EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 17, No. 4, 2024, 3093-3108
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Multivalued Almost JS-Contractions, Related Fixed Point Results in Complete $\mathfrak{b}$-Metric Spaces and Applications

Maroua Meneceur[1], Haitham Qawaqneh[2,*], Habes Alsamir[3], Gawhara Al-Musannef[4]

[1] *Department of Mathematics, Exact Sciences Faculty, University of El Oued, P.O.Box 789, El Oued 39000, Algeria*
[2] *Department of Mathematics,Faculty of Science and Information Technology, Al-Zaytoonah University of Jordan, Amman 11733, Jordan*
[3] *Finance and Banking Department, Business Administration College, Dar Aluloom University, riyadh, Saudi Arabia*
[3] *Faculty of business studies,Arab Open University, Jeddah, Saudi Arabia*

Abstract. In this paper, we introduce a new class of multivalued contractions and prove the existence of a fixed point for such contractions. Some consequences are presented in b-metric spaces endowed with partial order or with graph. To illustrate the applicability of our results, we offer an example and an application to the existence of solutions of an integral inclusions.

2020 Mathematics Subject Classifications: 47H10, 54H25

Key Words and Phrases: ϑ-contraction, fixed point, $\alpha_{\mathfrak{s}}$-admissible, Volterra integral inclusions.

1. Introduction and preliminaries

One of the key generalizations for metric spaces is the idea of a $\mathfrak{b}$-metric space. Bakhtin [4] first proposed the idea of building such spaces, and Czerwik[6] refined it. Several fixed-point results were provided in this way for single or set valued mappings, for instance, see[7, 13, 23, 24, 26]. One of the key generalizations for metric spaces is the idea of a $\mathfrak{b}$-metric space. Bakhtin [4] first proposed the idea of building such spaces, and Czerwik citesc1 refined it. Several fixed-point results were provided in this way for single or set valued mappings, for instance, see [7, 13, 23, 24, 26]. However, Samet et al. [27] presented the idea of α-admissible, and they established some results. Some results were reached by using this notion in conjunction with non-linear contractions; see [11, 12, 18, 23]. This

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v17i4.5466

Email addresses: chaimanoor9@gmail.com (M. Meneceur), h.alqawaqneh@zuj.edu.jo (H. Qawaqneh), habes@dau.edu.sa (H. Alsamir), G.almusannef@arabou.edu.sa (J.M. Al-musannef)

idea was later extended to α_s-admissible in the setting of $\mathfrak{b}$-metric spaces by Ali et al. [14]. Jleli and Samet [10] established a novel idea known as the ϑ-contraction and proved the existence of fixed points. Here, it is important to note that a contraction in in the sense of Banach, is a particular case of ϑ-contraction, while there are some ϑ-contractions which do not satisfy Banach contractive condition. Subsequently, several authors studied different variations of ϑ-contractions and other different contractions in single and set valued cases, for example, see [2, 3, 15–17, 19–22, 25, 28] In this work, we prove the existence of a fixed point for such a novel contraction type in complete $\mathfrak{b}$-metric spaces spaces by combining the notion of α_s-admissible mapping with ϑ-contraction in the case of multivalued mappings. In this work, we prove the existence of a fixed point for such a novel contraction type in complete $\mathfrak{b}$-metric spaces by combining the notion of α_s-admissible mapping with ϑ-contraction in the case of multivalued mappings. Using our major findings, we also infer the existence of fixed points in partially ordered metric spaces. Lastly, to demonstrate the applicability of our results, we offer an example and an application pertaining to an existence problem of solutions for a Volterra integral inclusion and for applications in factional equtions, see [1, 8]

Definition 1. *[14] Let* $\mathbf{X}$ *be a non-empty set and* $\mathfrak{s}$ *be a real number with* $\mathfrak{s} \geq 1$. *A function* $\mathfrak{d} : \mathbf{X} \times \mathbf{X} \to [0, \infty)$ *is a* $\mathfrak{b}$*-metric on* $\mathbf{X}$ *if for all* $\nu, \mu, \eta \in \mathbf{X}$*, it satisfies the following conditions:*

($\mathfrak{b}1$) $\mathfrak{d}(\nu, \mu) = 0$ *iff* $\nu = \mu$*,*

($\mathfrak{b}2$) $\mathfrak{d}(\nu, \mu) = \mathfrak{d}(\mu, \nu)$*,*

($\mathfrak{b}3$) $\mathfrak{d}(\nu, \eta) \leq \mathfrak{s}[\mathfrak{d}(\nu, \mu) + \mathfrak{d}(\mu, \eta)]$.

A triplet $(\mathbf{X}, \mathfrak{d}, \mathfrak{s})$ *is called a* $\mathfrak{b}$*-metric space.*

Every metric space is a $\mathfrak{b}$-metric space with $\mathfrak{s} = 1$.

Denote the family of non-empty, closed and bounded subsets of $\mathbf{X}$ by $\mathcal{CB}(\mathbf{X})$. For $\mathcal{A}, \mathcal{B} \in \mathcal{CB}(\mathbf{X})$, define $\mathcal{H}\colon \mathcal{CB}(\mathbf{X}) \times \mathcal{CB}(\mathbf{X}) \to [0, +\infty)$ by

$$\mathcal{H}(\mathcal{A}, \mathcal{B}) = \max\left\{\sup_{\mathfrak{a}\in\mathcal{A}} \mathfrak{d}(\mathfrak{a}, \mathcal{B}),\ \sup_{\mathfrak{b}\in\mathcal{B}} \mathfrak{d}(\mathfrak{b}, \mathcal{A})\right\}$$

where $\mathfrak{d}(\mathfrak{a}, \mathcal{B}) = \inf\{\mathfrak{d}(\mathfrak{b}, \nu)\colon\ \nu \in \mathcal{B}\}$. Such a function $\mathcal{H}$ is called the Pompeiu-Hausdorff metric induced by $\mathfrak{d}$, for more details, see [5]. Also, denote the family of non-empty and closed subsets of $\mathbf{X}$ by $\mathcal{CL}(\mathbf{X})$.

Lemma 1. *[14] Let* $(\mathbf{X}, \mathfrak{d}, \mathfrak{s})$ *be a* $\mathfrak{b}$*-metric space. The following properties are satisfied:*

1) $\mathfrak{d}(\nu, \mathcal{B}) \leq \mathfrak{d}(\nu, \mathfrak{b})$ *for all* $\nu \in \mathbf{X}$, $\mathfrak{b} \in \mathcal{B}$ *and* $\mathcal{B} \in \mathcal{CB}(\mathbf{X})$.

2) $\mathfrak{d}(\nu, \mathcal{B}) \leq \mathcal{H}(\mathcal{A}, \mathcal{B})$ *for all* $\nu \in \mathbf{X}$ *and* $\mathcal{A}, \mathcal{B} \in \mathcal{CB}(\mathbf{X})$.

3) $\mathfrak{d}(\nu, \mathcal{A}) \leq \mathfrak{s}(\mathfrak{d}(\nu, \mu) + \mathfrak{d}(\mu, \mathcal{B}))$ *for all* $\nu, \mu \in \mathbf{X}$ *and* $\mathcal{A}, \mathcal{B} \in \mathcal{CB}(\mathbf{X})$.

