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Generalized Linear Differential Equation using Hyers - Ulam Stability Approach

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Abstract. In this paper, We demonstrate the Hyers - Ulam stability of linear differential equation of fourth order. We interact with the differential equation

$$\gamma^{iv}(\omega) + \rho_1\gamma'''(\omega) + \rho_2\gamma''(\omega) + \rho_3\gamma'(\omega) + \rho_4\gamma(\omega) = \chi(\omega),$$

where $\gamma \in c^4[\alpha, \beta], \chi \in [\alpha, \beta]$. Hyers-Ulam stability concerns the robustness of solutions of functional equations under small perturbations, ensuring that a solution approximately satisfying the equation is close to an exact solution. We extend this concept to fourth-order linear differential equations and continuous functions. Using fixed-point methods and various norms, we establish conditions under which such equations exhibit Hyers-Ulam stability. Several illustrative examples are provided to demonstrate the application of these results in specific cases, contributing to the growing understanding of stability in higher-order differential equations. Our findings have implications in both theoretical research and practical applications in physics and engineering.

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Key Words and Phrases: Hyers-Ulam stability, Linear differential equation.

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1. Introduction

The stability issue of functional equation began from an issue of Ulam [18] concerning the strength of gathering homomorphisms.

Let G_1 be a group and let G_2 be a measurement group with the metric $d(.,.)$. Given $\in > 0$, does there exists a $\delta > 0$ to such an extent if a mapping $H : G_1 \to G_2$ fulfills the imbalance $d(H(\omega\nu), H(\omega)H(\nu)) < \delta$ with respect to $\omega, \nu \in G_1$, at that point there exists a homomorphism $h : G_1 \to G_2$ with $d(H(\omega), h(\omega)) <\in$ with respect to $\omega \in G_1$. As it were, if a mapping is almost a homomorphism, at that point there exists a true homomorphism were τ it with little blunder however much as could reasonably be expected.

The issue from the instance of roughly additive mappings was formed by Hyers [12] when G_1 and G_2 are Banach spaces also, the after effect of Hyers was summed up by Rassias (See [15]). From that point forward, the dependability issues of practical conditions have been broadly examined by a few mathematician (see [2–4, 9, 11]).

Supposedly, papers by Ozawa [7] were among the first commitments managing with $H-U$ stability of differential equations. Alsina [1] and Ger demonstrated $H-U$ stability of differential condition $\gamma'(\omega) = \gamma(\omega)$. Afterword, Takahasi et al. stretched out consequences of [16, 17] to the Banach space esteemed differential condition $\gamma'(\omega) = \lambda\gamma(\omega)$. Utilizing direct strategy, cycle technique, find point technique, and open mapping theorem, Huang and Li explored the $H - U$ stability of certain classes of useful fractional differential equations (see [11, 14, 16, 19]).

For higher-order differential equations, such as fourth-order, the characteristic equation can become quite complex. Solving for the roots, especially if they are non-real or repeated, can be tedious. While homogeneous equations can sometimes be solved through standard methods (like finding the roots of the characteristic polynomial), non-homogeneous equations require additional techniques such as variation of parameters or the method of undetermined coefficients, which are not always straightforward. Many fourth-order linear differential equations, especially those arising in physics and engineering (e.g., in beam theory or fluid dynamics), cannot be solved using elementary functions and require special functions (e.g., Bessel, Airy, or Legendre functions). These solutions can be difficult to interpret or manipulate further. A fourth-order equation requires four boundary or initial conditions to determine a unique solution. This increases the complexity of the problem, and choosing appropriate conditions can be tricky. In some cases, the boundary or initial conditions may be incompatible with the differential equation, leading to no solution or non-physical solutions.

Solving fourth-order differential equations numerically (e.g., with finite difference methods, finite element methods, or Runge-Kutta methods) can lead to instability, especially if the equation involves stiff terms. This requires careful attention to step sizes and discretization methods. Higher-order equations require more computational effort. Discretizing fourth-order equations often leads to larger, more complex systems of linear equations, which increases computational cost. Numerical methods are approximate by nature. For higher-order equations, truncation and rounding errors can accumulate, leading to reduced accuracy.

Higher-order terms in a differential equation might not always have clear physical meanings, making it harder to interpret the behavior of the system. In some cases, it may not be guaranteed that a solution exists or is unique, especially for non-linear fourth-order equations or equations with unusual boundary conditions. Analytical methods, in particular, may fail to provide a solution in such cases. While the equation is linear, real-world problems often involve non-linearities. Extending methods for linear equations to nonlinear fourth-order differential equations introduces significant complications, as many techniques for linear equations do not directly apply.

For instance, Farhan [10] studied the implementation of the one-step one-hybrid block method on the nonlinear equation. While these methods aim to provide more efficient solutions, they are sensitive to the formulation of boundary conditions and may encounter difficulties in ensuring stability and convergence. This highlights the limitation of standard linear methods when extended to nonlinear systems, as additional strategies must be employed to deal with nonlinearity and complex geometrical configurations. In fluid mechanics, Basha [6] and Shelly Arora [5] investigated the higher-order differential equations govern the behavior of non-Newtonian fluids and their heat transfer properties. The nonlinear expansion of the sheet and the specific boundary conditions introduce further complexity, requiring specialized numerical methods. This method provides excellent super convergence properties, its application to highly nonlinear systems reveal limitations in classical linear differential equation methods. Specifically, the wave behavior and chaotic nature of the Kuramoto-Shivashinsky equation push traditional numerical methods to their limits, often requiring adaptive meshes or hybrid approaches to maintain accuracy and computational efficiency.

In the context of the one-step one-hybrid block method on the nonlinear equation and the introduction of Hyers-Ulam stability provides a novel way to verify whether the numerical methods employed are capable of maintaining solution stability despite perturbations in initial or boundary data. The Hyers-Ulam framework allows for the quantification of stability in cases where small errors in modeling the oscillator geometry (e.g., boundary conditions of the circular sector) could lead to significant deviations in the oscillatory motion. Thus, incorporating Hyers-Ulam stability into the block method enhances the understanding of the method's robustness and ensures that solutions remain bounded even in the face of minor perturbations. TheKuramoto-Shivashinsky equation, with its chaotic and wave-like behavior, presents substantial challenges when using traditional fourth-order linear differential methods. The Super Convergence Analysis of Fully Discrete Hermite Splines is already a sophisticated method to address this. However, the application of Hyers-Ulam stability introduces an additional layer of novelty by ensuring that the solutions obtained remain stable under perturbations, which is crucial for chaotic systems. Small inaccuracies in initial conditions or computational errors could rapidly escalate into significant deviations in the wave dynamics. With Hyers-Ulam stability, researchers can provide stronger guarantees that the numerical approximations made using fully discrete Hermite splines remain valid and bounded, offering greater confidence in the accuracy and applicability of these methods for simulating complex wave behavior.

The integration of Hyers-Ulam stability into the study of fourth-order linear differential equations represents a novel contribution to the field. Traditional stability analyses often focus on whether solutions remain bounded based on specific methods, but they do not typically address how sensitive the solutions are to small perturbations in initial or boundary data. By applying Hyers-Ulam stability, this study provides a quantitative measure of how solutions to fourth-order linear differential equations respond to small deviations in data or boundary conditions. This robustness measure is particularly important for real-world applications, where exact data is rarely available, and errors in modeling are inevitable.Fourth-order linear differential equations are often applied in fields that require precise boundary conditions, such as beam theory, fluid dynamics, and elasticity theory. Hyers-Ulam stability expands the applicability of these equations by ensuring that even in the presence of small uncertainties or numerical errors, solutions remain within acceptable bounds.By integrating Hyers-Ulam stability into methods such as the One-Step One-Hybrid Block Method or Hermite splines, this study enhances the reliability of these numerical techniques. While traditional numerical methods focus on accuracy and convergence, Hyers-Ulam stability ensures that the solutions generated by these methods are not overly sensitive to perturbations, thus offering a more robust framework for practical applications. The novelty of applying Hyers-Ulam stability to nonlinear problems, such as those encountered in the Sutterby hybrid nanofluid flow or the Kuramoto-Shivashinsky equation, lies in its ability to provide stability guarantees in cases where linear methods struggle. Nonlinear systems are notoriously sensitive to perturbations, and the Hyers-Ulam framework provides a new tool to ensure that solutions remain bounded and reliable.

