**EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS**
2025, Vol. 18, Issue 1, Article Number 5470
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

# Cordial Labeling of Corona Product of Paths and Fourth Order of Lemniscate Graphs

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**Abstract.** A graph $G = (V, E)$ is called cordial if it is possible to label the vertex by the function $f : V \to 0, 1$ and label the edges by $f^* : E \to 0, 1$, where $f^*(uv) = (f(u) + f(v))mod2$, $u, v \in V$ so that $|v_0 - v_1| \leq 1$ and $|e_0 - e_1| \leq 1$.A lemniscate graph is a plane curve with a characteristic shape, consisting of two loops that meet at a central point as shown below. The curve is also known as the lemniscate of Bernoulli. A fourth order of lemniscate graph is a graph of two fourth order of circles that have two vertex in common. In this paper, we give the conditions that the corona product of paths and fourth order of lemniscate graphs be cordial.

**2020 Mathematics Subject Classifications**: 05C78,05C15

**Key Words and Phrases**: Cordial labeling, corona, lemniscate, Social Networking, Network security, Edge Computing

## 1. Introduction

Let G be a graph with $p$ vertices and $q$ edges. All graphs considered here are simple, finite, connected and undirected. The concept of graph labeling was introduced during the sixties' of the last century by Rosa [16]. Hundreds of researches have been working with different types of labeling graphs [5, 13, 14, 17]. A labeling of a graph G is a process of allocating numbers or labels to the nodes of $G$ or lines of $G$ or both through mathematical functions [1]. Labeling graphs are used for a wide range of applications in different subjects including astronomy, coding theory and communication networks. Cordial labeling is a weaker version of graceful labeling and harmonious labeling introduced by Cahit in [3]. In 1990, Cahit [4], proved the following: each tree is cordial; an Euerlian graph is not cordial if its size is congruent to $2(mod\ \ 4)$ ; a complete graph $K_n$ is cordial if and only if $n \leq 3$ and a complete bipartite graph $K_{n,m}$ is cordial for all positive integers $n$ and $m$. Let

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DOI: https://doi.org/10.29020/nybg.ejpam.v18i1.5470

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$G_1$, $G_2$ respectively be $(p_1, q_1)$, $(p_2, q_2)$ graphs. The fourth power of a lemniscate graph is defined as the union of fourth power of cycles where both have a common vertex; it is denoted by $L^4_{n,m} \equiv C^4_n \sharp C^4_m$ as shown in Fig.1. Obviously, $L^4_{n,m}$ has $n+m-1$ vertices and $4n+4m-18$ edges. For more details about the cordial labeling and types of labeling, the reader can refer to [2, 6–12, 15]. The corona $G_1 \odot G_2$ of two graphs $G_1$ (with $n_1$ vertices ,

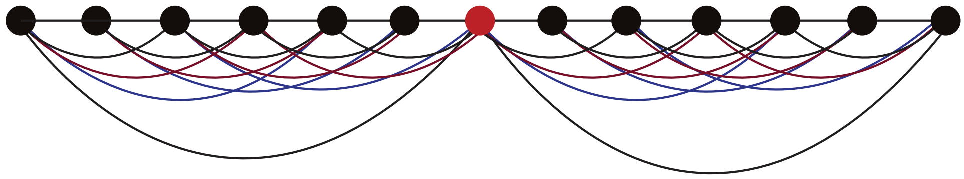

Figure 1: The fourth power of a lemniscate graph $L^4_{7,7}$.

$m_1$ edges) and $G_2$ (with $n_2$ vertices , $m_2$ edges) is defined as the graph obtained by taking one copy of $G_1$ and copies of $G_2$ , and then joining the $i^{th}$ vertex of $G_1$ with an edge to every vertex in the $i^{th}$ copy of $G_2$ . It is easy to see that the corona $G_1 \odot G_2$ that has $n_1 + n_1 n_2$ vertices and $m_1 + n_1 m_2 + n_1 n_2$ edges.