Lemma 2. *[6] Let* $(\mathbf{X}, \mathfrak{d}, \mathfrak{s})$ *be a* $\mathfrak{b}$*-metric space and* $\mathcal{A}, \mathcal{B} \in \mathcal{CL}(\mathbf{X})$ *with* $\mathcal{H}(\mathcal{A}, \mathcal{B}) > 0$. *Then, for each* $\mathfrak{b} \in \mathcal{B}$*, there exists* $\mathfrak{a} = \mathfrak{a}(\mathfrak{b}) \in \mathcal{A}$ *such that*

$$\mathfrak{d}(\mathfrak{a}, \mathfrak{b}) \leq \mathfrak{s}\mathcal{H}(\mathcal{A}, \mathcal{B}).$$

Definition 2. *[26] Consider a non-empty set* $\mathbf{X}$ *and two mappings* $\mathcal{T}:\mathbf{X}\to\mathbf{X}$ *and* $\alpha:\mathbf{X}\times\mathbf{X}\to[0,+\infty)$. *For a given real number* $\mathfrak{s}\geq 1$, $\mathcal{T}$ *is weak* α*-admissible of type* $\mathfrak{S}$ *if for* $\nu\in\mathbf{X}$ *and* $\alpha(\nu,\mathcal{T}\nu)\geq\mathfrak{s}$, *we have* $\alpha(\mathcal{T}\nu,\mathcal{T}\mathcal{T}\nu)\geq\mathfrak{s}$.

Definition 3. *[14] Let* $(\mathbf{X},\mathfrak{d},\mathfrak{s})$ *be a* $\mathfrak{b}$*-metric space. For a given function* $\alpha\colon\mathbf{X}\times\mathbf{X}\to[0,+\infty)$, *a multivalued mapping* $\mathcal{T}\colon\mathbf{X}\to\mathcal{CL}(\mathbf{X})$ *is*
(1) $\alpha_{\mathfrak{s}}$*-admissible, if for each* $\nu\in\mathbf{X}$ *and* $\mu\in\mathcal{T}\nu$ *with* $\alpha(\nu,\mu)\geq\mathfrak{s}^2$, *we have* $\alpha(\mu,\eta)\geq\mathfrak{s}^2$ *for each* $\eta\in\mathcal{T}\mu$.
(2) $\alpha^*_{\mathfrak{s}}$*-admissible, if for* $\nu,\mu\in\mathbf{X}$ *with* $\alpha(\nu,\mu)\geq\mathfrak{s}^2$ *we have* $\alpha^*(\mathcal{T}\nu,\mathcal{T}\mu)\geq\mathfrak{s}^2$, *where* $\alpha^*(\mathcal{T}\nu,\mathcal{T}\mu)=\inf\{\alpha(\mathfrak{a},\mathfrak{b}):\ \mathfrak{a}\in\mathcal{T}\nu,\mathfrak{b}\in\mathcal{T}\mu\}$.

Definition 4. *[9, 18] Let* $(\mathbf{X},\mathfrak{d})$ *be a metric space, and* $\mathcal{T}:\mathbf{X}\to\mathcal{CL}(\mathbf{X})$ *and* $\alpha:\mathbf{X}\times\mathbf{X}\to[0,+\infty)$ *be given maps. Then* $\mathcal{T}$ *is called an* $\alpha_{\mathfrak{s}}$*-lower semi-continuous if for* $\nu\in\mathbf{X}$ *and a sequence* $\{\nu_n\}$ *in* $\mathbf{X}$ *with* $\lim_{n\to\infty}\mathfrak{d}(\nu_n,\nu)=0$ *and* $\alpha(\nu_n,\nu_{n+1})\geq\mathfrak{s}^2$ *for all* $n\in\mathbb{N}$, *implies*

$$\liminf_{n\to\infty}\mathfrak{d}(\nu_n,\mathcal{T}\nu_n)\geq\mathfrak{d}(\nu,\mathcal{T}\nu).$$

Definition 5. *[10] Let* $\Theta_{\mathfrak{s}}$ *be the set of all functions* $\vartheta:(0,+\infty)\to(1,+\infty)$ *such that*

(ϑ_1) ϑ *is a strictly increasing function;*

(ϑ_2) *for each sequence* $\{\omega_n\}$ *of positive real numbers* $\lim_{n\to\infty}\vartheta(\omega_n)=1$ *iff* $\lim_{n\to\infty}\omega_n=0$*;*

(ϑ_3) *there exist* $\rho\in(0,1)$ *and* $\chi\in(0,+\infty]$ *such that* $\lim_{\omega\to0^+}\frac{\vartheta(\omega)-1}{\omega^\rho}=\chi$*;*

(ϑ_4) *for each sequence* $\{\omega_n\}$ *in* $\mathbb{R}_+$ *such that* $\vartheta(\mathfrak{s}\omega_n)\leq\Big[\vartheta(\omega_{n-1})\Big]^\rho$, *where* $\rho\in(0,1)$, *then* $\vartheta(\mathfrak{s}^n\omega_n)\leq\Big[\vartheta(\mathfrak{s}^{n-1}\omega_n)\Big]^\rho$.

Example 1. *The following functions* $\vartheta_i:(0,+\infty)\to(1,+\infty)$ *for* $i\in\{1,2,3,4\}$, *are the elements of* $\Theta_{\mathfrak{s}}$.

(i) $\vartheta_1(\omega)=e^\omega$*;*

(ii) $\vartheta_2(\omega)=e^{\omega e^\omega}$*;*

(iii) $\vartheta_3(\omega)=e^{\sqrt{\omega}}$*;*

(iv) $\vartheta_4(\omega)=e^{\sqrt{\omega}e^\omega}$.

2. Main results

We begin this section with the following definition.

Definition 6. *Let* $(\mathbf{X}, \mathfrak{d}, \mathfrak{s})$ *be a* $\mathfrak{b}$*-metric space and* $\alpha : \mathbf{X} \times \mathbf{X} \to [0, +\infty)$ *be given. A map* $\mathcal{T} : \mathbf{X} \to \mathcal{CL}(\mathbf{X})$ *is called multivalued almost* $(\alpha_{\mathfrak{s}}, \vartheta, \kappa)$*-contraction of Hardy-Rogers type if there exist* $\vartheta \in \Theta_{\mathfrak{s}}$, $\mathcal{L} \geq 0$ *and* $\kappa : (0, +\infty) \to [0, 1)$ *satisfies* $\lim_{\omega \to z^+} \sup \kappa(\omega) < 1$ *for all* $z \in (0, +\infty)$ *and non-negative real numbers* $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \mathfrak{a}_4, \mathfrak{a}_5$ *with* $\mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{a}_3 + 2\mathfrak{s}\mathfrak{a}_4 = 1$, *and* $\mathfrak{a}_3 \neq 1$ *such that*

$$\vartheta(\mathfrak{s}^3 \mathcal{H}(\mathcal{T}\nu, \mathcal{T}\mu)) \leq \Big[\vartheta(N_{\mathfrak{s}}(\nu, \mu))\Big]^{\kappa(\mathfrak{d}(\nu,\mu)} + \mathcal{L} \min\{\mathfrak{d}(\nu, \mathcal{T}\mu), \mathfrak{d}(\mu, \mathcal{T}\nu)\}, \tag{2.1}$$

for all $\nu, \mu \in \mathbf{X}$ *with* $\alpha(\nu, \mu) \geq \mathfrak{s}^2$ *and* $\mathcal{H}(\mathcal{T}\nu, \mathcal{T}\mu) > 0$ *where*

$$N_{\mathfrak{s}}(\nu, \mu) = \mathfrak{a}_1 \mathfrak{d}(\nu, \mu) + \mathfrak{a}_2 \mathfrak{d}(\nu, \mathcal{T}\nu) + \mathfrak{a}_3 \mathfrak{d}(\mu, \mathcal{T}\mu) + \mathfrak{a}_4 \mathfrak{d}(\nu, \mathcal{T}\mu) + \mathfrak{a}_5 \mathfrak{d}(\mu, \mathcal{T}\nu).$$

If $\alpha(\nu, \mu) = \mathfrak{s}^2$, $\mathcal{T}$ is said to be an almost (ϑ, κ)-contraction of Hardy-Rogers type.