In this paper, we demonstrate the Hyers - Ulam stability of linear differential equation of fourth order. That is, γ is an interact arrangement of the differential equation

$$\gamma^{iv}(\omega) + \rho_1\gamma'''(\omega) + \rho_2\gamma''(\omega) + \rho_3\gamma'(\omega) + \rho_4\gamma(\omega) = \chi(\omega)$$

Where $\gamma \in c^4[\alpha, \beta], \chi \in [\alpha, \beta]$, we demonstrate that $\gamma^{iv}(\omega)+\rho_1\gamma'''(\omega)+\rho_2\gamma''(\omega)+\rho_3\gamma'(\omega)+\rho_4\gamma(\omega) = \chi(\omega)$ has the Hyers - Ulam stability. An example is provided to illustrate the theory.

Moreover, the after effect of $H - U$ Stability for first order differential conditions has been summed up by Miara, Miyajima and Takahasi [17] by Takahasi, Takagi, Miara and Miyajima [8], and furthermore by jung [13]. They managed the non homogeneous straight differential equation of first order

$$\gamma' + \rho(\tau)\gamma + \sigma(\tau) = 0. \tag{1}$$

Jung [13] demonstrated the summed up $H - U$ Stability of differential condition of the structure

$$\tau\gamma'(\tau) = \alpha\gamma(\tau) + \beta\tau^{\gamma}\omega_0 = 0$$

and furthermore applied this out come to the examination of the $H - U$ stability of the differential equation

$$\tau^2\gamma''(\tau) + a\tau\gamma'(\tau) + b\gamma(\tau) = 0. \tag{2}$$

As of late, Wang, Zhon and sun [19] examined the $h - U$ Stability of the first order linear differential condition

$$\rho(\omega)\gamma' + \sigma(\omega)\gamma + \eta(\omega) = 0. \tag{3}$$

As a matter of first importance, we give the meaning of the $H - U$ stability.

Definition 1. *We say that Equation 2 has the $H - U$ Stability if there exists a steady $\kappa > 0$ with accompanying property, for every $\in > 0, \gamma \in c^2[\alpha, \beta]$, if*

$$|\gamma'' + a\gamma' + b\gamma| \leq \in, \tag{4}$$

at the point there exists some $U \in c^2[\alpha, \beta]$ fulfilling

$$|u'' + au' + bu| \leq \chi(\omega) \tag{5}$$

such that $|\gamma(\omega) - u(\omega)| < \kappa \in$. We call such κ a $H - U$ Stability constant for equation 2.

Definition 2. *We say that equation 2 extend has the $H - U$ Stability, if there exists a steady $\kappa > 0$ with accompanying property: for every $\in > 0, \gamma \in c^3[\alpha, \beta]$, if*

$$|\gamma''' + a\gamma'' + b\gamma' + c\gamma| \leq \in, \tag{6}$$

at the point there exists some $U \in c^3[\alpha, \beta]$ fulfilling

$$|u''' + au'' + bu' + cu| = 0 \tag{7}$$

such that $|\gamma(\omega) - u(\omega)| < \kappa \in$. We call such κ a $H - U$ Stability constant for Equation 6.

Definition 3. *We say that equation 6 extend has the $H - U$ Stability, if there exists a steady $\kappa > 0$ with accompanying property: for every $\in > 0, \gamma \in c^4[\alpha, \beta]$, if*

$$|\gamma^{iv} + \rho_1\gamma''' + \rho_2\gamma'' + \rho_3\gamma' + \rho_4\gamma| \leq \in, \tag{8}$$

at the point there exists some $U \in c^4[\alpha, \beta]$ fulfilling

$$|u^{iv} + \rho_1 u''' + \rho_2 u'' + \rho_3 u' + \rho_4 u| = 0 \tag{9}$$

such that $|\rho(\omega) - u(\omega)| < \kappa \in$. We call such κ a $H - U$ Stability constant for equation 8.

2. Main results

Now, fundamental consequence of this work is given in the accompanying hypothesis.

Lemma 1. *The differential equation j $\gamma^{iv}(\omega) + \rho_1\gamma'''(\omega) + \rho_2\gamma''(\omega) + \rho_3\gamma'(\omega) + \rho_4\gamma(\omega) = \chi(\omega)$ has the Hyers -Ulam Stability, where $\gamma \in c^4[\alpha, \beta]$ and $\chi \in [\alpha, \beta]$.*

Proof. Assume that u_1, u_2, u_3 and u_4 are the roots of $\nu^4 + \rho_1\nu^3 + p_2\nu^2 + p_3\nu + p_4 = 0$ with $q_1 = \mathbb{R}u_1, q_2 = \mathbb{R}u_2, q_3 = \mathbb{R}u_4$ and $q_4 = \mathbb{R}u_3$. Here $\mathbb{R}$ means the real parts.
Let $\in > 0$ and $\gamma \in c^4[\alpha, \beta]$

$$|\gamma^{iv}(\omega) + \rho_1\gamma'''(\omega) + \rho_2\gamma''(\omega) + \rho_3\gamma'(\omega) + \rho_4\gamma(\omega) - \chi(\omega)| \leq \in \tag{10}$$

and let

$$g_1(\omega) = \gamma'''(\omega) + (u_1 + \rho_1)\gamma''(\omega) + (u_1^2 + \rho_1 u_1 + \rho_2)\gamma'(\omega) + (u_1^3 + \rho_1 u_1^2 + \rho_2 u_1 + \rho_3)\gamma(\omega),$$

we acquire

$$g_1'(\omega) = \gamma'''(\omega) + (u_1 + \rho_1)\gamma'''(\omega) + (u_1^2 + \rho_1 u_1 + \rho_2)\gamma''(\omega) + (u_1^3 + \rho_1 u_1^2 + \rho_2 u_1 + \rho_3)\gamma'(\omega) + (u_1^4 + \rho_1 u_1^3 + \rho_2 u_1^2 + \rho_3 u_1 + \rho_4)\gamma(\omega) \tag{11}$$

with respect to $\omega \in [\alpha, \beta]$, at that point

$$|g_1'(\omega) - u_1 g_1(\omega) - \chi(\omega)| \leq \in \tag{12}$$

with respect to $\omega \in [\alpha, \beta]$, yields that

$$\begin{aligned} |g_1'(\omega) - u_1 g_1(\omega) - \chi(\omega)| \leq {} & |\gamma'''(\omega) + (u_1 + \rho_1)\gamma'''(\omega) + (u_1^2 + \rho_1 u + \rho_2)\gamma''(\omega) \\ & + (u_1^3 + \rho_1 u_1^2 + \rho_2 u_2 + \rho_3)\gamma'(x) \\ & + (u_1^4 + \rho_1 u_1^3 + \rho_2 u_1^2 + \rho_3 u_1 + \rho_4)\gamma(\omega) \\ & - u_1(\gamma'''(\omega) + (u_1 + \rho)\gamma''(\omega) + u_1^2 + (\rho_1 u_1 + \rho_2)\gamma'(\omega) \\ & + \big((u_1^3 + \rho_1 u_1^2 + \rho_2 u_1 + \rho_3)\gamma(\omega)\big) - \chi(\omega)| \end{aligned} \tag{13}$$

with respect to $\omega \in [\alpha, \beta]$. Utilizing the above condition, we get

$$\begin{aligned} |g_1'(\omega) - u_1 g_1(\omega) - \chi(\omega)| &= |\gamma^{iv}(\omega) + \rho_1\gamma'''(\omega) + \rho_2\gamma^n(\omega) + \rho_3\gamma'(\omega) + \rho_4\gamma(\omega)| \\ &< \in . \end{aligned}$$

with respect to $\omega \in [\alpha, \beta]$. Equally g_1 fulfills

$$- \in \leq g_1'(\omega) - u_1 g_1(\omega) - \chi(\omega) \leq \in \tag{14}$$

with respect to $\omega \in [\alpha, \beta]$. Multiplying by $e^{-u_1(\omega-\alpha)}$ the above condition, shown up

$$\in e^{-u_1(\omega-\alpha)} \leq g_1'(\omega)e^{-u_1(\omega-\alpha)} - u_1 g_1(\omega)e^{-u_1(\omega-\alpha)} - \chi(\omega)e^{-u_1(\omega-\alpha)} \leq \in e^{-u_1(\omega-\alpha)} \tag{15}$$

with respect to $\omega \in [\alpha, \beta]$. Without loss of all inclusive statement we may accept that $u_1 > 1$, thus

$$-u_1 \in e^{-u_1(\omega-\alpha)} \leq g_1'(\omega)e^{-u_1(\omega-\alpha)} - u_1 g_1(\omega)e^{-u_1(\omega-\alpha)} - \chi(\omega)e^{-u_1(\omega-\alpha)}$$