## 2. Terminology and Notation

Given a path or a cycle with $4r$ vertices, We let $L_4 r$ denote the labeling 0011...0011 (repeated $r$-times), let $L'_{4r}$ denote the labeling 1100...1100 (repeated $r$ times). The labeling 1001 1001...1001 (repeated $r$ times) and 0110...0110 (repeated $r$ times) are denoted by $S_{4r}$ and $S'_4 r$. Let $M_{2r}$ denote the labeling 0101...01, zero-one repeated $r-$times if $r$ is even and 0101...010 if $r$ is odd. Sometimes, we modify labeling by adding symbols at one end or the other (or both). If $G$ and $H$ are two graphs, where $G$ has $n$ vertices, the labeling of the corona $G \odot H$ is often denoted by $[A{:}B_1, B_2, B_3, ..., B_n]$, where $A$ is the labeling of the $n$ vertices of $G$, and $B_i$, $1 \leq i \leq n$ is the labeling of the vertices of the copy of $H$ that is connected to the $i-^{th}$ vertex of $G$. For a given labeling of the corona $G \odot H$, we denote $v_i$ and $e_i$ ($i$= $0, 1$) to represent the numbers of vertices and edges, respectively, labeled by $i$. Let us denote $x_i$ and $a_i$ to be the numbers of vertices and edges labeled by $i$ for the graph $G$. Also, we let $y_i$ and $b_i$ be those for $H$, which are connected to the vertices labeled 0 of $G$. Likewise, let $y'_i$ and $b'_i$ be those for $H$, which are connected to the vertices labeled 1 of $G$. It is easily to verify that $v_0{=}x_0 + x_0 y_0 + x_1 y'_0$, $v_1{=}x_1 + x_0 y_1 + x_1 y'_1$, $e_0{=}a_0 + x_0 b_0 + x_1 b'_0 + x_0 y_1 + x_1 y'_0$ and $e_1{=}a_1 + x_0 b_1 + x_1 b'_1 + x_0 y_0 + x_1 y'_1$. Thus $v_0 - v_1{=}\ (x_0 - x_1) + x_0(y_0 - y_1) + x_1(y'_0 - y'_1)$ and $e_0 - e_1{=}\ (a_0 - a_1) + x_0(b_0 - b_1) + x_1(b'_0 - b'_1) + x_0(y_0 - y_1) - x_1(y'_0 - y'_1)$ In particular, if we have only one labeling for all copies of $H$, i.e., $y_i{=}y'_i$ and $b_i{=}b'_i$, then $v_0{=}x_0 + n y_0$, $v_1{=}x_1 + n y_1$, $e_0{=}a_0 + n b_0 + x_0 y_1 + x_1 y_0$ and $e_1{=}a_1 + n b_1 + x_0 y_0 + x_1 y_1$. Thus $v_0 - v_1{=}\ (x_0 - x_1) + n(y_0 - y_1)$ and $e_0 - e_1{=}\ (a_0 - a_1) + n(b_0 - b_1) + (x_1 - x_0)(y_0 - y_1)$, where $n$ is the order of $G$. Section one contains a brief literary analysis of the topic of this work, and Section Two deals with the terminology employed throughout. section three examines and study the cordiality of the corona product $P_k \odot L^4_{n,m}$ of paths and fourth power of lemniscate graphs, and show that this is cordial for all positive integers

$k \geq 1, n, m \geq 3$. The last section, is the conclusion which summarize the important points of our finding in this paper

## 3. Main results.

In this section, we show that the corona product of paths and fourth power of lemniscate graphs, $P_k \odot L^4_{n,m}$ , is cordial for all $k \geq 1$, $n, m \geq 3$.Throughout our proofs, the way of labelling $L^4_{n,m}$ starts always from a vertex that next the common vertex and go further opposite to this common vertex. Before considering the general form of the final result, let us first prove it in the following specific case.

**Theorem 3.1**. *The corona $P_k \odot L^4_{n,m}$ between paths $P_k$ and fourth power of lemniscate graphs $L^4_{n,m}$ is cordial for al $k \geq 1, n, m \geq 3$.* In order to prove this theorem, we will introduce a number of lemmas as follows.

**Lemma 1.** *$P_k \odot L^4_{3,m}$ is cordial for all $k \geq 1$ and $m \geq 3$.*

Proof. *We need to examine the following cases :*

**Case 1.** At $m$=3, we consider the following subcases.

**subcase 1.1.** $k$ is even.

Let $k$= $2r$, $r \geq 1$. Then, one can choose the labelling$[M_{2r}; 00100, 11011, ..., (r - times)]$ for $P_{2r} \odot L^4_{3,3}$. Therefore $x_0$=$x_1$=$r$, $a_0$= 0,$a_1$= $2r - 1$,$y_0$= 4,

$y_1$= 1,$b_0$= 2,$b_1$= 4,$y'_0$= 1,$y'_1$= 4,$b'_0$= 2 and $b'_1$= 4. Hence, it is easy to show that $|v_0 - v_1| = 0$ and $|e_0 - e_1| = 1$. Thus $P_{2r} \odot L^4_{3,3}$ ,$r \geq 1$ is cordial.