Theorem 1. *Let* $(\mathbf{X}, \mathfrak{d}, \mathfrak{s})$ *be a complete* $\mathfrak{b}$*-metric space and* $\mathcal{T} : \mathbf{X} \to \mathcal{CB}(\mathbf{X})$ *be a multivalued almost* $(\alpha_{\mathfrak{s}}, \vartheta, \kappa)$*-contraction of Hardy-Rogers type. Assume that the following conditions are satisfied:*

(i) $\mathcal{T}$ *is* $\alpha_{\mathfrak{s}}$*-admissible;*

(ii) there exist $\nu_0 \in \mathbf{X}$ *and* $\nu_1 \in \mathcal{T}\nu_0$ *such that* $\alpha(\nu_0, \nu_1) \geq \mathfrak{s}^2$*;*

(iii) $\mathcal{T}$ *is* $\alpha_{\mathfrak{s}}$*-lower semi-continuous, or* $\mathbf{X}$ *is* $\alpha_{\mathfrak{s}}$*-regular, that is, for every sequence* $\{\nu_n\}$ *in* $\mathbf{X}$ *such that* $\nu_n \to \nu^* \in \mathbf{X}$ *and* $\alpha(\nu_n, \nu_{n+1}) \geq \mathfrak{s}^2$ *for all* $n \in \mathbb{N}$, *then* $\alpha(\nu_n, \nu^*) \geq \mathfrak{s}^2$, *for all* $n \in \mathbb{N}$.

Then $\mathcal{T}$ *has a fixed point.*

Proof. From the hypothesis (2), there exist $\nu_0 \in \mathbf{X}$ and $\nu_1 \in \mathcal{T}\nu_0$ such that $\alpha(\nu_0, \nu_1) \geq \mathfrak{s}^2$. If $\nu_0 = \nu_1$ or $\nu_1 \in \mathcal{T}\nu_1$, then ν_1 is a fixed point of $\mathcal{T}$ and the proof is completed. Assume that $\nu_0 \neq \nu_1$ and $\nu_1 \notin \mathcal{T}\nu_1$, then $\mathcal{H}(\mathcal{T}\nu_0, \mathcal{T}\nu_1) \geq \mathfrak{d}(\nu_1, \mathcal{T}\nu_1) > 0$. From Lemma 2, there exists $\nu_2 \in \mathcal{T}\nu_1$ such that

$$\mathfrak{d}(\nu_1, \nu_2) \leq \mathfrak{s}\mathcal{H}(\mathcal{T}\nu_0, \mathcal{T}\nu_1) \leq \mathfrak{s}^2 \mathcal{H}(\mathcal{T}\nu_0, \mathcal{T}\nu_1),$$

which implies

$$\mathfrak{s}\mathfrak{d}(\nu_1, \nu_2) \leq \mathfrak{s}^3 \mathcal{H}(\mathcal{T}\nu_0, \mathcal{T}\nu_1).$$

Since ϑ is strictly increasing, we get

$$\vartheta(\mathfrak{s}\mathfrak{d}(\nu_1, \nu_2)) \leq \vartheta(\mathfrak{s}^3 \mathcal{H}(\mathcal{T}\nu_0, \mathcal{T}\nu_1)).$$

Then by using (2.1) we get

$$\begin{aligned} \vartheta(\mathfrak{s}\mathfrak{d}(\nu_1, \nu_2)) &\leq \vartheta(\mathfrak{s}^3 \mathcal{H}(\mathcal{T}\nu_0, \mathcal{T}\nu_1)) \\ &\leq \Big[\vartheta(N_{\mathfrak{s}}(\nu_0, \nu_1))\Big]^{\kappa(\mathfrak{d}(\nu_0,\nu_1))} + L \min\{\mathfrak{d}(\nu_0, \mathcal{T}\nu_1), \mathfrak{d}(\nu_1, \mathcal{T}\nu_0)\} \end{aligned}$$

$$< [\vartheta(N_{\mathfrak{s}}(\nu_0,\nu_1))]^{\kappa(\mathfrak{d}(\nu_0,\nu_1))}$$
$$< \vartheta(N_{\mathfrak{s}}(\nu_0,\nu_1)),$$

which gives

$$\vartheta(\mathfrak{s}\mathfrak{d}(\nu_1,\nu_2)) < \vartheta(N_{\mathfrak{s}}(\nu_0,\nu_1)).$$

Since ϑ is increasing, we get

$$\mathfrak{s}\mathfrak{d}(\nu_1,\nu_2) < N_{\mathfrak{s}}(\nu_0,\nu_1),$$

where

$$\begin{aligned} N_{\mathfrak{s}}(\nu_0,\nu_1) &= \mathfrak{a}_1\mathfrak{d}(\nu_0,\nu_1) + \mathfrak{a}_2\mathfrak{d}(\nu_0,\mathcal{T}\nu_0) + \mathfrak{a}_3\mathfrak{d}(\nu_1,\mathcal{T}\nu_1) + \mathfrak{a}_4\mathfrak{d}(\nu_0,\mathcal{T}\nu_1) + \mathfrak{a}_5\mathfrak{d}(\nu_1,\mathcal{T}\nu_0) \\ &\le \mathfrak{a}_1\mathfrak{d}(\nu_0,\nu_1) + \mathfrak{a}_2\mathfrak{d}(\nu_0,\nu_1) + \mathfrak{a}_3\mathfrak{d}(\nu_1,\nu_2) + \mathfrak{a}_4\mathfrak{d}(\nu_0,\nu_2) \\ &\le \mathfrak{a}_1\mathfrak{d}(\nu_0,\nu_1) + \mathfrak{a}_2\mathfrak{d}(\nu_0,\nu_1) + \mathfrak{a}_3\mathfrak{d}(\nu_1,\nu_2) + \mathfrak{s}\mathfrak{a}_4(\mathfrak{d}(\nu_0,\nu_1) + \mathfrak{d}(\nu_1,\nu_2)) \\ &\le (\mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{s}\mathfrak{a}_4)\mathfrak{d}(\nu_0,\nu_1) + (\mathfrak{a}_3 + \mathfrak{s}\mathfrak{a}_4)\mathfrak{d}(\nu_1,\nu_2), \end{aligned}$$

which implies that

$$\mathfrak{d}(\nu_1,\nu_2) \le \mathfrak{s}\mathfrak{d}(\nu_1,\nu_2) \le (\mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{s}\mathfrak{a}_4)\mathfrak{d}(\nu_0,\nu_1) + (\mathfrak{a}_3 + \mathfrak{s}\mathfrak{a}_4)\mathfrak{d}(\nu_1,\nu_2).$$

Then,

$$\mathfrak{d}(\nu_1,\nu_2) \le \frac{\mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{s}\mathfrak{a}_4}{1 - \mathfrak{a}_3 - \mathfrak{s}\mathfrak{a}_4}\mathfrak{d}(\nu_0,\nu_1).$$

Since $\mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{a}_3 + 2\mathfrak{s}\mathfrak{a}_4 = 1$, we get

$$\mathfrak{d}(\nu_1,\nu_2) < \mathfrak{d}(\nu_0,\nu_1).$$

Thus,

$$\mathfrak{s}\mathfrak{d}(\nu_1,\nu_2) < (\mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{s}\mathfrak{a}_4)\mathfrak{d}(\nu_0,\nu_1) + (\mathfrak{a}_3 + \mathfrak{s}\mathfrak{a}_4)\mathfrak{d}(\nu_0,\nu_1) = \mathfrak{d}(\nu_0,\nu_1),$$

and so

$$\vartheta(\mathfrak{s}\mathfrak{d}(\nu_1,\nu_2)) \le \Big[\vartheta(\mathfrak{d}(\nu_0,\nu_1))\Big]^{\kappa(\mathfrak{d}(\nu_0,\nu_1))}.$$

