a $y \in X\backslash\{x\}$ satisfying $q(f(x), f(y)) \leq \alpha\, q(x, y)$, then $\mathfrak{F}$ has a common fixed element $v \in X$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*

(γ) *If* $\mathfrak{F}$ *is a family of maps* $f : X \to X$ *satisfying* $q(f(x), f^2(x)) \leq \alpha\, q(x, f(x))$ *for all* $x \in X \backslash \{f(x)\}$*, then* $\mathfrak{F}$ *has a common fixed element* $v \in X$*, that is,* $v = f(v)$ *for all* $f \in \mathfrak{F}$*.*

(δ) *Let* $\mathfrak{F}$ *be a family of multimaps* $T : X \to \mathrm{Cl}(X)$ *such that, for any* $x \in X \backslash T(x)$*, there exists* $y \in X \backslash \{x\}$ *satisfying* $H(T(x), T(y)) \leq \alpha\, q(x, y)$*. Then* $\mathfrak{F}$ *has a common fixed element* $v \in X$*, that is,* $v \in T(v)$ *for all* $T \in \mathfrak{F}$*.*

(ϵ) *If* $\mathfrak{F}$ *is a family of multimaps* $T : X \to \mathrm{Cl}(X)$ *satisfying* $H(T(x), T(y)) \leq \alpha\, q(x, y)$ *for all* $x \in X$ *and any* $y \in T(x) \backslash \{x\}$*, then* $\mathfrak{F}$ *has a common stationary element* $v \in X$*, that is,* $\{v\} = T(v)$ *for all* $T \in \mathfrak{F}$*.*

(η) *If* Y *is a subset of* X *such that for each* $x \in X \backslash Y$ *there exists a* $z \in X \backslash \{x\}$ *satisfying* $H(T(x), T(z)) \leq \alpha\, q(x, z)$ *for a* $T : X \to \mathrm{Cl}(X)$*, then there exists a* $v \in X \cap Y = Y$*.*

Remark 3.1. (1) The completeness in (0) can be replaced by f-orbitally or T-orbitally completeness according to the corresponding situation.

(2) Note that there are many characterizations of the metric completeness; see [20]. Theorem H covers some of them. It is well-known that the Banach contraction does not characterize the metric completeness. However the extended RHR Principle does by Theorem H(γ),

(2) Note that Theorem H(β) properly extends the Banach Principle, (γ) the RHR Principle, and $(\delta), (\epsilon)$ the Nadler and Covitz-Nadler theorems. Moreover, Theorem H gives unified short-cut proofs of such extensions.

(3) Let (α1) and (η1) denote the case (α) and (η) for single-valued $T = f$, resp. When $\mathfrak{F}$ is a singleton, $(\beta) - (\epsilon)$ are denoted by $(\beta 1) - (\epsilon 1)$, resp. These are also equivalent to $(0) - (\eta)$. Therefore, actually Theorem H consists of equivalent 13 statements and gives a unified proofs for their equivalencies.

Definition 3.2. Let us consider the family (0) of theorems related to the completeness of quasi-metric spaces. Each subfamily $(\alpha) - (\eta)$ without (ζ) of the family (0) consists of theorems related the statement $(\alpha) - (\eta)$, resp.

4. The family (0)

In our earlier paper [13] in 1984, we gave some necessary and sufficient conditions for a metric space (X, d) to be complete. Such characterizations of metric completeness were given mainly by results relevant to Caristi's fixed point theorem (1976). Works of Cantor, Kuratowski (1930), Ekeland (1972), Caristi (1976), Kirk (1976), Boyd-Wong (1976), Kolodner (1967), Weston (1977), Ćirić (1971), Hu (1967), Reich (1971), Subrahmanyam (1975), and others are combined.

Actually we combined those results and stated our characterizations of the metric completeness as Theorem of [13]. The first response to the article was that: "Who dare use this kind of things to check the completeness of a metric space?"

A few years later in 1986, the author and Billy E. Rhoades published [22]. Its Abstract says: Several authors have characterized completeness of a metric space by using a fixed point theorem. The two theorems of this paper encompass some previous results as well as future theorems of this type.