$$\leq u_1 e^{-u_1(\omega-\alpha)} \tag{16}$$

with respect to $\omega \in [\alpha, \beta]$, integrating 16 from ω to β, we achieve

$$-\in \left(-e^{-u_1(\beta-\alpha)} + e^{-u_1(\omega-\alpha)}\right) \leq g_1(\beta)e^{-u_1(\beta\alpha)} - g_1(\omega)e^{-u_1(\omega-\alpha)} - \int_\omega^\beta \chi(\tau)e^{-u_1(\tau-\alpha)}d\tau$$
$$\leq \in \left(-e^{-u_1(\beta-\alpha)} + e^{-u_1(\omega-\alpha)}\right) \tag{17}$$

with respect to $\omega \in [\alpha, \beta]$, thus

$$-\in e^{-u_1(\omega-\alpha)} \leq g_1(\beta)e^{-u_1(\omega-\alpha)} - \in e^{-u_1(\beta-\alpha)} - g_1(\omega)e^{-u_1(\omega-\alpha)} - \int_\omega^\beta \chi(\tau)e^{-u_1(\tau-\alpha)}d\tau$$
$$\leq \in \left(-e^{-u_1(\omega-\alpha)} + e^{-u_1(\beta-\alpha)}\right) \tag{18}$$

with respect to $\omega \in [\alpha, \beta]$, the above condition shown up

$$\in -e^{-u_1(\omega-\alpha)} \leq g_1(\beta) - e^{-u_1(\omega-\alpha)} - \in -e^{-u_1(\beta-\alpha)} - g_1(\omega) - e^{-u_1(\omega-\alpha)}$$
$$-\int_\omega^\beta \chi(\tau)e^{-u_1(\tau-\alpha)}d\tau \leq \in e^{-u_1(\omega-\alpha)} \tag{19}$$

with respect to $\omega \in [\alpha, \beta]$. Multiplying 19 by $e^{u_1(\omega-\alpha)}$ on both sides, we get

$$-\in \leq g_1(\beta)e^{-u_1(\beta-\omega)} - \in e^{-u_1(\beta-\omega)} - g_1(\omega) - e^{-u_1\omega} \int_\omega^\beta \chi(\tau)e^{-u_1\tau}d\tau$$
$$\leq \in \tag{20}$$

thus

$$-\in \leq g_1(\beta)e^{u_1(\omega-\beta)} - \in e^{u_1(\omega-\beta)} - g_1(\omega) - e^{u_1\omega} \int_\omega^\beta \chi(\tau)e^{-u_1\tau}d\tau$$
$$\leq \in \tag{21}$$

with respect to $\omega \in [\alpha, \beta]$. Let

$$\zeta(\omega) = g_1(\beta)e^{u_1(\omega-\beta)} - e^{u_1(\omega)} \int_\omega^\beta \chi(\tau)e^{-u_1\tau}d\tau,$$

then $\zeta(\omega)$ fulfilling $\zeta'(\omega) = u_1\zeta(\omega) + \chi(\omega)$ with respect to $\omega \in [\alpha, \beta]$, then the satisfies inequality that

$$|\zeta(\omega) - g_1(\omega)| = |g_1(\beta)e^{u_1(\omega-\beta)} - g_1(\omega) - e^{u_1\omega} \int_\omega^\beta \chi(\tau)e^{-u_1\tau}d\tau|$$
$$= e^{\rho\omega}| \int_\omega^\beta [e^{-u_1\tau}g_1(\tau)]^l d\tau - \int_\omega^\beta \chi(\tau)e^{-u_1\tau}d\tau|$$

$$\leq e^{\rho\omega} \int_{\omega}^{\beta} e^{-\rho\tau} |g_1'(\tau) - u_1 g_1(\tau) - \chi(\tau)| d\tau$$
$$\leq \in e^{\rho\omega} \int_{\omega}^{\beta} e^{-\rho\tau} d\tau \tag{22}$$

with respect to $\omega \in [\alpha, \beta]$. If $\rho \neq 0$, then

$$|\zeta(\omega) - g_1(\omega)| \leq \in e^{\rho\omega} \int_{\omega}^{\beta} e^{-\rho\tau} d\tau$$
$$\leq \frac{\in}{\rho} \left(1 - e^{-\rho(\beta-\alpha)}\right) \tag{23}$$

with respect to $\omega \in [\alpha, \beta]$. If $\rho = 0$, then

$$|\zeta(\omega) - g_1(\omega)| \leq \in e^{\rho\omega} \int_{\omega}^{\beta} e^{-\rho\tau} d\tau$$
$$\leq \in (\beta - \alpha) \tag{24}$$

with respect to $\omega \in [\alpha, \beta]$. Therefore

$$|\zeta(\omega) - g_1(\omega)| \leq \begin{cases} \frac{1-e^{-\rho(\beta-\alpha)}}{\rho}; & if \quad \rho \neq 0 \\ (\beta - \alpha) \in; & if \quad \rho = 0. \end{cases} \tag{25}$$

Theorem 1. *The differential equation*
$\gamma^{iv}(\omega) + \rho_1\gamma'''(\omega) + \rho_2\gamma''(\omega) + \rho_3\gamma'(\omega) + \rho_4\gamma(\omega) = \chi(\omega)$ *has the $H - U$ Stability, where* $\gamma \in c^4[\alpha, \beta]$ *and* $\chi \in [\alpha, \beta]$*. Therefore*

$$|\kappa(\omega) - h(\omega)| \leq \begin{cases} \frac{\left(1-e^{-\gamma(\beta-\alpha)}\right)\left(1-e^{-\rho(\beta-\alpha)}\right)\in}{\gamma\rho}; & if \quad \rho, \gamma \neq 0 \\ \frac{1-e^{-\gamma(\beta-\alpha)}(\beta-\alpha)\in}{\gamma}; & if \quad \rho \neq 0, \gamma \neq 0 \\ \frac{1-e^{-\rho(\beta-\alpha)}(\beta-\alpha)\in}{\rho}; & if \quad \rho \neq 0, \gamma = 0 \\ (\beta - \alpha)^2 \in; & if \quad \rho = 0, \gamma = 0 \end{cases}$$

with respect to $\omega \in [\alpha, \beta]$.

Proof. Similar to the proof of Lemma 1. Let $H(\omega) = \gamma'(\omega) - u_2\gamma(\omega)$ by $H'(\omega) = \gamma''(\omega) - u_1\gamma'(\omega)$ and let $\in > 0; \gamma \in c^4[\alpha, \beta]$.
Also

$$|H'(\omega) - u_4 H(\omega) - \zeta(\omega)| = |\zeta(\omega) - g(\omega)| \tag{26}$$

with respect to $\omega \in [\alpha, \beta]$. Thus

$$|H'(\omega) - u_4 H(\omega) - \zeta(\omega)| \leq \in \tag{27}$$

with respect to $\omega \in [\alpha, \beta]$. Equivalently H fulfilling

$$|H'(\omega) - u_4 H(\omega) - \zeta(\omega)| = |\gamma^{''}(\omega) - (u_1 + u_4)\gamma'(\omega) + u_1 u_4 \gamma(\omega) - \zeta(\omega)|$$
$$= |\gamma''(\omega) + \rho_1 \gamma'(\omega) + \rho_2 \gamma(\omega) - \zeta(\omega)| < \in \tag{28}$$

with respect to $\omega \in [\alpha, \beta]$. Multiplying 28 by $e^{-u_4(\omega-\alpha)}$ on both sides, shown up

$$- \in e^{-u_4(\omega-\alpha)} \leq H'(\omega) e^{-u_4(\omega-\alpha)} - u_4 H(\omega) e^{-u_4(\omega-\alpha)} - \zeta(\omega) e^{-u_4(\omega-\alpha)}$$
$$\leq \in e^{-u_4(\omega-\alpha)} \tag{29}$$

with respect to $\omega \in [\alpha, \beta]$. Without loss of all inclusive statement we may accept that $u_4 > 1$, thus

$$u_4 \in e^{-u_4(\omega-\alpha)} \leq H'(\omega) e^{-u_4(\omega-\alpha)} - H(\omega) e^{-u_4(\omega-\alpha)} - \zeta(\omega) e^{-u_4(\omega-\alpha)}$$
$$\in u_4 e^{-u_4(\omega-\alpha)} \tag{30}$$

with respect to $\omega \in [\alpha, \beta]$, integrating 30 from ω to β, we achieve