Assume that $\nu_1 \neq \nu_2$, then $\nu_2 \notin \mathcal{T}\nu_2$ and $\mathfrak{d}(\nu_2,\mathcal{T}\nu_2) > 0$ so $\mathcal{H}(\mathcal{T}\nu_1,\mathcal{T}\nu_2) > 0$. From Lemma 2, there exists $\nu_3 \in \mathcal{T}\nu_2$ such that

$$\begin{aligned} \vartheta(\mathfrak{s}\mathfrak{d}(\nu_2,\nu_3)) &\le \vartheta(\mathfrak{s}^3\mathcal{H}(\mathcal{T}\nu_1,\mathcal{T}\nu_2)) \\ &\le \Big[\vartheta(N_{\mathfrak{s}}(\nu_1,\nu_2))\Big]^{\kappa(\mathfrak{d}(\nu_1,\nu_2))} + L\min\{\mathfrak{d}(\nu_1,\mathcal{T}\nu_2),\mathfrak{d}(\nu_2,\mathcal{T}\nu_1)\} \\ &< [\vartheta(N_{\mathfrak{s}}(\nu_1,\nu_2))]^{\kappa(\mathfrak{d}(\nu_1,\nu_2))} \\ &< \vartheta(N_{\mathfrak{s}}(\nu_1,\nu_2)). \end{aligned}$$

Then,

$$\vartheta(\mathfrak{s}\mathfrak{d}(\nu_2,\nu_3)) \le \vartheta(N_{\mathfrak{s}}(\nu_1,\nu_2)),$$

which gives

$$\mathfrak{s}\mathfrak{d}(\nu_2,\nu_3) < N_{\mathfrak{s}}(\nu_1,\nu_2),$$

where

$$\begin{aligned} N_{\mathfrak{s}}(\nu_1,\nu_2) &= \mathfrak{a}_1\mathfrak{d}(\nu_1,\nu_2) + \mathfrak{a}_2\mathfrak{d}(\nu_1,\mathcal{T}\nu_1) + \mathfrak{a}_3\mathfrak{d}(\nu_2,\mathcal{T}\nu_2) + \mathfrak{a}_4\mathfrak{d}(\nu_1,\mathcal{T}\nu_2) + \mathfrak{a}_5\mathfrak{d}(\nu_2,\mathcal{T}\nu_1). \\ &\le \mathfrak{a}_1\mathfrak{d}(\nu_1,\nu_2) + \mathfrak{a}_2\mathfrak{d}(\nu_1,\nu_2) + \mathfrak{a}_3\mathfrak{d}(\nu_2,\nu_3) + \mathfrak{s}\mathfrak{a}_4(\mathfrak{d}(\nu_1,\nu_2) + \mathfrak{d}(\nu_2,\nu_3)) \\ &\le (\mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{s}\mathfrak{a}_4)\mathfrak{d}(\nu_1,\nu_2) + (\mathfrak{a}_3 + \mathfrak{s}\mathfrak{a}_4)\mathfrak{d}(\nu_2,\nu_3). \end{aligned}$$

Hence,

$$\mathfrak{d}(\nu_2,\nu_3) \le \mathfrak{s}\mathfrak{d}(\nu_2,\nu_3) \le (\mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{s}\mathfrak{a}_4)\mathfrak{d}(\nu_1,\nu_2) + (\mathfrak{a}_3 + \mathfrak{s}\mathfrak{a}_4)\mathfrak{d}(\nu_2,\nu_3),$$

and so

$$\mathfrak{d}(\nu_2,\nu_3) \le \frac{\mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{s}\mathfrak{a}_4}{1 - \mathfrak{a}_3 - \mathfrak{s}\mathfrak{a}_4}\mathfrak{d}(\nu_1,\nu_2).$$

Since $\mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{a}_3 + 2\mathfrak{s}\mathfrak{a}_4 = 1$, we get

$$\mathfrak{d}(\nu_2,\nu_3) < \mathfrak{d}(\nu_1,\nu_2).$$

Then, we infer that

$$\vartheta(\mathfrak{s}\mathfrak{d}(\nu_2,\nu_3) \le \left[\vartheta(\mathfrak{d}(\nu_1,\nu_2))\right]^{\kappa(\mathfrak{d}(\nu_1,\nu_2))}.$$

By continuing in this manner, we construct a sequence $\{\nu_n\}$ in $\mathbf{X}$, if there exists n_0 such that $\nu_{n_0} = \nu_{n_0+1}$, or $\nu_{n_0+1} \in \mathcal{T}\nu_{n_0+1}$ then ν_{n_0+1} is fixed point. If $\nu_n \neq \nu_{n+1}$ and $\nu_{n+1} \notin \mathcal{T}\nu_{n+1}$, then $\mathcal{H}(\mathcal{T}\nu_n,\mathcal{T}\nu_{n+1}) > 0$. From Lemma 2, there exists $\nu_{n+1} \in \mathcal{T}\nu_n$ such that

$$\theta(\mathfrak{s}\mathfrak{d}(\nu_n,\nu_{n+1})) \le \left[\theta(\mathfrak{d}(\nu_{n-1},\nu_n))\right]^{\kappa(\mathfrak{d}(\nu_{n-1},\nu_n))}, \quad \text{for all } n \in \mathbb{N}. \tag{2.2}$$

It follows by (2.2) and (ϑ_4) that

$$\theta(\mathfrak{s}^n\mathfrak{d}(\nu_n,\nu_{n+1})) \le \left[\theta(\mathfrak{s}^{n-1}\mathfrak{d}(\nu_{n-1},\nu_n))\right]^{\kappa(\mathfrak{d}(\nu_{n-1},\nu_n))}, \quad \text{for all } n \in \mathbb{N}. \tag{2.3}$$

Since ϑ is increasing, then the sequence $\{\mathfrak{d}(\nu_n,\nu_{n+1})\}$ is decreasing and so convergent. By the property of κ, there exist $\delta \in (0,1)$ and $n_0 \in \mathbb{N}$ such that $\kappa(\mathfrak{d}(\nu_n,\nu_{n+1})) < \delta$, for all $n \ge n_0$. Thus, from (2.3), we deduce

$$\begin{aligned} 1 &< \vartheta(\mathfrak{s}^n\mathfrak{d}(\nu_n,\nu_{n+1})) \\ &\le \left[\vartheta(\mathfrak{s}^{n-1}\mathfrak{d}(\nu_{n-1},\nu_n))\right]^{\kappa(\mathfrak{d}(\nu_{n-1},\nu_n)} \\ &\le \left[\vartheta(\mathfrak{s}^{n-2}\mathfrak{d}(\nu_{n-2},\nu_{n-1}))\right]^{\kappa(\mathfrak{d}(\nu_{n-2},\nu_{n-1})\kappa(\mathfrak{d}(\nu_{n-1},\nu_n)} \\ &\vdots \\ &\le \left[\vartheta(\mathfrak{d}(\nu_0,\nu_1))\right]^{\delta^{n-n_0}}, \end{aligned} \tag{2.4}$$

for all $n \geq n_0$. On taking the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \vartheta(\mathfrak{s}^n \mathfrak{d}(\nu_n, \nu_{n+1})) = 1,$$

and from (ϑ_2),

$$\lim_{n\to\infty} \mathfrak{s}^n \mathfrak{d}(\nu_n, \nu_{n+1}) = 0.$$

Now, we prove $\{\nu_n\}$ is a Cauchy sequence, by (ϑ_3) there exist $\rho \in (0,1)$ and $\chi \in (0, +\infty]$ such that

$$\lim_{n\to\infty} \frac{\vartheta(\mathfrak{s}^n \mathfrak{d}(\nu_n, \nu_{n+1})) - 1}{(\mathfrak{s}^n \mathfrak{d}(\nu_n, \nu_{n+1}))^\rho} = \chi.$$