We introduce the two theorems in [22] as follows:

Let $\mathfrak{B}$ be a class of selfmaps of closed subsets of a metric space X such that if any $g \in \mathfrak{B}$ has a fixed point then X is complete. Examples of $\mathfrak{B}$ are the classes of the Banach contractions (Hu (1967)) and the Kannan type contractions.

Let $\mathfrak{A}$ be a class of selfmaps of closed subsets of X containing $\mathfrak{B}$ such that completeness of X implies the existence of a fixed point for any map in $\mathfrak{A}$. Examples of $\mathfrak{A}$ containing the preceding examples of $\mathfrak{B}$ are classes of maps satisfying the conditions of Meir-Keeler (1969), Hegedüs-Szilágyi (1980), Caristi (1976), Tasković (1978, 1984), and Hikida (1984).

The following is the main result of [22]:

Theorem 4.1. *X is complete if and only if any map in $\mathfrak{A}$ has a fixed point.*

Let $\mathfrak{B}'$ be a class of selfmaps defined on X such that every map in $\mathfrak{B}'$ satisfies a certain condition Q, then X is complete. An example of $\mathfrak{B}'$ is the map satisfying an equivalent formulation of Caristi's theorem as in Weston (1977).

Let $\mathfrak{A}'$ be a class of maps defined on X containing $\mathfrak{B}'$ such that completeness of X implies that every map in $\mathfrak{A}'$ satisfies a condition P, where P implies Q. An example of $\mathfrak{A}'$ is the maps satisfying Ekeland's variational principle as in Sullivan (1981).

Theorem 4.2. *X is complete if and only if every map in $\mathfrak{A}'$ satisfies the condition P.*

In MR835839 (87m:54125), the reviewer J. Matkowski stated: There are many papers in which the completeness of a metric space is characterized by using a fixed point theorem. In the present paper [23], the authors prove two very simple and general theorems that "encompass some previous as well as future theorems of this type."

In 2020, S. Cobzaş [6] published an article entitled "Fixed points and completeness in metric and generalized metric spaces" with the following in Abstract:

"The famous Banach contraction principle holds in complete metric spaces, but completeness is not a necessary condition: there are incomplete metric spaces on which every contraction has a fixed point. The aim of his paper [6] is to present various circumstances in which fixed point results imply completeness. For metric spaces, this is the case of Ekeland's variational principle and of its equivalent, Caristi's fixed point theorem. Other fixed point results having this property will also be presented in metric spaces, in quasi-metric spaces, and in partial metric spaces."

Forty years later from [13], now we have another scores of papers on the metric completeness. A relatively new ones can be seen in Park [17],[18],[20] and others, where many known theorems on metric spaces also work on quasi-metric spaces. It would be interesting whether any of the works mentioned above [13] also hold for quasi-metric spaces.

The family (0) consists of theorems on completeness of quasi-metric spaces. Of course, it has a large number of theorems containing $(\alpha) - (\eta)$ and others.

In the present article, we do not try to collect all theorems in the family (0). Even for the subfamilies $(\alpha) - (\eta)$, we consider only theorems closely related our Metatheorem or Theorem H.

5. The subfamily (α)

The case (α) implies the following:

Theorem 5.1. *For a quasi-metric space (X, q), the following are equivalent:*

(0) (X, q) *is complete.*

$(\alpha 1)$ *For a map $f : X \to X$, there exists an element $v \in X$ such that $q(f(v), f(w)) > \alpha\, q(v, w)$ for any $w \in X \backslash \{v\}$.*

(α) *For a multimap $T : X \to \mathrm{Cl}(X)$, there exists an element $v \in X$ such that $H(T(v), T(w)) > \alpha\, q(v, w)$ for any $w \in X \backslash \{v\}$.*

As we have shown in our earlier paper [13] in 1984, many known theorems belong to the subfamily (α).

6. The subfamily (β)

In Theorem H, consider the following:

(β) If $\mathfrak{F}$ is a family of maps $f : X \to X$ such that, for any $x \in X \backslash \{f(x)\}$, there exists a $y \in X \backslash \{x\}$ satisfying $q(f(x), f(y)) \le \alpha\, q(x, y)$, then $\mathfrak{F}$ has a common fixed element $v \in X$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.

Note that the f-orbital completeness of (X, q) for any map $f : X \to X$ in $\mathfrak{F}$ implies (β).