$$- \in \left(e^{-u_4(\omega-\alpha)} - e^{-u_4(\beta-\alpha)}\right) \leq H(\beta) e^{-u_4(\beta-\alpha)} - H(\omega) e^{-u_4(\omega-\alpha)} - \int_\omega^\beta \zeta(\tau) e^{-u_4(\omega-\alpha)} d\tau$$
$$\leq \in \left(e^{-u_4(\omega-\alpha)} - e^{-u_4(\beta-\alpha)}\right) \tag{31}$$

with respect to $\omega \in [\alpha, \beta]$. It follows from 31, we get

$$- \in e^{-u_4(\omega-\alpha)} \leq H(\beta) e^{-u_4(\beta-\alpha)} - \in e^{-u_4(\beta-\alpha)} - H(\omega) e^{-u_4(\omega-\alpha)} - \int_\omega^\beta \zeta(\tau) e^{-u_4(\omega-\alpha)} d\tau$$
$$\leq \in \left(e^{-u_4(\omega-\alpha)}\right) \tag{32}$$

with respect to $\omega \in [\alpha, \beta]$. Multiplying the formula by the function $e^{-u_4(\omega-\alpha)}$ in 32, we get

$$- \in \leq H(\beta) e^{-u_4(\beta-\omega)} - \in e^{-u_4(\beta-\omega)} - H(\omega) - e^{u_4 \omega} \int_\omega^\beta \zeta(\tau) e^{u_4 \tau} d\tau$$
$$\leq \in \tag{33}$$

with respect to $\omega \in [\alpha, \beta]$. It follows from 33, we get

$$- \in \leq H(\beta) e^{u_4(\Omega-\beta)} - \in e^{u_4(\omega-\beta)} - H(\omega) - e^{u_4 \omega} \int_\omega^\beta \zeta(\tau) e^{u_4 \tau} d\tau$$
$$\leq \in \tag{34}$$

with respect to $\omega \in [\alpha, \beta]$. Let $\kappa(\omega) = H(\beta) e^{-u_4(\omega-\beta)} - e^{u_4 \omega} \int_\omega^\beta \zeta(\tau) e^{-u_4 \tau} d\tau$. with respect to $\omega \in [\alpha, \beta]$. Then

$$\kappa'(\omega) - u_4 \kappa(\omega) - \zeta(\omega) = 0 \quad by$$

$$\kappa'(\omega) = u_4\kappa(\omega) + \zeta(\omega).$$

Thus

$$\begin{aligned}
|\kappa(\omega) - H(\omega)| &= e^{u_4(\omega-\beta)}H(\beta) - H(\omega) - e^{u_4\omega}\int_{\alpha}^{\beta}\zeta(\tau)e^{-u_4\tau}d\tau \\
&= e^{\gamma\omega}|\int_{\alpha}^{\beta}\left[e^{-u_4\tau}H(\tau)-\right] - \int_{\alpha}^{\beta}\zeta(\tau)e^{-u_4\tau}d\tau| \\
&\leq e^{\gamma\omega}\int_{\omega}^{\beta}|e^{-u_4\tau}||H'(\tau) - u_4H(\tau) - \zeta(t)|d\tau \\
&\leq e^{\gamma\omega}\int_{\omega}^{\beta}e^{-\gamma\tau}|H'(\tau) - u_4H(\tau) - \zeta(t)|d\tau \\
|\kappa(\omega) - H(\omega)| &\leq \in e^{\gamma\omega}\int_{\omega}^{\beta}e^{-\gamma\tau}d\tau \qquad (35)
\end{aligned}$$

with respect to $\omega \in [\alpha, \beta]$. If $\gamma \neq 0$, then

$$\begin{aligned}
|\kappa(\omega) - H(\omega)| &\leq \in e^{\gamma\omega}\int_{\omega}^{\beta}e^{-\gamma\tau}d\tau \\
&\leq \frac{\in}{\gamma}[1 - e^{-\gamma(\beta-\omega)}] \\
|\kappa(\omega) - H(\omega)| &\leq \frac{\in}{\gamma}[1 - e^{-\gamma(\beta-\alpha)}] \qquad (36)
\end{aligned}$$

with respect to $\omega \in [\alpha, \beta]$. If $\gamma = 0$, then

$$|\kappa(\omega) - H(\omega)| \leq \in (\beta - \alpha) \qquad (37)$$

with respect to $\omega \in [\alpha, \beta]$. If follows from 25, shown up

$$|\kappa(\omega) - H(\omega)| \leq \begin{cases} \frac{\left(1-e^{-\gamma(\beta-\alpha)}\right)\left(1-e^{-\rho(\beta-\alpha)}\right)\in}{\gamma\rho}; & if \quad \rho, \gamma \neq 0 \\ \frac{1-e^{-\gamma(\beta-\alpha)}(\beta-\alpha)\in}{\gamma}; & if \quad \rho = 0, \gamma \neq 0 \\ \frac{1-e^{-\rho(\beta-\alpha)}(\beta-\alpha)\in}{\rho}; & if \quad \rho \neq 0, \gamma = 0 \\ (\beta-\alpha)^2 \in; & if \quad \rho = 0, \gamma = 0 \end{cases} \qquad (38)$$

with respect to $\omega \in [\alpha, \beta]$.

Theorem 2. *The DE* $\gamma^{iv}(\omega) + \rho_1\gamma'''(\omega) + \rho_2\gamma''(\omega) + \rho_3\gamma'(\omega) + \rho_4\gamma(\omega) = \chi(\omega)$ *has the Hyers Ulam Stability, where* $\gamma \in c^4[\alpha, \beta]$ *and with respect to* $\omega \in [\alpha, \beta]$, $|u(\omega) - \gamma(\omega)| \leq T$

where

$$T = \begin{cases} \frac{\left(1-e^{-\gamma(\beta-\alpha)}\right)\left(1-e^{-\rho(\beta-\alpha)}\right)\left(1-e^{-\sigma(\beta-\alpha)}\right)\in}{\gamma\rho\sigma}; & if \quad (\rho,\gamma,\sigma) \neq 0 \\ \frac{\left(1-e^{-\gamma(\beta-\alpha)}\right)\left(1-e^{-\rho(\beta-\alpha)}\right)(\beta-\alpha)\in}{\gamma\rho}; & if \quad \sigma = 0; (\rho,\gamma) \neq 0 \\ \frac{\left(1-e^{-\gamma(\beta-\alpha)}\right)\left(1-e^{-\sigma(\beta-\alpha)}\right)(\beta-\alpha)\in}{\sigma\gamma}; & if \quad \rho = 0; (\sigma,\gamma) \neq 0 \\ \frac{\left(1-e^{-\rho(\beta-\alpha)}\right)\left(1-e^{-\sigma(\beta-\alpha)}\right)(\beta-\alpha)\in}{\rho\sigma}; & if \quad \gamma = 0; (\rho,\sigma) \neq 0 \\ \frac{\left(1-e^{-\rho(\beta-\alpha)}\right)(\beta-\alpha)^2\in}{\rho}; & if \quad (\sigma,\gamma) = 0; \rho \neq 0 \\ \frac{\left(1-e^{-\sigma(\beta-\alpha)}\right)(\beta-\alpha)^2\in}{\sigma}; & if \quad (\rho,\gamma) = 0; \sigma \neq 0 \\ \frac{\left(1-e^{-\gamma(\beta-\alpha)}\right)(\beta-\alpha)^2\in}{\gamma}; & if \quad (\rho,\sigma) = 0; \gamma \neq 0 \\ (\beta-\alpha)^3 \in; & if \quad (\rho,\sigma,\gamma) = 0 \end{cases}$$

with respect to $\omega \in [\alpha, \beta]$.