Take $\delta \in (0, \chi)$. By the definition of limit, there exists $n_1 \in \mathbb{N}$ such that

$$(\mathfrak{s}^n \mathfrak{d}(\nu_n, \nu_{n+1}))^\rho \leq \delta^{-1}[\theta(\mathfrak{s}^n \mathfrak{d}(\nu_n, \nu_{n+1})) - 1], \quad \text{for all } n \geq n_1.$$

Using (2.4) and the above inequality, we deduce

$$n(\mathfrak{s}^n \mathfrak{d}(\nu_n, \nu_{n+1}))^\rho \leq \delta^{-1} n([\vartheta(\mathfrak{d}(\nu_0, \nu_1))]^{\delta^{n-n_0}} - 1), \quad \text{for all } n \geq n_1.$$

This implies that

$$\lim_{n\to\infty} n(\mathfrak{s}^n \mathfrak{d}(\nu_n, \nu_{n+1}))^\rho = 0.$$

Thence, there exists $n_2 \in \mathbb{N}$ such that

$$\mathfrak{s}^n \mathfrak{d}(\nu_n, \nu_{n+1}) \leq \frac{1}{n^{\frac{1}{\rho}}}, \quad \text{for all } n \geq n_2. \tag{2.5}$$

Let $m > n \geq \max\{n_0, n_1, n_2\}$. Then, using the triangular inequality and (2.5), we have

$$\mathfrak{d}(\nu_n, \nu_m) \leq \sum_{j=n}^{m-1} \mathfrak{d}(\nu_j, \nu_{j+1}) \leq \sum_{j=n}^{m-1} \mathfrak{s}^n \mathfrak{d}(\nu_j, \nu_{j+1}) \leq \sum_{j=n}^{m-1} \frac{1}{j^{\frac{1}{p}}} \leq \sum_{j=n}^{\infty} \frac{1}{j^{\frac{1}{p}}} < \infty,$$

and so $\{\nu_n\}$ is a Cauchy sequence. Since $(\mathbf{X}, \mathfrak{d}, \mathfrak{s})$ is complete, so $\{\nu_n\}$ converges to some $\nu^* \in \mathbf{X}$.

If $\mathcal{T}$ is $\alpha_{\mathfrak{s}}$-lower semi-continuous, then for all $n \in \mathbb{N}$, we have

$$\mathfrak{d}(\nu_n, \mathcal{T}\nu_n) \leq \mathfrak{d}(\nu_n, \nu_{n+1}).$$

Passing to the limit, we get

$$\lim_{n\to\infty} \mathfrak{d}(\nu_n, \mathcal{T}\nu_n) = 0.$$

Then taken in the account $\mathcal{T}$ is $\alpha_{\mathfrak{s}}$-lower semi-continuous, we obtain

$$0 < \mathfrak{d}(\nu^*, \mathcal{T}\nu^*) \leq \liminf_{n\to\infty} \mathfrak{d}(\nu_n, \mathcal{T}\nu_n) = 0,$$

which gives $\mathfrak{d}(\nu^*, \mathcal{T}\nu^*) = 0$. Hence, ν^* is a fixed point of $\mathcal{T}$.
If $\mathbf{X}$ is $\alpha_{\mathfrak{s}}$-regular, so for every sequence $\{\nu_n\}$ converges to ν^* with $\alpha(\nu_n, \nu_{n+1}) \geq s^2$, then $\alpha(\nu_n, \nu^*) \geq \mathfrak{s}^2$ so $\mathfrak{d}(\nu_{n+1}, \mathcal{T}\nu^*) > 0$, which implies $H(\mathcal{T}\nu_n, \mathcal{T}\nu^*) > 0$, then by (2.1), we get

$$\begin{aligned} 1 < \vartheta(\mathfrak{s}d(\nu_{n+1}, \mathcal{T}\nu^*)) &\leq \vartheta(\mathfrak{s}^3\mathcal{H}(\mathcal{T}\nu_n, \mathcal{T}\nu^*) \\ &\leq [\vartheta(N_{\mathfrak{s}}(d(\nu_0, \nu_1)))]^{\delta^{n-n_0}} \\ &< [\vartheta(\mathfrak{d}(\nu_0, \nu_1))]^{\delta^{n-n_0}}. \end{aligned}$$

Passing to the limit, we have

$$\lim_{n\to\infty} \vartheta(\mathfrak{d}(\nu_{n+1}, \mathcal{T}\nu^*)) = 1,$$

then (ϑ_2) gives

$$\lim_{n\to\infty} \mathfrak{d}(\nu_{n+1}, \mathcal{T}\nu^*) = 0,$$

which implies $\mathfrak{d}(\nu^*, \mathcal{T}\nu^*) = 0$. Hence ν^* is a fixed point of $\mathcal{T}$.

Since each α_s^*-admissible mapping is also α_s-admissible, we obtain the following result.

Corollary 1. *Let $(\mathbf{X}, \mathfrak{d}, \mathfrak{s})$ be a complete $\mathfrak{b}$-metric space and $\mathcal{T} : \mathbf{X} \to \mathcal{CB}(\mathbf{X})$ be a multivalued almost $(\alpha_{\mathfrak{s}}, \vartheta, \kappa)$-contraction of Hardy-Rogers type. Assume that the following conditions are satisfied:*

(i) *$\mathcal{T}$ is an α_s^*-admissible;*

(ii) *there exist $\nu_0 \in \mathbf{X}$ and $\nu_1 \in \mathcal{T}\nu_0$ such that $\alpha(\nu_0, \nu_1) \geq \mathfrak{s}^2$;*

(iii) *$\mathcal{T}$ is $\alpha_{\mathfrak{s}}$-lower semi-continuous, or for every sequence $\{\nu_n\} \subset \mathbf{X}$ converges to some ν^* in $\mathbf{X}$ and $\alpha^*(\nu_n, \nu_{n+1}) \geq s^2$, for all $n \in \mathbb{N}$. Then $\alpha^*(\nu_n, \nu^*) \geq \mathfrak{s}^2$, for all $n \in \mathbb{N}$.*

Then $\mathcal{T}$ has a fixed point.

Corollary 2. *Let $(\mathbf{X}, \mathfrak{d}, \mathfrak{s})$ be a complete $\mathfrak{b}$-metric space, $\alpha\colon \mathbf{X}\times\mathbf{X} \to [0, +\infty)$ be a function and $\mathcal{T}\colon \mathbf{X} \to \mathcal{CB}(\mathbf{X})$ be a multivalued mapping. Assume that the following conditions are satisfied:*

(i) *$\mathcal{T}$ is an $\alpha_{\mathfrak{s}}$-admissible;*

(ii) *there exist $\nu_0 \in \mathbf{X}$ and $\nu_1 \in \mathcal{T}\nu_0$ such that $\alpha(\nu_0, \nu_1) \geq \mathfrak{s}^2$;*

(iii) *$\mathcal{T}$ is $\alpha_{\mathfrak{s}}$-lower semi-continuous, or $\mathbf{X}$ is $\alpha_{\mathfrak{s}}$-regular;*