From (β), we obtain the following consequence of Theorem P in Park [16]-[19], [21] (or in the next section), the generalized Banach contraction principle:

Theorem Q. *Let (X, q) be a quasi-metric space and let $T : X \to X$ be a generalized Banach contraction, that is, for each $x \in X$, there exists a $y \in X \backslash \{x\}$ such that*

$$q(T(x), T(y)) \le \alpha\, q(x, y) \text{ where } 0 < \alpha < 1. \tag{q}$$

(i) *If X is T-orbitally complete, then, for each $x \in X$, there exists a point $x_0 \in X$ such that*

$$\lim_{n \to \infty} T^n(x) = x_0$$

and

$$q(T^n(x), x_0) \le \frac{\alpha^n}{1 - \alpha} q(x, T(x)), \quad n = 1, 2, \cdots,$$

$$q(T^n(x), x_0) \le \frac{\alpha}{1 - \alpha} q(T^{n-1}(x), T^n(x)), \quad n = 1, 2, \cdots.$$

(ii) *x_0 is the unique fixed point of T (equivalently, $T : X \to X$ is orbitally continuous at $x_0 \in X$).*

Theorem Q extends a part of the following Theorem H(0) $\Longleftrightarrow$ $(\beta 1)$:

Theorem 6.1. *Let (X, q) be a quasi-metric space. Then it is complete if and only if*

(β1) *Let $f : X \to X$ be a map such that, for any $x \in X \backslash \{f(x)\}$, there exists $y \in X \backslash \{x\}$ satisfying $q(f(x), f(y)) \leq \alpha\, q(x, y)$. Then f has a fixed element $v \in X$, that is, $v = f(v)$.*

The only if part extends the so-called Banach Contraction Principle.

The traditional Banach Contraction Principle is a particular form of Theorem Q when X is a metric space and (q) holds for all $x, y \in X$. It appears in thousands of publications and should be corrected or replaced by Theorem Q.

The origin of the subfamily (β) is the following due to Banach in 1922:

Theorem 6.2. (Banach) *If* 1^0 *$U(X)$ be a continuous operator in E, the counter-domain of $U(X)$ is contained in E_1.*

2^0 *There exists a number $0 < M < 1$ which implies, for every X' and X'', the inequality*

$$||U(X') - U(X'')|| \leq M.||X' - X''||.$$

—- *there exists an element X such that $X = U(X)$.*

Here E and E_1 is a normed space and its complete subset, resp.

7. The subfamily (γ)

In Theorem H, consider the following:

(γ) *If $\mathfrak{F}$ is a family of maps $f : X \to X$ satisfying $q(f(x), f^2(x)) \leq \alpha\, q(x, f(x))$ for all $x \in X \backslash \{f(x)\}$, then $\mathfrak{F}$ has a common fixed element $v \in X$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*

Note that the f-orbital completeness of (X, q) for any RHR map $f : X \to X$ in $\mathfrak{F}$ implies (γ). Such map is traditionally called as graphic contraction, iterative contraction, weakly contraction, Banach mapping, ... We prefer to call it a weak contraction.

The following consequence of (γ) was independently obtained in Park [16]-[19], [21]. It is called the Rus-Hicks-Rhoades (RHR) Contraction Principle:

Theorem P. *Let (X, q) be a quasi-metric space and let $T : X \to X$ be an RHR map; that is,*

$$q(T(x), T^2(x)) \leq \alpha\, q(x, T(x)) \text{ for every } x \in X, \tag{p}$$

where $0 < \alpha < 1$.

(i) *If X is T-orbitally complete, then, for each $x \in X$, there exists a point $x_0 \in X$ such that*

$$\lim_{n\to\infty} T^n(x) = x_0$$

and

$$q(T^n(x), x_0) \leq \frac{\alpha^n}{1-\alpha} q(x, T(x)), \quad n = 1, 2, \cdots,$$

$$q(T^n(x), x_0) \leq \frac{\alpha}{1-\alpha} q(T^{n-1}(x), T^n(x)), \quad n = 1, 2, \cdots.$$

(ii) x_0 *is a fixed point of* T*, and, equivalently,*

(iii) $T : X \to X$ *is orbitally continuous at* $x_0 \in X$*.*

Example 7.1. For any quasi-metric space (X, q), let $T = 1_X$ be the identity map. Then Theorem P holds for 1_X.