**subcase 1.2.** $k$ is odd.

Let $k$= $2r$+1 where $r \geq 0$. Then take the labeling $[M_{2r+1}; 00100, 11011, 00100, 11011, ..., (r-times), 11100]$ for $P_{2r+1} \odot L^4_{3,3}$. Therefore $x_0$=$r$+1,$x_1$=$r$, $a_0$= 0,$a_1$= $2r$, $y_0$= 4,$y_1$= 1,$b_0$= 2,$b_1$= 4, $y'_0$= 1,$y'_1$= 4,$b'_0$= 2,$b'_1$= 4,$y^*_0$= 2,$y^*_1$= 3,$b^*_0$= 4 and $b^*_1$= 2, where $y^*_i$ and $b^*_i$ are the numbers of vertices and edges labelled $i$ in $L^4_{3,3}$ that are connected to the last zero in $P_{4r+3}$. Consequently, it is easy to show that $|v_0 - v_1| = 0$ and $|e_0 - e_1| = 1$. Thus $P_{2r+1} \odot L^4_{3,3}$, $r \geq 0$, is cordial and the lemma follows.

**Case 2.** At $m \equiv 0(mod4)$, we consider the following subcases.

**subcase 2.1.** $k \equiv 0(mod4)$

Let $k$= $4r, r \geq 1$ and $m$= $4t$, $t$> 1. Then, the labelling $[L_{4r}; 0_3\ 1_3 M_{4t-4}, 0_3\ \ 1_3 M_{4t-4}, 01 L_4 M'_{4t-4}, 01 L_4 M'_{4t-4}, ..., (r-times)]$ for $P_{4r} \odot L^4_{3,4t}$ can be applied. Therefore $x_0$=$x_1$= $2r$, $a_0$= $2r$, $a_1$= $2r - 1$,$y_0$=$y_1$= $2t + 1$,$b_0$= $8t - 3$,$b_1$= $8t - 3$,$y'_0$=$y'_1$= $2t + 1$,$b'_0$= $8t - 3$ and $b'_1$= $8t - 3$. So, $|v_0 - v_1| = 0$ and $|e_0 - e_1| = 1$. For the case $P_{4r} \odot L^4_{3,4}$, the labeling $[L_{4r}; 0_3\ 1_3, 0_3\ 1_3, 01L_4, 01L_4, ..., (r - times)]$ is sufficient and thus $P_{4r} \odot L^4_{3,4t}$ is cordial.

**subcase 2.2.**$k \equiv 1(mod4)$

Let $k$= $4r$+1,$r \geq 0$ and $m$= $4t$, $t$> 1. Then, the labelling $[L_{4r}0; 0_3\ 1_3 M_{4t-4}, 0_3\ 1_3 M_{4t-4}, 01 L_4 M'_{4t-4}, 01 L_4 M'_{4t-4}, ..., (r - times), 0_3 L_3 M_{4t-4}]$ for $P_{4r+1} \odot L^4_{3,4t}$ is considered. Therefore $x_0$= $2r$+1,$x_1$= $2r$, $a_0$=$a_1$= $2r$, $y_0$=$y_1$= $2t$+1,$b_0$= $8t$−3,$b_1$= $8t$−3,$y'_0$=$y'_1$= $2t$+1,$b'_0$= $8t$−3 and $b'_1$= $8t$−3. Hence, $|v_0 - v_1| = 1$ and $|e_0 - e_1| = 0$. For the case $P_{4r+1} \odot L^4_{3,4}$, the labeling $[L_{4r}0; 0_3\ 1_3, 0_3\ 1_3, 01L_4, 01L_4, ..., (r - times), 1_3 0_3]$ is sufficient and thus $P_{4r+1} \odot L^4_{3,4t}$ is cordial.