(iv) *there exist $\vartheta \in \Theta_{\mathfrak{s}}$, $\mathcal{L} \geq 0$ and $\kappa : (0, +\infty) \to [0, 1)$ satisfies $\lim_{\omega\to z^+} \sup \kappa(\omega) < 1$ for all $z \in (0, +\infty)$ and nonnegative real numbers $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \mathfrak{a}_4, \mathfrak{a}_5$ with $\mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{a}_3 + 2\mathfrak{s}\mathfrak{a}_4 = 1$, and $\mathfrak{a}_3 \neq 1$ such that*

$$\vartheta(\mathfrak{s}^3\alpha(\nu, \mu)H(\mathcal{T}\nu, \mathcal{T}\mu)) \leq \left[\vartheta(N_{\mathfrak{s}}(\nu, \mu))))\right]^{\kappa(\mathfrak{d}(\nu,\mu)} + L\min\{\mathfrak{d}(\nu, \mathcal{T}\mu), \mathfrak{d}(\mu, \mathcal{T}\nu)\},$$

for all $\nu, \mu \in \mathbf{X}$ with $\mathcal{H}(\mathcal{T}\nu, \mathcal{T}\mu) > 0$, where

$$N_{\mathfrak{s}}(\nu, \mu) = \mathfrak{a}_1\mathfrak{d}(\nu, \mu) + \mathfrak{a}_2\mathfrak{d}(\nu, \mathcal{T}\nu) + \mathfrak{a}_3\mathfrak{d}(\mu, \mathcal{T}\mu) + \mathfrak{a}_4\mathfrak{d}(\nu, \mathcal{T}\mu) + \mathfrak{a}_5\mathfrak{d}(\mu, \mathcal{T}\nu).$$

Then $\mathcal{T}$ has a fixed point.

Proof. For all $\nu, \mu \in \mathbf{X}$, we have

$$\mathcal{H}(\mathcal{T}\nu, \mathcal{T}\mu) \leq \alpha(\nu, \mu)\mathcal{H}(\mathcal{T}\nu, \mathcal{T}\mu),$$

since ϑ is increasing function, we get

$$\begin{aligned}\vartheta(\mathfrak{s}^3\mathcal{H}(\mathcal{T}\nu, \mathcal{T}\mu)) &\leq \vartheta(\mathfrak{s}^3\alpha(\nu, \mu)\mathcal{H}(\mathcal{T}\nu, \mathcal{T}\mu)) \\ &\leq \left[\vartheta(N_{\mathfrak{s}}(\nu, \mu))))\right]^{\kappa(\mathfrak{d}(\nu,\mu)} + L\min\{\mathfrak{d}(\nu, \mathcal{T}\mu), \mathfrak{d}(\mu, \mathcal{T}\nu)\}.\end{aligned}$$

So this result is a consequence of Theorem 1.

Corollary 3. *Let $(\mathbf{X}, \mathfrak{d}, \mathfrak{s})$ be a complete $\mathfrak{b}$-metric space, $\alpha\colon \mathbf{X}\times\mathbf{X} \to [0, +\infty)$ be a function and $\mathcal{T}\colon \mathbf{X} \to \mathcal{CB}(\mathbf{X})$ be a multivalued mapping. Assume that the following conditions hold:*

- *(i) $\mathcal{T}$ is almost (ϑ, κ)-contraction of Hardy Rogers type.*
- *(i) $\mathcal{T}$ is lower semi continuous.*

Then $\mathcal{T}$ has a fixed point.

Proof. It suffices to take $\alpha(\nu, \mu) = \mathfrak{s}^2$ for all $\nu, \mu \in \mathbf{X}$ in Theorem1.

Example 2. *Let $\mathbf{X} = [0, 2]$ be a set endowed with a $\mathfrak{b}$-metric $\mathfrak{d}(\nu_1, \nu_2) = |\nu_1 - \nu_2|^2$. Define $\mathcal{T} : \mathbf{X} \to CB(\mathbf{X})$ and $\alpha : \mathbf{X} \times \mathbf{X} \to [0, \infty)$ by*

$$\mathcal{T}\nu = \begin{cases} [0, \frac{\nu}{4}], & \nu \in [0, 2) \\ \{2\}, & \nu = 2 \end{cases}$$

and

$$\alpha(\nu, \mu) = \begin{cases} 4, & (\nu, \mu) \in [0, 2) \\ 0, & otherwise. \end{cases}$$

Taking $\vartheta(\omega) = e^{\omega}$, $\kappa = 3/4$, $s = 2, a_1 = \frac{4}{5}$, $a_2 = a_4 = a_5 = 0$ and $a_3 = 1/8$.
For all $\nu, \mu \in (0, 2)$, we have $\alpha(\nu, \mu) = 4$, $\mathcal{H}(\mathcal{T}\nu, \mathcal{T}\mu) > |\frac{\nu-\mu}{4}|^2 > 0$ and $\mathfrak{d}(\nu, \mu) = |\nu - \mu|^2$. Then

$$8\mathcal{H}(\mathcal{T}\nu, \mathcal{T}\mu) = \frac{1}{2}|\nu - \mu|^2 \leq \frac{3}{4}|\nu - \mu|^2 \leq \frac{3}{4}N_{\mathfrak{s}}(\nu, \mu),$$

which implies that

$$e^{8\mathcal{H}(\mathcal{T}\nu,\mathcal{T}\mu)} \leq e^{\frac{9}{16}\mathfrak{d}(\nu,\mu)} \leq e^{\frac{9}{16}N_{\mathfrak{s}}(\nu,\mu)}.$$

$\mathcal{T}$ is α-continuous, since if (ν_n) is a sequence in $\mathbf{X}$ converges to ν^* with $\alpha(\nu_n, \nu_{n+1}) \geq 4$, then $(\nu_n) \subset [0, 2)$ which implies $T\nu_n = [0, \frac{\nu_n}{4}]$ and $\lim_{n\to\infty} \mathcal{T}\nu_n = [0, \frac{\nu}{4}] = \mathcal{T}\nu^*$.
Consequently, all conditions of Theorem 1 are satisfied. Then $\mathcal{T}$ has a fixed point which is 2.

Now, we give some consequences concerning, two existence theorems of fixed point in metric space endowed with a graph and other in partially order metric spaces.

Theorem 2. *Let* $(\mathbf{X}, \preceq, \mathfrak{d})$ *be a complete ordered* $\mathfrak{b}$*-metric space and* $\mathcal{T}\colon \mathbf{X} \to \mathcal{CB}(\mathbf{X})$ *be a multivalued mapping. Assume that the following assertions hold.*

(i) For each $\nu \in \mathbf{X}$ *and* $\mu \in \mathcal{T}\nu$ *with* $\nu \preceq \mu$*, we have* $\mu \preceq \eta$ *for all* $\nu_3 \in \mathcal{T}\mu$*.*

(ii) There exist $\nu_0 \in \mathbf{X}$ *and* $\nu_0 \in \mathcal{T}\nu_0$ *such that* $\nu_0 \preceq \nu_1$*;*

(iii) For $\nu^* \in \mathbf{X}$ *and a sequence* $\{\nu_n\}$ *in* $\mathbf{X}$ *with* $\lim_{n\to\infty} \mathfrak{d}(\nu_n, \nu^*) = 0$ *and* $\nu_n \preceq \nu_{n+1}$ *for all* $n \in \mathbb{N}$*, implies*

$$\lim\inf_{n\to\infty} \mathfrak{d}(\nu_n, \mathcal{T}\nu_n) \geq \mathfrak{d}(\nu^*, \mathcal{T}\nu^*)$$

or, for every sequence $\{\nu_n\}$ *in* $\mathbf{X}$ *such that* $\nu_n \to \nu^* \in \mathbf{X}$ *and* $\nu_n \preceq \nu_{n+1}$ *for all* $n \in \mathbb{N}$*, we have* $\nu_n \preceq \nu^*$ *for all* $n \in \mathbb{N}$*.*