Proof. If follows from Theorem 1, let us choose

$$\gamma(\omega) = u''{}_3(\omega) + (u_2 + \rho_1)u_3'(\omega) + (u_2^2 + \rho_1 u_2 + \rho_2)u_2(\omega)$$

by

$$\begin{aligned} \gamma'(\omega) = u'''{}_3(\omega) + (u_2 + \rho_1)u''{}_3(\omega) + (u_2^2 + \rho_1 u_2 + \rho_2)u_3'(\omega) \\ + (u_2^3 + \rho_1 u_2^2 + \rho_2 u_2 + \rho_3)u_2(\omega). \end{aligned}$$

Then

$$\begin{aligned} |\gamma'(\omega) - u_2\gamma(\omega) - \kappa(\omega)| &= |u'''{}_3(\omega) + (u_2 + \rho_1)u''{}_3(\omega) + (u_2^2 + \rho_1 u_2 + \rho_2)u_3'(\omega) \\ &\quad + (u_2^3 + \rho_1 u_2^2 + \rho_2 u_2 + \rho_3)u_3(\omega) - u_2(u_3''(\omega) \\ &\quad + (u_2 + \rho_1)u_3'(\omega) + (u_2^2 + \rho_1 u_2 + \rho_2)u_3(\omega) - \kappa(\omega)| \\ &= |u_3'''(\omega) + \rho_1 u_2''(\omega) + \rho_2 u_2'(\omega) + \rho_3 u + 3(\omega) - \kappa(\omega)| \\ &\leq \in \end{aligned}$$

with respect to $\omega \in [\alpha, \beta]$. So we have

$$|\gamma'(\omega) - u_2\gamma(\omega) - \kappa(\omega)| \leq \in \tag{39}$$

with respect to $\omega \in [\alpha, \beta]$. Equivalently γ fulfilling

$$- \in \leq \gamma'(\omega) - u_2\gamma(\omega) - \kappa(\omega) \leq \in \tag{40}$$

with respect to $\omega \in [\alpha, \beta]$. Multiplying the condition by the function $e^{-u_3(\omega-\alpha)}$

$$- \in e^{-u_3(\omega-\alpha)} \leq \gamma'(\omega)e^{-u_3(\omega-\alpha)} - u_2\gamma(\omega)e^{-u_3(\omega-\alpha)} - \kappa(\omega)e^{-u_3(\omega-\alpha)} \leq \in e^{-u_3(\omega-\alpha)} \tag{41}$$

with respect to $\omega \in [\alpha, \beta]$. Without loss of inclusive statement we may accept that $u_3 > 1$. Then

$$-u_3 \in e^{-u_3(\omega-\alpha)} \leq \gamma'(\omega)e^{-u_3(\omega-\alpha)} - u_3\gamma(\omega)e^{-u_3(\omega-\alpha)} - \kappa(\omega)e^{-u_3(\omega-\alpha)} \leq \in e^{-u_3(\omega-\alpha)} \tag{42}$$

with respect to $\omega \in [\alpha, \beta]$. Integrating 42 from ω to β, we get

$$\begin{aligned} - \in (e^{-u_3(\omega-\alpha)} - e^{-u_3(\beta-\alpha)}) \leq e^{-u_3(\beta-\alpha)}\gamma(\alpha) - \gamma(\omega)e^{-u_3(\omega-\alpha)} - \int_\omega^\beta \kappa(\tau)e^{-u_3(\tau-\alpha)}d\tau \\ \leq \in (e^{-u_3(\omega-\alpha)} - e^{-u_3(\beta-\alpha)}) \end{aligned} \tag{43}$$

with respect to $\omega \in [\alpha, \beta]$. If follows from 43, shown up

$$\begin{aligned} - \in e^{-u_3(\omega-\alpha)} \leq e^{-u_3(\beta-\alpha)}\gamma(\alpha) - \in e^{-u_3(\beta-\alpha)} - \gamma(\omega)e^{-u_3(\omega-\alpha)} - \int_\omega^\beta \kappa(\tau)e^{-u_3(\tau-\alpha)}d\tau \\ \leq \in e^{-u_3(\omega-\alpha)} \end{aligned} \tag{44}$$

with respect to $\omega \in [\alpha, \beta]$. Again multiplying the condition by function $e^{-u_3(\omega-\alpha)}$ that

$$\begin{aligned} - \in \leq e^{-u_3(\beta-\alpha)}\gamma(\alpha) - \in e^{-u_3(\beta-\omega)} - \gamma(\omega) - \int_\omega^\beta \kappa(\tau)e^{-u_3(\tau-\alpha)}d\tau \\ \leq \in \end{aligned} \tag{45}$$

with respect to $\omega \in [\alpha, \beta]$. From 45 that

$$\begin{aligned} - \in \leq e^{-u_3(\omega-\beta)}\gamma(\alpha) - \in e^{-u_3(\beta-\alpha)} - \gamma(\omega) - e^{u_3\omega} \int_\omega^\beta \kappa(\tau)e^{-u_3(\tau-\alpha)}d\tau \\ \leq \in \end{aligned} \tag{46}$$

for all $\omega \in [\alpha, \beta]$. Let $u_2(\omega) = \gamma(\beta)e^{-u_3(\omega-\beta)} - e^{u_3\omega} \int_\omega^\beta \kappa(\tau)e^{-u_3(\tau-\alpha)}d\tau$, then $u_2'(\omega) - u_3u_2(\omega) - \kappa(\omega) = 0$ by $u_2'(\omega) = u_3u_2(\omega) + \kappa(\omega)$, for all $\omega \in [\alpha, \beta]$. Thus

$$\begin{aligned} |u_2(\omega) - \gamma(\omega)| &= |\gamma(\beta)e^{-u_3(\omega-\beta)} - \gamma(\omega) - e^{u_3\omega} \int_\omega^\beta \kappa(\tau)e^{-u_3(\tau-\alpha)}d\tau| \\ &\leq e^{\sigma\omega} \int_\omega^\beta e^{-\sigma\tau}|\gamma'(\tau) - u_3\gamma(\tau) - \kappa(\tau)|d\tau \\ |u_2(\omega) - \gamma(\omega)| &\leq \in e^{\sigma\omega} \int_\omega^\beta e^{-\sigma\tau}d\tau \end{aligned} \tag{47}$$

with respect to $\omega \in [\alpha, \beta]$. If $\sigma \neq 0$, then

$$|u_2(\omega) - \gamma(\omega)| \leq \frac{\in}{\sigma}(1 - e^{-\sigma(\beta-\omega)})$$

$$\leq \frac{\in}{\sigma}(1 - e^{-\sigma(\beta-\alpha)}) \tag{48}$$

with respect to $\omega \in [\alpha, \beta]$. If $\sigma = 0$, then

$$\begin{aligned} |u_2(\omega) - \gamma(\omega)| &\leq \in e^{\sigma\omega} \int_{\omega}^{\beta} e^{-\sigma\tau} d\tau \\ &\leq \in (\beta - \omega) \\ &\leq \in (\beta - \alpha) \end{aligned} \tag{49}$$

with respect to $\omega \in [\alpha, \beta]$. t follows from 49, then
$|u(\omega) - \gamma(\omega)| \leq T$, where

$$T = \begin{cases} \frac{(1-e^{-\gamma(\beta-\alpha)})(1-e^{-\rho(\beta-\alpha)})(1-e^{-\sigma(\beta-\alpha)})\in}{\gamma\rho\sigma}; & if \ \ (\rho,\gamma,\sigma) \neq 0 \\ \frac{(1-e^{-\gamma(\beta-\alpha)})(1-e^{-\rho(\beta-\alpha)})(\beta-\alpha)\in}{\gamma\rho}; & if \ \ \sigma = 0; (\rho,\gamma) \neq 0 \\ \frac{(1-e^{-\gamma(\beta-\alpha)})(1-e^{-\sigma(\beta-\alpha)})(\beta-\alpha)\in}{\sigma\gamma}; & if \ \ \rho = 0; (\sigma,\gamma) \neq 0 \\ \frac{(1-e^{-\rho(\beta-\alpha)})(1-e^{-\sigma(\beta-\alpha)})(\beta-\alpha)\in}{\rho\sigma}; & if \ \ \gamma = 0; (\rho,\sigma) \neq 0 \\ \frac{(1-e^{-\rho(\beta-\alpha)})(\beta-\alpha)^2\in}{\rho}; & if \ \ (\sigma,\gamma) = 0; \rho \neq 0 \\ \frac{(1-e^{-\sigma(\beta-\alpha)})(\beta-\alpha)^2\in}{\sigma}; & if \ \ (\rho,\gamma) = 0; \sigma \neq 0 \\ \frac{(1-e^{-\gamma(\beta-\alpha)})(\beta-\alpha)^2\in}{\gamma}; & if \ \ (\rho,\sigma) = 0; \gamma \neq 0 \\ (\beta-\alpha)^3 \in; & if \ \ (\rho,\sigma,\gamma) = 0 \end{cases} \tag{50}$$

with respect to $\omega \in [\alpha, \beta]$.