**subcase 2.3.**$k \equiv 2(mod4)$

Let $k= 4r+2, r \geq 0$ and $m= 4t$, $t> 1$. Then, the labelling $[L_{4r}10;0_3\ 1_3M_{4t-4}, 0_3\ 1_3M_{4t-4}, 01L_4M'_{4t-4}, 01L_4M'_{4t-4}, ..., (r-times), 01L_4M'_{4t-4}, 0_3\ 1_3M_{4t-4}]$ for $P_{4r+2}\odot L^4_{3,4t}$ is applied. Therefore $x_0=x_1= 2r+1, a_0= 2r+1, a_1= 2r, y_0=y_1= 2t+1, b_0= 8t-3, b_1= 8t-3, y'_0=y'_1= 2t+1, b'_0= 8t-3$ and $b'_1= 8t-3$. So, $|v_0-v_1| = 0$ and $|e_0-e_1| = 1$. For the case $P_{4r+2}\odot L^4_{3,4}$, the labeling $[L_{4r}10;0_3\ 1_3, 0_3\ 1_3, 01L_4,\ 01L_4, ..., (r-times), 01L_4, 0_3\ 1_3]$ is sufficient and thus $P_{4r+2}\odot L^4_{3,4t}$ is cordial.

**subcase 2.4.**$k \equiv 3(mod4)$

Let $k= 4r+3, r \geq 0$ and $m= 4t$, $t> 1$. Then, one can select the labelling $[L_{4r}001;0_31_3M_{4t-4}, 0_3\ 1_3M_{4t-4}, 01L_4M'_{4t-4}, 01L_4M'_{4t-4}, ..., (r-times),$
$0_3\ 1_3M_{4t-4}, 0_3\ 1_3M_{4t-4}, 01L_4M'_{4t-4}]$ for $P_{4r+3}\odot L^4_{3,4t}$. Therefore $x_0= 2r+2, x_1= 2r+1, a_0=a_1= 2r+1, y_0=y_1= 2t+1, b_0= 8t-3, b_1= 8t-3, y'_0=y'_1= 2t+1, b'_0= 8t-3$ and $b'_1= 8t-3$. Hence, one can easily show that $|v_0-v_1| = 1$ and $|e_0-e_1| = 0$. For the case $P_{4r+3}\odot L^4_{3,4}$, the labeling $[L_{4r}001;0_3\ 1_3, 0_3\ 1_3, 01L_4, 01L_4..., (r-times)]$ is sufficient and thus $P_{4r+3}\odot L^4_{3,4t}$ is cordial.

**Case 3.** At $m \equiv 1(mod4)$, we consider the following subcases.

**subcase 3.1.** $k$ even

Let $k= 2r, r \geq 1$ and $m= 4t+1$, $t> 1$. Then, one can choose the labelling $[M_{2r};10\ 1_3L'_{4t-4}1_2, 10\ 1_3S_{4t-4}0_2, ..., (r-times)]$ for $P_{2r}\odot L^4_{3,4t+1}$. Therefore $x_0=x_1=r, a_0= 0, a_1= 2r-1, y_0= 2t+2, y_1= 2t+1, b_0= 8t-1, b_1= 8t-1, y'_0= 2t+1, y'_1= 2t+2, b'_0= 8t-1$ and $b'_1= 8t-1$. Hence, one can easily show that $|v_0-v_1| = 0$ and $|e_0-e_1| =1$. For the special case $P_{4r}\odot L^4_{3,5}$and $P_{4r+2}\odot L^4_{3,5}$, the labeling $[L_{4r};\ 1_40_3,\ 1_40_3, 0_4\ 1_3, 0_4\ 1_3, ..., (r-times)]$ and $[L_{4r}01;\ 1_40_3,\ 1_40_3, 0_4\ 1_3, 0_4\ 1_3, ..., (r-times),\ 1_40_3, 0_4\ 1_3, 0_4\ 1_3]$ is sufficient and thus $P_{2r}\odot L^4_{3,4t+1}$ is cordial.