(iv) There exist $\vartheta \in \Theta_{\mathfrak{s}}$*,* $L \geq 0$ *and* $\kappa : (0, \infty) \to [0, 1)$ *satisfies* $\lim_{\omega\to z^+} \sup \kappa(\omega) < 1$ *for all* $z \in (0, \infty)$ *such that*

$$\vartheta(\mathfrak{s}^3\mathcal{H}(\mathcal{T}\nu, \mathcal{T}\mu)) \leq \Big[\vartheta(N_{\mathfrak{s}}(\nu, \mu))\Big]^{\kappa(\mathfrak{d}(\nu,\mu))} + \mathcal{L}\min\{\mathfrak{d}(\nu, \mathcal{T}\mu), \mathfrak{d}(\mu, \mathcal{T}\nu)\},$$

where

$$N_{\mathfrak{s}}(\nu, \mu) = \mathfrak{a}_1\mathfrak{d}(\nu, \mu) + \mathfrak{a}_2\mathfrak{d}(\nu, \mathcal{T}\nu) + \mathfrak{a}_3\lceil(\mu, \mathcal{T}\mu) + \mathfrak{a}_4\mathfrak{d}(\nu, \mathcal{T}\mu) + \mathfrak{a}_5\mathfrak{d}(\mu, \mathcal{T}\nu).$$

Then $\mathcal{T}$ *has a fixed point.*

Proof.
Define

$$\alpha\colon \mathbf{X} \times \mathbf{X} \to [0, +\infty), \quad \alpha(\nu, \mu) = \begin{cases} \mathfrak{s}^2, & \text{if } \nu \preceq \mu, \\ 0, & \text{otherwise.} \end{cases}$$

The rest of proof is like the proof of Theorem 1.

Nextly, we present an existence theorem of a fixed point for multivalued ϑ-contractions in a $\mathfrak{b}$-metric space $\mathbf{X}$, endowed with a graph, into the space of nonempty closed and bounded subsets of the metric space. Consider a graph $\widetilde{G}$ such that the set $\mathcal{V}\left(\widetilde{G}\right)$ of its vertices coincides with $\mathbf{X}$ and the set $\mathcal{E}\left(\widetilde{G}\right)$ of its edges contains all loops; that is, $\mathcal{E}\left(\widetilde{G}\right) \supseteq \Delta$, where $\widetilde{\Delta} = \{(\nu, \nu), \nu \in \mathbf{X}\}$. We assume $\widetilde{G}$ has no parallel edges, so we can identify $\widetilde{G}$ with the pair $\left(\mathcal{V}\left(\widetilde{G}\right), \mathcal{E}\left(\widetilde{G}\right)\right)$.

We define the function

$$\alpha\colon \mathbf{X} \times \mathbf{X} \to [0, +\infty), \quad \alpha(\nu, \mu) = \begin{cases} \mathfrak{s}^2, & \text{if } (\nu, \mu) \in \mathcal{E}\left(\widetilde{G}\right), \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 3. *Let* $(\mathbf{X}, \mathfrak{d}, s)$ *be a complete* $\mathfrak{b}$*-metric space endowed with a graph* $\widetilde{G}$ *and* $\mathcal{T}\colon \mathbf{X} \to CB(\mathbf{X})$ *be a multivalued mapping. Assume that the following conditions hold:*

(i) For each $\nu \in \mathbf{X}$ *and* $\mu \in \mathcal{T}\nu$ *with* $(\nu, \mu) \in \mathcal{E}(\widetilde{G})$*, we have* $(\mu, \eta) \in \mathcal{E}(\widetilde{G})$ *for all* $\eta \in T\mu$*;*

(ii) There exist $\nu_0 \in \mathbf{X}$ *and* $\nu_1 \in \mathcal{T}\nu_0$ *such that* $(\nu_0, \nu_1) \in \mathcal{E}(\widetilde{G})$*;*

(iii) For every sequence $\{\nu_n\}$ *in* $\mathbf{X}$ *such that* $\nu_n \to \nu^* \in \mathbf{X}$ *and* $(x_n, x_{n+1}) \in \mathcal{E}(\widetilde{G})$ *for all* $n \in \mathbb{N}$*, we have* $(\nu_n, \nu^*) \in \mathcal{E}(\widetilde{G})$ *for all* $n \in \mathbb{N}$*;*

(iv) There exist $\vartheta \in \Theta_s$ *and* $\kappa : (0, \infty) \to [0, 1)$ *satisfies* $\lim_{\omega \to z^+} \sup \kappa(\omega) < 1$ *such that*

$$\vartheta(\mathfrak{s}^3 \mathcal{H}(\mathcal{T}\nu, \mathcal{T}\mu)) \leq \Big[\vartheta(N_{\mathfrak{s}}(\nu, \mu))\Big]^{\kappa(\mathfrak{d}(\nu,\mu))} + L \min\{\mathfrak{d}(\nu, \mathcal{T}\mu), \mathfrak{d}(\mu, \mathcal{T}\nu)\}, \tag{2.6}$$

where

$$N_{\mathfrak{s}}(\nu, \mu) = \mathfrak{a}_1 \mathfrak{d}(\nu, \mu) + \mathfrak{a}_2 \mathfrak{d}(\nu, T\mu) + \mathfrak{a}_3 \mathfrak{d}(\mu, \mathcal{T}\mu) + \mathfrak{a}_4 \mathfrak{d}(\nu, \mathcal{T}\mu) + \mathfrak{a}_5 \mathfrak{d}(\mu, \mathcal{T}\nu).$$

Then $\mathcal{T}$ *has a fixed point.*

Proof. It suffices to consider

$$\alpha\colon \mathbf{X} \times \mathbf{X} \to [0, +\infty), \quad \alpha(\nu, \mu) = \begin{cases} \mathfrak{s}^2, & \text{if } (\nu, \mu) \in \mathcal{E}\left(\widetilde{G}\right), \\ 0, & \text{otherwise.} \end{cases}$$

.

3. Application

In this section, we apply our obtained results to prove existence theorem of solution for an integral inclusion of Volterra-type. For this purpose, let $\mathbf{X} := \mathcal{C}([\mathfrak{a}, \mathfrak{b}], \mathbb{R})$ be the space of all continuous real valued functions on $[\mathfrak{a}, \mathfrak{b}]$. Note that $\mathbf{X}$ is $\mathfrak{b}$-complete $\mathfrak{b}$-metric space by considering $d(\nu, \mu) = \sup_{\omega \in [\mathfrak{a}, \mathfrak{b}]} |\nu(\omega) - \mu(\omega)|^2$ with $\mathfrak{s} = 2$ and define $\alpha : \mathbf{X} \times \mathbf{X} \to \mathbb{R}_+$ by $\alpha(\nu, \mu) = 4$, for all $\nu, \mu \in \mathbf{X}$.
Consider now the following problem

$$\nu(t) \in \mathfrak{p}(\omega) + \int_{\mathfrak{a}}^{\omega} \mathcal{F}(\omega, \tau, \nu(\tau)) d\tau, \qquad \omega \in \mathcal{J} = [\mathfrak{a}, \mathfrak{b}]. \tag{3.1}$$

where $\mathfrak{p} \in \mathbf{X}$ and $\mathcal{F} : \mathcal{J} \times \mathcal{J} \times \mathbb{R} \to \mathcal{K}(\mathbb{R})$.
Consider the set-valued operator $\mathcal{T} : \mathbf{X} \to CL(\mathbf{X})$ as follows

$$\mathcal{T}\nu(\omega) = \left\{ \mu \in \mathbf{X}\colon \mu \in \mathfrak{p}(\omega) + \int_{\mathfrak{a}}^{\omega} \mathcal{F}(\omega, \tau, \nu(\tau)) d\tau, \ \ \omega \in J \right\}.$$

We consider the following hypotheses:

$(\mathcal{A}_1)$: For each $\nu \in \mathbf{X}$, the multivalued operator $\mathcal{F}_\nu : (\omega,\tau) \mapsto \mathcal{F}(\omega,\tau,\nu(\tau))$, is lower semi continuous.