Theorem 3. *The differential equation*
$\gamma^{iv}(\omega) + \rho_1\gamma'''(\omega) + \rho_2\gamma''(\omega) + \rho_3\gamma'(\omega) + \rho_4\gamma(\omega) = \chi(\omega)$ *has the Hyers Ulam Stability, where* $\gamma \in c^4[\alpha, \beta]$ *and with respect to* $\omega \in [\alpha, \beta]$*, at the point*

$|\chi(\omega) - \zeta(\omega)| \leq \Theta$,*where*

$$\Theta = \begin{cases} \frac{\left(1-e^{-\gamma(\beta-\alpha)}\right)\left(1-e^{-\rho(\beta-\alpha)}\right)\left(1-e^{-\sigma(\beta-\alpha)}\right)\left(1-e^{-\eta(\beta-\alpha)}\right)}{\gamma\rho\sigma\eta} \in; & if \quad (\rho,\gamma,\sigma,\eta) \neq 0 \\ \frac{\left(1-e^{-\gamma(\beta-\alpha)}\right)\left(1-e^{-\rho(\beta-\alpha)}\right)\left(1-e^{-\sigma(\beta-\alpha)}\right)}{\gamma\rho\sigma}(\beta-\alpha) \in; & if \quad \rho \neq \gamma \neq \sigma \neq 0, \eta = 0 \\ \frac{\left(1-e^{-\gamma(\beta-\alpha)}\right)\left(1-e^{-\rho(\beta-\alpha)}\right)\left(1-e^{-\eta(\beta-\alpha)}\right)}{\gamma\rho\eta}(\beta-\alpha) \in; & if \quad \rho \neq \gamma \neq \eta \neq 0, \sigma = 0 \\ \frac{\left(1-e^{-\gamma(\beta-\alpha)}\right)\left(1-e^{-\eta(\beta-\alpha)}\right)\left(1-e^{-\sigma(\beta-\alpha)}\right)}{\gamma\eta\sigma}(\beta-\alpha) \in; & if \quad \eta \neq \gamma \neq \sigma \neq 0, \rho = 0 \\ \frac{\left(1-e^{-\eta(\beta-\alpha)}\right)\left(1-e^{-\rho(\beta-\alpha)}\right)\left(1-e^{-\sigma(\beta-\alpha)}\right)}{\eta\rho\sigma}(\beta-\alpha) \in; & if \quad \rho \neq \eta \neq \sigma \neq 0, \gamma = 0 \\ \frac{\left(1-e^{-\rho(\beta-\alpha)}\right)\left(1-e^{-\sigma(\beta-\alpha)}\right)}{\rho\sigma}(\beta-\alpha)^2 \in; & if \quad \rho \neq \sigma \neq 0, \gamma = \eta = 0 \\ \frac{\left(1-e^{-\rho(\beta-\alpha)}\right)\left(1-e^{-\gamma(\beta-\alpha)}\right)}{\rho\sigma}(\beta-\alpha)^2 \in; & if \quad \rho \neq \gamma \neq 0, \sigma = \eta = 0 \\ \frac{\left(1-e^{-\rho(\beta-\alpha)}\right)\left(1-e^{-\eta(\beta-\alpha)}\right)}{\rho\eta}(\beta-\alpha)^2 \in; & if \quad \rho \neq \eta \neq 0, \gamma = \sigma = 0 \\ \frac{\left(1-e^{-\sigma(\beta-\alpha)}\right)\left(1-e^{-\gamma(\beta-\alpha)}\right)}{\sigma\gamma}(\beta-\alpha)^2 \in; & if \quad \sigma \neq \gamma \neq 0, \rho = \eta = 0 \\ \frac{\left(1-e^{-\rho(\beta-\alpha)}\right)\left(1-e^{-\eta(\beta-\alpha)}\right)}{\rho\eta}(\beta-\alpha)^2 \in; & if \quad \rho \neq \eta \neq 0, \gamma = \sigma = 0 \\ \frac{\left(1-e^{-\gamma(\beta-\alpha)}\right)\left(1-e^{-\eta(\beta-\alpha)}\right)}{\gamma\eta}(\beta-\alpha)^2 \in; & if \quad \gamma \neq \eta \neq 0, \rho = \sigma = 0 \\ \frac{\left(1-e^{-\rho(\beta-\alpha)}\right)}{\rho}(\beta-\alpha)^3 \in; & if \quad \rho \neq 0, \sigma = \gamma = \eta = 0 \\ \frac{\left(1-e^{-\gamma(\beta-\alpha)}\right)}{\gamma}(\beta-\alpha)^3 \in; & if \quad \gamma \neq 0, \sigma = \rho = \eta = 0 \\ \frac{\left(1-e^{-\sigma(\beta-\alpha)}\right)}{\sigma}(\beta-\alpha)^3 \in; & if \quad \sigma \neq 0, \rho = \gamma = \eta = 0 \\ \frac{\left(1-e^{-\eta(\beta-\alpha)}\right)}{\eta}(\beta-\alpha)^3 \in; & if \quad \eta \neq 0, \sigma = \gamma = \rho = 0 \\ (\beta-\alpha)^4 \in; & if \quad \rho = \sigma = \gamma = \eta = 0 \end{cases}$$

with respect to $\omega \in [\alpha, \beta]$.

Proof. Like the verification of theorem 2. Let $\in > 0$ and $\gamma \in c^4[\alpha, \beta]$.
Allow us the consider

$$\zeta(\omega) = \gamma'''(\omega) + (u_2 + \rho_1)\gamma''(\omega) + (u_2^2 + \rho_1 u_2 + \rho_2)\gamma'(\omega) + (u_2^3 + \rho_1 u_2^2 + \rho_2 u_2 + \rho_3)\gamma(\omega),$$

we acquire

$$\begin{aligned} \zeta'(\omega) = \gamma^{iv}(\omega) + (u_2 + \rho_1)\gamma'''(\omega) + (u_2^2 + \rho_1 u_2 + \rho_3)\gamma''(\omega) \\ + (u_2^3 + \rho_1 u^2 + \rho_2 u_2 + \rho_3)\gamma'(\omega) + (u_2^4 + \rho_1 u_2^3 + \rho_2 u_2^2 + \rho_3 u_2 + \rho_4)\gamma(\omega) \end{aligned} \tag{51}$$

with respect to $\omega \in [\alpha, \beta]$, at the point

$$|\zeta'(\omega) - u_2\zeta(\omega) - H(\omega)| < \in \tag{52}$$

with respect to $\omega \in [\alpha, \beta]$. If follows from 51 that

$$|\zeta'(\omega) - u_2\zeta(\omega) - H(\omega)| = |\gamma^{iv}(\omega) + (u_2 + \rho_1)\gamma'''(\omega) + (u_2^2 + \rho_1 u_2 + \rho_3)\gamma''(\omega)$$

$$
\begin{aligned}
&+ (u_2^3 + \rho_1 u^2 + \rho_2 u_2 + \rho_3)\gamma'(\omega) \\
&+ (u_2^4 + \rho_1 u_2^3 + \rho_2 u_2^2 + \rho_3 u_2 + \rho_4)\gamma(\gamma) \\
&- u_2(\gamma'''(\omega) + (u_2 + \rho_1)\gamma''(\omega) \\
&+ (u_2^2 + \rho_1 u_2 + \rho_2)\gamma'(\omega) \\
&+ (u_2^3 + \rho_1 u_2^2 + \rho_2 u_2 + \rho_3)\gamma(\omega)) - H(\omega)| \\
= &|\gamma^{iv}(\omega) + \rho_1\gamma'''(\omega) + \rho_2\gamma''(\omega) + \rho_3\gamma'(\omega) + \rho_4\gamma(\omega) - H(\omega)| \\
\leq &\in .
\end{aligned}
$$