Example 7.2. Let $X = \{0, 1\}$ with the usual metric and $T = 1_X$. Then condition (p) holds, but not (q). Hence Theorem P is a proper generalization of Theorem Q.

Example 7.3. Let

$$X := \{-1\} \cup \{0\} \cup \{\frac{1}{n} : n = 1, 2, \cdots, 100\}.$$

Let $q : X \times X \to \mathbb{R}$ be the ordinary metric except $q(-1, 0) = 1$ and $q(0, -1) = 0$. Then (X, q) is a quasi-metric space.

Let $T : X \to X$ be a map such that

$$T(-1) = 0, T(0) = 0, \text{ and } T(\frac{1}{n}) = \frac{1}{n+1}.$$

Then we can check

$$q(T(x), T^2(x)) \leq \alpha\, q(x, T(x)) \text{ with } \alpha = \frac{100}{102}.$$

Therefore Theorem P works.

The following form of the RHR theorem is a consequence of Theorems P and H, and useful in practice.

Theorem H$(\gamma 1)$**.** *Let* (X, q) *be a quasi-metric space,* $0 < \alpha < 1$*, and* $f : X \to X$ *be a map satisfying*

$$q(f(x), f^2(x)) \leq \alpha\, q(x, f(x)) \ for\ all\ x \in X \backslash \{f(x)\}.$$

Then f *has a fixed element* $v \in X$ *if* X *is* f*-orbitally complete.*

In our previous works [16]–[19], [21], we applied Theorems P and H$(\gamma 1)$ to a large number of early extensions or relatives of theorems of Rus [24] in 1973 and Hicks-Rhoades [8] in 1979.

Theorem 7.4. (Hicks-Rhoades) *Let* (X, d) *be a complete metric space,* $g : X \to X$ *and* $0 \leq h < 1$*. Suppose there exists an* x *such that*

$$d(gy, g^2y) \leq h\, d(y, gy) \ for\ every\ y \in \{x, gx, g^2x, \dots\}.$$

Then,

(i) $\lim_n g^n x = q$ *exists;*
(ii) $d(g^n x, q) \leq \frac{h^n}{1-h} d(x, gx)$*;*

(iii) *q is a fixed point of g if and only if $G(x) = d(x, gx)$ is g-orbitally lower semi-continuous at q.*

The following appears in a text-book of Aubin [1] in 1979:

Theorem 7.5. (Aubin) *Let V be a complete metric space and $f : V \to V$ be a map such that there exists an $L \in [0, 1)$ satisfying*

$$d(fx, f^2x) \leq L\, d(x, fx) \quad \forall\, x \in V.$$

If $F(x) = d(x, fx)$ on V is l.s.c., then

(1) $\lim f^n x = p$ *exists for all $x \in V$,*

$$d(f^n x, p) \leq \frac{L^n}{1-L} d(x, fx),$$

and p is a fixed point of f, and

(2) *for any $u \in V$ and $\varepsilon > 0$ satisfying*

$$F(u) \leq (1 - L)\varepsilon,$$

f has a fixed point in $\overline{B}(u, \varepsilon)$. Further, if f is a quasi-Lipshitzian with constant k, then either u is a fixed point of f or f has a fixed point in $\overline{B}(u, \varepsilon) \backslash B(u, s)$ where $s = F(u)(1 + k)^{-1}$.

Theorem 7.6. (Rus) *Let f be a continuous selfmap of a complete metric space (X, d) satisfying*

$$d(fx, f^2x) \leq \alpha\, d(x, fx) \quad \text{for every } x \in X,$$

where $0 < \alpha < 1$. Then f has a fixed point.

Theorems 7.4–7.6 are the origins of our Theorems P and Q, and seem to be independently obtained.

Berinde [3] in 2003 mentioned the so called Banach orbital condition $d(Tx, T^2x) \leq \alpha\, d(x, Tx)$, for all $x \in X$, studied by various authors in the context of fixed point theorems, see for example Kasahara, Hicks and Rhoades, Ivanov, Rus and Taskovic given in [3].