$(\mathcal{A}_2)$: There exists a continuous function $\xi : \mathcal{J} \times \mathcal{J} \to [0,+\infty)$ such that

$$|\mathfrak{q}_\nu(\omega,\tau) - \mathfrak{q}_\mu(\omega,\tau)| \leq \xi(\omega,\tau)|\nu(\tau) - \mu(\tau)|.$$

For all $\nu,\mu \in \mathbf{X}$, all $\mathfrak{q}_\nu \in \mathcal{F}_\nu, \mathfrak{q}_\mu \in \mathcal{F}_\mu$ and for each $(\omega,\tau) \in \mathcal{J} \times \mathcal{J}$.

$(\mathcal{A}_3)$: There exists $\gamma > 0$ such that

$$\sup_{\omega\in\mathcal{J}} \int_a^\omega |\xi(\omega,\tau)|d\tau \leq (\frac{e^{-\gamma}}{8})^{\frac{1}{2}}.$$

Theorem 4. *The integral inclusion* (3.1) *has a solution in* $\mathbf{X}$ *provided the assumptions* $(\mathcal{A}_1) - (\mathcal{A}_3)$ *hold.*

Proof. The set-valued operator $\mathcal{F}_\nu(\omega,\tau) : \mathcal{J} \times \mathcal{J} \to \mathcal{K}(\mathbb{R})$ is lower semi continuous, then from Michael's selection theorem, for $\nu \in \mathbf{X}$ there exists a continuous function $q_\nu : \mathcal{J} \times \mathcal{J} \to \mathbb{R}$ such that $q_\nu(\omega,\tau) \in F_\nu(\omega,\tau)$ for all $\omega,\tau \in \mathcal{J}$. It follows that $p(\omega) + \int_\mathfrak{a}^\omega \mathfrak{q}_\nu(\omega,\tau)ds \in \mathcal{T}\nu$, so $\mathcal{T}\nu$ is non-empty for all $\nu \in \widetilde{\mathbf{X}}$. Since $\mathfrak{p}$ and q_ν are continuous on $\mathcal{J}$, resp. $\mathcal{J}^2$, their ranges are bounded and closed and hence $\mathcal{T}\nu$ is bounded, i.e., $\mathcal{T} : \mathbf{X} \to \mathcal{K}(\mathbf{X})$.
Let $\nu,\mu \in \mathbf{X}$ and let $\vartheta \in \mathcal{T}\nu$. Then

$$\vartheta(\omega) \in \mathfrak{p}(\omega) + \int_\mathfrak{a}^\omega \mathcal{F}(\omega,\tau,\nu(\tau))d\tau, \quad \omega \in \mathcal{J}.$$

It follows that there exists $\mathfrak{q}_\nu \in \mathcal{F}(\omega,\tau)$ such that

$$\vartheta(\omega) = \mathfrak{p}(\omega) + \int_\mathfrak{a}^\omega \mathfrak{q}_\nu(\omega,\tau)d\tau, \quad (\omega,\tau) \in \mathcal{J} \times \mathcal{J},$$

From $(\mathcal{A}_2)$, there exists $\varsigma(\omega,\tau) \in \mathcal{F}_\mu(\omega,\tau)$ such that

$$|\mathfrak{q}_\nu(\omega,\tau) - \varsigma(\omega,\tau)| \leq \ \xi(\omega,\tau) \cdot |\nu(\tau) - \mu(\tau)|^2,$$

for all $(\omega,\tau) \in \mathcal{J} \times \mathcal{J}$. Let $\mathcal{P}$ be a multi valued operator defined by

$$\mathcal{P}(\omega,\tau) = \mathcal{F}_\mu(\omega,\tau) \cap \{\mathfrak{z} \in \mathbb{R}\colon |\mathfrak{q}_\nu(\omega,\tau) - \mathfrak{z}| \leq \xi(\omega,\tau) \cdot |\nu(\tau) - \mu(\tau)|\},$$

for all $(\omega,\tau) \in \mathcal{J} \times \mathcal{J}$. Since, by $(\mathcal{A}_1)$, $\mathcal{P}$ is lower semi-continuous, there exists a continuous function $\mathfrak{q}_\mu(\omega,\tau) \in \mathcal{P}(\omega,\tau)$. Then we have

$$\zeta(\omega) = \mathfrak{p}(\omega) + \int_a^\omega \mathfrak{q}_\mu(\omega,\tau)d\tau \in \mathfrak{p}(\omega) + \int_a^\omega \mathcal{F}(\omega,\tau,\mu(\tau))d\tau, \quad \omega \in \mathcal{J}$$

and

$$\begin{aligned}
\mathfrak{d}(\vartheta, \mathcal{T}\mu) &\leq |\vartheta(\omega,\tau)-\zeta(\omega,\tau)|^2 \leq \Big(\int_{\mathfrak{a}}^{\omega} |\mathfrak{q}_{\nu}(\omega,\tau)-\mathfrak{q}_{\mu}(\omega,\tau)|d\tau\Big)^2 \\
&\leq \Big(\int_{a}^{\omega} \xi(\omega,\tau)|\nu(\tau)-\mu(\tau)|d\tau\Big)^2 \\
&\leq \sup_{\tau\in[\mathfrak{a},\mathfrak{b}]} |\nu(\tau)-\mu(\tau)|^2 \Big(\int_{a}^{\omega} \xi(\omega,\tau)d\tau\Big)^2 \\
&= \mathfrak{d}(\nu,\mu)(\int_{a}^{\omega} \xi(\omega,\tau)d\tau)^2 \\
&\leq \frac{e^{-\gamma}}{8}\mathfrak{d}(\nu,\mu).
\end{aligned}$$

Consequently, we have

$$8\mathfrak{d}(\vartheta, \mathcal{T}\mu) \leq e^{-\gamma}\mathfrak{d}(\nu,\mu),$$

interchanging the role of ν and μ, we get

$$8\mathcal{H}(\mathcal{T}\nu, \mathcal{T}\mu) \leq e^{-\tau}\mathfrak{d}(\nu,\mu).$$

Taking exponents we get

$$e^{(8\mathcal{H}(\mathcal{T}\nu,\mathcal{T}\mu)} \leq \Big[e^{\mathfrak{d}(\nu,\mu)}\Big]^{e^{-\gamma}}$$

Then, the mapping $\mathcal{T}$ satisfies all the conditions of Corollary 3 with $\vartheta(\omega) = e^{\omega}$, $\mathfrak{a}_1 = 1$, $\mathfrak{a}_i = L = 0, i = 2,3,4,5$ and $\kappa = e^{-\gamma}$. So, $\mathcal{T}$ has a fixed point, which implies that the integral inclusion (3.1) has a solution in $\mathbf{X}$.

4. Conclusion

In presented paper we have introduced a new class of multivalued contractions, by combining some concepts, as generalized Berinde type contractive conditions and α_s-admissible mappings due to [14] with JS- contractions type due to [10] also we have proved the existence of a fixed point for such novel contractions under some conditions. An example is given to support the validity of our results and an application to the existence of solutions for integral inclusions.

Acknowledgements

We extend their appreciation to Al-Zaytoonah University of Jordan (ZUJ) and to the Deanship of Post Graduate and Scientific Research at Dar Al Uloom University for funding this work.

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