So

$$|\zeta'(\omega) - u_2\zeta(\omega) - H(\omega)| < \in$$

for all $\omega \in [\alpha, \beta]$. Equivalently ζ fulfilling

$$- \in \leq \zeta'(\omega) - u_2\zeta(\omega) - H(\omega) < \in \tag{53}$$

with respect to $\omega \in [\alpha, \beta]$. Multiplying the formula by the function $e^{-u_2(\omega-\alpha)}$ satisfies

$$- \in e^{-u_2(\omega-\alpha)} \leq \zeta'(\omega)e^{-u_2(\omega-\alpha)} - u_2\zeta(\omega)e^{-u_2(\omega-\alpha)} - H(\omega)e^{-u_2(\omega-\alpha)} \tag{54}$$

$$\leq \in e^{-u_2(\omega-\alpha)} \tag{55}$$

with respect to $\omega \in [\alpha, \beta]$. without loss of inclusive statement we may accept that $u_2 > 1$, at the point

$$
\begin{aligned}
- \in u_2 e^{-u_2(\omega-\alpha)} &\leq \zeta'(\omega)e^{-u_2(\omega-\alpha)} - u_2\zeta(\omega)e^{-u_2(\omega-\alpha)} - H(\omega)e^{-u_2(\omega-\alpha)} \\
&\leq \in u_2 e^{-u_2(\omega-\alpha)}
\end{aligned} \tag{56}
$$

for all $\omega \in [\alpha, \beta]$. Integrating 54 from ω to β, at the point

$$
\begin{aligned}
- \in \left(e^{-u_2(\omega-\alpha)} - e^{-u_2(\beta-\alpha)}\right) &\leq \zeta(\beta)e^{-u_2(\beta-\alpha)} - \zeta(\omega)e^{-u_2(\omega-\alpha)} - \int_\omega^\beta H(\tau)e^{-u_2(\tau-\alpha)}d\tau \\
&\leq \in \left(e^{-u_2(\omega-\alpha)} - e^{-u_2(\beta-\alpha)}\right)
\end{aligned} \tag{57}
$$

with respect to $\omega \in [\alpha, \beta]$, at the point 57 that

$$
\begin{aligned}
- \in \left(e^{-u_2(\omega-\alpha)}\right) &\leq \zeta(\beta)e^{-u_2(\beta-\alpha)} - \in e^{-u_2(\beta-\alpha)} - \zeta(\omega)e^{-u_2(\omega-\alpha)} - \int_\omega^\beta H(\tau)e^{-u_2(\tau-\alpha)}d\tau \\
&\leq \in \left(e^{-u_2(\omega-\alpha)}\right)
\end{aligned} \tag{58}
$$

for all $\omega \in [\alpha, \beta]$. Multiplying the formula by the function $e^{-u_2(\omega-\alpha)}$, we acquire

$$- \in \leq \zeta(\beta)e^{-u_2(\omega-\beta)} - \in e^{-u_2(\omega-\beta)} - \zeta(\omega) - e^{u_2\omega}\int_\omega^\beta H(\tau)e^{-u_2(\tau-\alpha)}d\tau$$

$$\leq \in \tag{59}$$

with respect to $\omega \in [\alpha, \beta]$.
Let $\chi(\omega) = \zeta(\beta)e^{-u_2(2H} - e^{u_2\omega}\int_{\omega}^{\beta} H(\tau)e^{-u_2(\tau-\alpha)}d\tau$, at the point $\chi(\omega)$ satisfies $\chi'(\omega) - u_2\chi(\omega) - H(\omega) = 0$ by

$$\chi'(\omega) = u_2\chi(\omega) + H(\omega) \tag{60}$$

with respect to $\omega \in [\alpha, \beta]$, at the point

$$\begin{aligned} |\chi(\omega) - \zeta(\omega)| &= |\zeta(\beta)e^{-u_2(\omega-\beta)} - \zeta(\omega) - e^{u_2\omega}\int_{\omega}^{\beta} H(\tau)e^{-u_2\tau}d\tau| \\ &\leq e^{\eta\omega}|\int_{\omega}^{\beta}[e^{-u_2\tau\zeta(\tau)}]" d\tau - \int_{\omega}^{\beta} H(\tau)e^{-u_2\tau}d\tau| \\ &\leq \in e^{\eta\omega}\int_{\omega}^{\beta} e^{-\eta\tau}d\tau \\ |\chi(\omega) - \zeta(\omega)| &\leq e^{\eta\omega}\int_{\omega}^{\beta} e^{-\eta\tau} \in d\tau \end{aligned} \tag{61}$$

with respect to $\omega \in [\alpha, \beta]$. If $\eta \neq 0$, at the point

$$\begin{aligned} |\chi(\omega) - \zeta(\omega)| &\leq \frac{\in}{\eta}(1 - e^{-\eta(\beta-\omega)} \\ &\leq \frac{\in}{\eta}(1 - e^{-\eta(\beta-\alpha)} \end{aligned}$$

with respect to $\omega \in [\alpha, \beta]$. If $\eta = 0$, then

$$\begin{aligned} |\chi(\omega) - \zeta(\omega)| &\leq \in (\beta - \omega) \\ &\leq \in (\beta - \alpha) \end{aligned}$$

with respect to $\omega \in [\alpha, \beta]$. If follows from 50, thus

$$|\chi(\omega) - \zeta(\omega)| \leq \Theta, \quad where \tag{62}$$

$$\Theta = \begin{cases} \frac{\left(1-e^{-\gamma(\beta-\alpha)}\right)\left(1-e^{-\rho(\beta-\alpha)}\right)\left(1-e^{-\sigma(\beta-\alpha)}\right)\left(1-e^{-\eta(\beta-\alpha)}\right)}{\gamma\rho\sigma\eta} \in; & if \;\; (\rho,\gamma,\sigma,\eta) \neq 0 \\ \frac{\left(1-e^{-\gamma(\beta-\alpha)}\right)\left(1-e^{-\rho(\beta-\alpha)}\right)\left(1-e^{-\sigma(\beta-\alpha)}\right)}{\gamma\rho\sigma}(\beta-\alpha) \in; & if \;\; \rho \neq \gamma \neq \sigma \neq 0, \eta = 0 \\ \frac{\left(1-e^{-\gamma(\beta-\alpha)}\right)\left(1-e^{-\rho(\beta-\alpha)}\right)\left(1-e^{-\eta(\beta-\alpha)}\right)}{\gamma\rho\eta}(\beta-\alpha) \in; & if \;\; \rho \neq \gamma \neq \eta \neq 0, \sigma = 0 \\ \frac{\left(1-e^{-\gamma(\beta-\alpha)}\right)\left(1-e^{-\eta(\beta-\alpha)}\right)\left(1-e^{-\sigma(\beta-\alpha)}\right)}{\gamma\eta\sigma}(\beta-\alpha) \in; & if \;\; \eta \neq \gamma \neq \sigma \neq 0, \rho = 0 \\ \frac{\left(1-e^{-\eta(\beta-\alpha)}\right)\left(1-e^{-\rho(\beta-\alpha)}\right)\left(1-e^{-\sigma(\beta-\alpha)}\right)}{\eta\rho\sigma}(\beta-\alpha) \in; & if \;\; \rho \neq \eta \neq \sigma \neq 0, \gamma = 0 \\ \frac{\left(1-e^{-\rho(\beta-\alpha)}\right)\left(1-e^{-\sigma(\beta-\alpha)}\right)}{\rho\sigma}(\beta-\alpha)^2 \in; & if \;\; \rho \neq \sigma \neq 0, \gamma = \eta = 0 \\ \frac{\left(1-e^{-\rho(\beta-\alpha)}\right)\left(1-e^{-\gamma(\beta-\alpha)}\right)}{\rho\sigma}(\beta-\alpha)^2 \in; & if \;\; \rho \neq \gamma \neq 0, \sigma = \eta = 0 \\ \frac{\left(1-e^{-\rho(\beta-\alpha)}\right)\left(1-e^{-\eta(\beta-\alpha)}\right)}{\rho\eta}(\beta-\alpha)^2 \in; & if \;\; \rho \neq \eta \neq 0, \gamma = \sigma = 0 \\ \frac{\left(1-e^{-\sigma(\beta-\alpha)}\right)\left(1-e^{-\gamma(\beta-\alpha)}\right)}{\sigma\gamma}(\beta-\alpha)^2 \in; & if \;\; \sigma \neq \gamma \neq 0, \rho = \eta = 0 \\ \frac{\left(1-e^{-\rho(\beta-\alpha)}\right)\left(1-e^{-\eta(\beta-\alpha)}\right)}{\rho\eta}(\beta-\alpha)^2 \in; & if \;\; \rho \neq \eta \neq 0, \gamma = \sigma = 0 \\ \frac{\left(1-e^{-\gamma(\beta-\alpha)}\right)\left(1-e^{-\eta(\beta-\alpha)}\right)}{\gamma\eta}(\beta-\alpha)^2 \in; & if \;\; \gamma \neq \eta \neq 0, \rho = \sigma = 0 \\ \frac{\left(1-e^{-\rho(\beta-\alpha)}\right)}{\rho}(\beta-\alpha)^3 \in; & if \;\; \rho \neq 0, \sigma = \gamma = \eta = 0 \\ \frac{\left(1-e^{-\gamma(\beta-\alpha)}\right)}{\gamma}(\beta-\alpha)^3 \in; & if \;\; \gamma \neq 0, \sigma = \rho = \eta = 0 \\ \frac{\left(1-e^{-\sigma(\beta-\alpha)}\right)}{\sigma}(\beta-\alpha)^3 \in; & if \;\; \sigma \neq 0, \rho = \gamma = \eta = 0 \\ \frac{\left(1-e^{-\eta(\beta-\alpha)}\right)}{\eta}(\beta-\alpha)^3 \in; & if \;\; \eta \neq 0, \sigma = \gamma = \rho = 0 \\ (\beta-\alpha)^4 \in; & if \;\; \rho = \sigma = \gamma = \eta = 0 \end{cases} \tag{63}$$

with respect to $\omega \in [\alpha, \beta]$, and $\alpha \neq 0, \beta \neq 0$.