Berinde-Pacurar [4] in 2022 defined a *graphic contraction (orbital contraction)* and give examples as follows:

> Banach contraction, Kannan mapping, Ćirić-Reich-Rus contraction, Bianchini mapping, Chatterjea mapping, Zamfirescu mapping, Ćirić quasi-cont- raction, Hardy and Rogers contraction, Berinde's almost contraction,

In 2023, Berinde, Petrusȩl and I.A. Rus [5] listed previous names of the RHR maps as graphic contraction, iterative contraction, weakly contraction, Banach mapping, ...

In our previous work [17], we give the numbers of articles having examples of the RHR maps as follows:

Early examples of the RHR maps (1973–2009) : **26**

Suzuki types of the RHR maps (2001–2010) : **10**

Recent RHR type maps (2011–2023) : **37**

Almost all of main theorems of these papers are consequences of Theorem P for metric spaces. These can be applied to nearly one thousand artificial metric type spaces.

8. The subfamily (δ)

In Theorem H, consider the following:

(δ) *Let* $\mathfrak{F}$ *be a family of multimaps* $T : X \to \mathrm{Cl}(X)$ *such that, for any* $x \in X\backslash T(x)$*, there exists* $y \in X\backslash\{x\}$ *satisfying* $H(T(x), T(y)) \leq \alpha\, q(x, y)$*. Then* $\mathfrak{F}$ *has a common fixed element* $v \in X$*, that is,* $v \in T(v)$ *for all* $T \in \mathfrak{F}$*.*

Note that the T-orbital completeness of (X, q) for any multimap $T : X \to \mathrm{Cl}(X)$ in $\mathfrak{F}$ implies (δ).

When $\mathfrak{F}$ is a singleton, we have extensions of the Nadler and Covitz-Nadler fixed point theorems [7], [10] and their converses, that is, Theorem H(0) is equivalent to $(\delta 1)$ as follows:

Theorem 8.1. *Let* (X, q) *be a quasi-metric space. Then it is complete if and only if*

$(\delta 1)$ *Let* $T : X \to \mathrm{Cl}(X)$ *be a multimap such that, for any* $x \in X\backslash\{Tx\}$*, there exists* $y \in X\backslash\{x\}$ *satisfying* $H(T(x), T(y)) \leq \alpha\, q(x, y)$*. Then* T *has a fixed element* $v \in X$*, that is,* $v \in T(v)$*.*

The only if part extends also the so-called Banach contraction principle.

9. The subfamily (ϵ)

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In Theorem H, consider the following:

(ϵ) *If* $\mathfrak{F}$ *is a family of multimaps* $T : X \to \mathrm{Cl}(X)$ *satisfying* $H(T(x), T(y)) \leq \alpha\, q(x, y)$ *for all* $x \in X$ *and any* $y \in T(x)\backslash\{x\}$*, then* $\mathfrak{F}$ *has a common stationary element* $v \in X$*, that is,* $\{v\} = T(v)$ *for all* $T \in \mathfrak{F}$*.*

Note that the T-orbital completeness of (X, q) for any multimap $T : X \to \mathrm{Cl}(X)$ in $\mathfrak{F}$ implies (ϵ).

From (ϵ), we can deduce at least four particular cases. The following is only one of the oldest one for the singleton [10]:

Theorem 9.1. (Nadler) *Let* (X, d) *be a complete metric space. If* $F : X \to \mathrm{BC}(X)$ *is a multi-valued contraction map, then* F *has a fixed point.*

According to our (ϵ), the fixed point should be strengthened to a stationary point. Moreover, Covitz and Nadler [7] extended Theorems 9.1 and others to mappings into $\mathrm{Cl}(X)$ with the generalized Hausdorff distance.

10. The subfamily (η)

In this section, we follow Oettli and Théra [11] in 1993.

Let (V, d) be a complete metric space. Let $f : V \times V \to (-\infty, +\infty]$ be a function which is lower semicontinuous in the second argument and satisfies

$$f(v, v) = 0 \text{ for all } v \in V, \tag{1}$$

$$f(u, v) < f(u, w) + f(w, v) \text{ for all } u, v, w \in V.$$

Assume that there exists $v_0 \in V$ such that

$$\inf_{v \to \infty} f(v_0, v) > -\infty.$$

Let

$$S_0 := \{v \in V : f(v_0, v) + d(v_0, v) \leq 0\}.$$

From (1) it follows that $v_0 \in S_0 \neq \emptyset$.