3. Examples

Finally, we give some examples to illustrate the results in this paper.

Example 1. *Consider the following differential equation of the form* $\sigma^{iv}(\omega) + 2\sigma'''(\omega) + \sigma''(\omega) = \chi(\omega); \omega \in [2,3]$. *Set* $\in > 0$*, at the point*

$$|\sigma^{iv}(\omega) + 2\sigma'''(\omega) + \sigma''(\omega) - \chi(\omega)| \leq \in .$$

with respect to $\omega \in [2,3]$*. Let* $\lambda = 1$*, then*

$$g(\omega) = \sigma'''(\omega) + 3\sigma''(\omega) + 4\sigma'(\omega) + 4\sigma(\omega) \quad and$$
$$g'(\omega) = \sigma^{iv}(\omega) + 3\sigma'''(\omega) + 4\sigma''(\omega) + 4\sigma'(\omega) + 4\sigma(\omega)$$

with respect to $\omega \in [2,3]$*. Thus the condition 25, 27 and 39 of Theorem 3 are satisfied. Hence there is a function* $\omega \in c^4[2,3]$ *which is a mild solution of* $u^{iv}(\omega) + 2u'''(\omega) + u''(\omega) = \chi(\omega)$ *is satisfied by 63.*

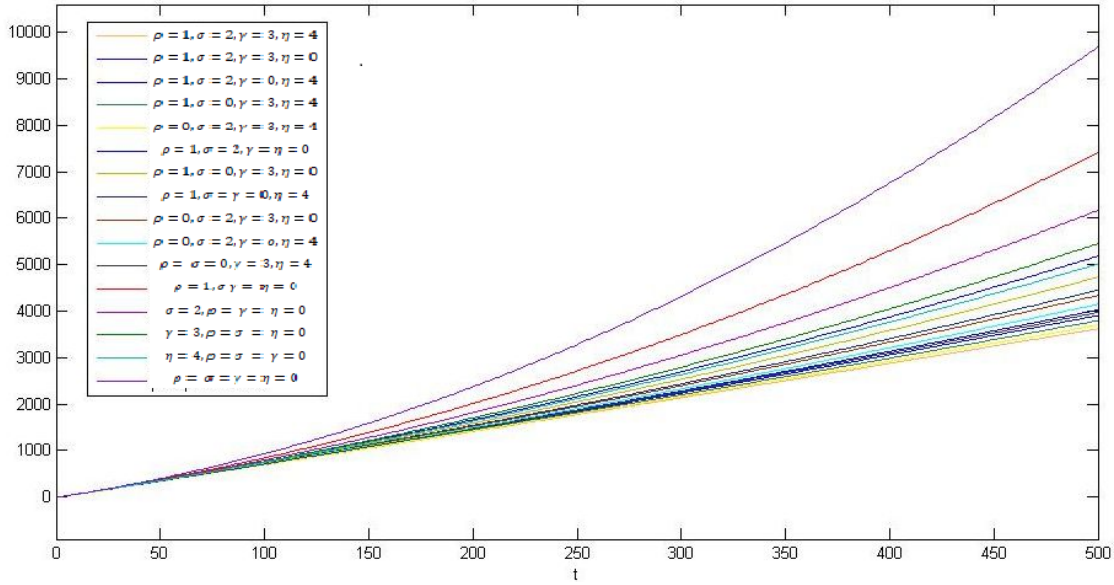

Figure 1: Graph solution $\chi(\omega)$ and $\zeta(\omega)$ for Equation 63

Example 2. *Consider the accompanying differential equation* $\sigma^{iv}(\omega) + \sigma'''(\omega) + \sigma''(\omega) = \chi(\omega); \omega \in [3, 2]$.
Let $\in > 0$, *and* $\gamma \in [3, 2]$. *such that*

$$|\sigma^{iv}(\omega) + \sigma'''(\omega) + \sigma''(\omega) - \chi(\omega)| \leq \in .$$

with respect to $\omega \in [3, 2]$. *we take*

$$g(\omega) = \sigma'''(\omega) + 2\sigma''(\omega) + 3\sigma'(\omega) + 3\sigma(\omega)$$

with respect to $\omega \in [3, 2]$. *Then*

$$g'(\omega) = \sigma^{iv}(\omega) + 2\sigma'''(\omega) + 3\sigma''(\omega) + 3\sigma'(\omega) + 3\sigma(\omega)$$

with respect to $\omega \in [3, 2]$. *At the point*

$$|g'(\omega) - g(\omega) - \chi(\omega)| = |\sigma^{iv}(\omega) + \sigma'''(\omega) + \sigma''(\omega) - \chi(\omega)| \leq \in$$

with respect to $\omega \in [\alpha, \beta]$. *Thus the conditions 25, 27 and 39 of Theorem 3 are satisfied. Subsequently there is a function* $\omega \in c^4[3, 2]$ *which is a mellow solution of* $u^{iv}(\omega) + u'''(\omega) + u''(\omega) = \chi(\omega)$ *is satisfied by 63.*

4. Conclusion

In this study, we have successfully demonstrated the Hyers-Ulam stability of fourth-order linear differential equations. By employing fixed-point theory and various norms,

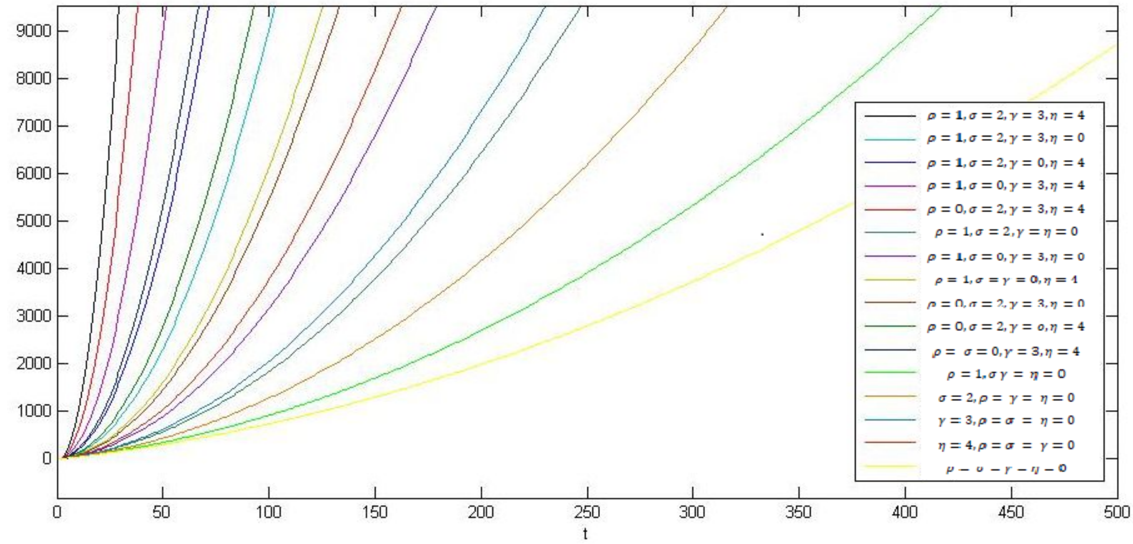

Figure 2: Plots of solution $\chi(\omega)$ and $\zeta(\omega)$ for Equation 63

we derived sufficient conditions that guarantee the stability of solutions to these higher-order equations under small perturbations. Our results show that for a wide class of fourth-order linear differential equations, if an approximate solution exists, there is a corresponding exact solution that is close to the approximate one, thereby confirming the equation's stability in the Hyers-Ulam sense. The extension of Hyers-Ulam stability to fourth-order equations enriches the understanding of the robustness of solutions in more complex systems, which is crucial in theoretical research as well as in practical applications in areas such as engineering, physics, and applied mathematics. The examples provided highlight the practical relevance of these theoretical findings, showcasing the broad applicability of Hyers-Ulam stability in various contexts. Future research can focus on extending these results to nonlinear or variable-coefficient systems, as well as exploring applications in more specialized fields.

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