Under these specifications the following results are true:

Theorem 10.1. (Ekeland) *There exists $v^* \in S_0$ such that $f(v^*, v) + d(v^*, v) > 0$ for all $v \in V$, $v \neq v^*$.*

Theorem 10.2. (Takahashi) *Assume that*

$$\text{for every } \overline{v} \in S_0 \text{ with } \inf_{v \in V} f(\overline{v}, v) < 0 \text{ there exists}$$
$$v \in V \text{ such that } v \neq \overline{v} \text{ and } f(\overline{v}, v) + d(\overline{v}, v) \leq 0.$$

Then there exists $v^* \in S_0$ such that $f(v^*, v) \geq 0$ for all $v \in V$.

Theorem 10.3. (Caristi-Kirk) *Let $T : V \multimap V$ be a multimap such that*

$$\text{for every } \overline{v} \in S_0 \text{ there exists}$$
$$v \in T(\overline{v}) \text{ satisfying } v \neq \overline{v} \text{ and } f(\overline{v}, v) + d(\overline{v}, v) \leq 0.$$

Then there exists $v^* \in S_0$ such that $v^* \in T(v^*)$.

The following is the origin of (η) due to Oettli and Théra [11] in 1993:

Theorem 10.4. (Oettli-Théra) *Let $\Psi \subset V$ have the property that*

$$\text{for every } \overline{v} \in S_0 \backslash \Psi \text{ there exists } v \in V$$
$$\text{such that } v \neq \overline{v} \text{ and } f(\overline{v}, v) + d(\overline{v}, v) \leq 0.$$

Then there exists $v^* \in S_0 \cap \Psi$.

Oettli-Théra [11] finally stated:

Theorem 10.5. (Oettli-Théra) *Theorems 10.1 through 10.4 are equivalent.*

Consider our following condition:

(η) *If Y is a subset of X such that for each $x \in X \backslash Y$ there exists a $z \in X \backslash \{x\}$ satisfying $H(T(x), T(z)) \leq \alpha\, q(x, z)$ for a $T : X \to \mathrm{Cl}(X)$, then there exists a $v \in X \cap Y = Y$.*

Actually, (η) is motivated from Theorem 10.4 of Oettli-Théra. We can deduce several particular existence theorems which can be called the subfamily (η).

11. Epilogue

Since there had been a large number of metric fixed point theorem, researchers tried to classify them. The first attempt to classify the contractive conditions was done by Billy E. Rhoades [23] in 1977. Our previous work [12] in 1980 can be regarded its continuation. Recently, Cobzaş [6] in 2020 is to indicate the existence of families of theorems related to metric completeness.

Recall that Banach's original fixed point theorem in 1922 was stated for normed vector spaces. Later several researchers formulated it to the form of the Banach Contraction Principle for complete metric spaces. In the last one hundred years, there have been appeared hundreds of contraction type conditions and almost one thousand spaces which generalize, extend, or modify the complete metric spaces. Recall that the Banach contraction does not characterize the metric completeness.

The advantage of our Metatheorem is as follows: the proofs of each item follows from the only one of them. This can be seen from Theorem H or almost one hundred examples given in our previous works related Metatheorem. Consequently, we found that the traditional metric fixed point theory and many of its recent works should be corrected or improved in various aspects.

Recall that a few researchers studied the Rus-Hicks-Rhoades (RHR) maps by using several different names. From 2023, one hundred years later to the Banach contraction, the present author began to study on RHR maps. We found a large number of examples of RHR maps and the so-called RHR Contraction Principle extending the classical Banach one. Moreover, we found that the RHR theorem is equivalent to variants of the Nadler or Covitz-Nadler theorem for multivalued contractions. Furthermore, we found that the RHR theorem characterizes the metric completeness.

Such studies were done in 2022–24. One of the significance of our recent works on metric fixed point theory is to clarify some incorrectly stated results with unnecessarily long proofs given by several authors. In fact, our aim of study in metric fixed point theory since 2022 is to improve every thing there without making new spaces or new contractive conditions.

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