**subcase 3.2.** $k$ odd

Let $k= 2r+1, r \geq 1$ and $m= 4t+1$, $t> 1$. Then, one can choose the labelling $[M_{2r+1};10\ 1_3L'_{4t-4}\ 1_2, 10\ 1_3S_{4t-4}0_2, ..., (r-times), 10\ 1_3S_{4t-4}0_2]$ for $P_{2r+1}\odot L^4_{3,4t+1}$. Therefore $x_0=r+1, x_1=r, a_0= 0, a_1= 2r, y_0= 2t+2, y_1= 2t+1, b_0= 8t-1, b_1= 8t-1, y'_0=y''_0= 2t+1, y'_1=y''_1= 2t+2, b'_0=b''_0= 8t-1$ and $b'_1=b''_1= 8t-1$. Hence, one can easily show that $|v_0-v_1| = 0$ and $|e_0-e_1| = 1$. For the special case $P_{4r+1}\odot L^4_{3,5}$and $P_{4r+3}\odot L^4_{3,5}$, the labeling $[L_{4r}0;\ 1_40_3,\ 1_40_3, 0_4\ 1_3, 0_4\ 1_3, ..., (r-times), 0_4\ 1_3]$ and $[L_{4r}010;\ 1_40_3,\ 1_40_3, 0_4\ 1_3, 0_4\ 1_3, ..., (r-times),\ 1_40_3, 0_4\ 1_3, 0_4\ 1_3, 0_4\ 1_3]$ and thus $P_{4r}\odot L^4_{3,4t+1}$ is cordial.

**Case 4.** At $m \equiv 2(mod4)$, we consider the following subcases.

**subcase 4.1.** $k \equiv 0(mod4)$

Let $k= 4r, r \geq 1$ and $m= 4t+2$, $t> 1$. Then, the labelling $[L_{4r};010_310\ 1_3M_{4t-6}, 010_310\ 1_3M_{4t-6},\ 010_310\ 1_3M_{4t-6}, 010_310\ 1_3M_{4t-6}, ..., (r-times)]$ for $P_{4r}\odot L^4_{3,4t+2}$ can be applied. Therefore $x_0=x_1= 2r, a_0= 2r, a_1= 2r-1, y_0=y_1= 2t+1, b_0=b_1= 8t+1, y'_0=y'_1= 2t+1, b'_0= 8t+1$ and $b'_1= 8t+1$. So, $|v_0-v_1| = 0$ and $|e_0-e_1| = 1$. For the case $P_{4r}\odot L^4_{3,6}$, the labeling $[L_{4r};0_3\ 1_20\ 1_2, 0_3\ 1_20\ 1_2, 0_3\ 1_20\ 1_2, 0_3\ 1_20\ 1_2, ..., (r-times)]$ is sufficient and thus $P_{4r}\odot L^4_{3,4t+2}$ is cordial.

**subcase 4.2.**$k \equiv 1(mod4)$

Let $k= 4r+1, r \geq 0$ and $m= 4t+2$, $t> 1$. Then, the labelling $[L_{4r}0;010_310\ 1_3M_{4t-6}, 010_310\ 1_3M_{4t-6}, 010_310\ 1_3M_{4t-6}, 010_310\ 1_3M_{4t-6}, ..., (r-times), 010_310\ 1_3M_{4t-6}]$ for $P_{4r+1}\odot L^4_{3,4t+2}$ is considered. Therefore $x_0= 2r+1, x_1= 2r, a_0=a_1= 2r, y_0=y_1= 2t+1, b_0=b_1= 8t+1, y'_0=y'_1= 2t+$

$1$,$b'_0$= $8t+1$ and $b'_1$= $8t+1$. Hence, $|v_0-v_1| = 1$ and $|e_0-e_1| = 0$. For the case $P_{4r+1}\odot L^4_{3,6}$, the labeling $[L_{4r}0;0_3\ 1_20\ 1_2, 0_3\ 1_20\ 1_2, 0_3\ 1_20\ 1_2, 0_3\ 1_20\ 1_2, ..., (r-times), 0_3\ 1_20\ 1_2]$ is sufficient and thus $P_{4r+1}\odot L^4_{3,4t+2}$ is cordial.

**subcase 4.3.** $k \equiv 2(mod4)$

Let $k$= $4r + 2$,$r \geq 0$ and $m$= $4t + 2$, $t>1$. Then, the labelling $[L_{4r}10;010_3101_3M_{4t-6}, 010_3101_3M_{4t-6}, 010_3101_3M_{4t-6}, 010_3101_3M_{4t-6}, ..., (r-times), 010_3101_3M_{4t-6}, 010_3101_3M_{4t-6}]$ for $P_{4r+2}\odot L^4_{3,4t+2}$ is applied. Therefore $x_0$=$x_1$= $2r + 1$,$a_0$= $2r + 1$,$a_1$= $2r$, $y_0$=$y_1$= $2t + 1$,$b_0$=$b_1$= $8t+1$,$y'_0$=$y'_1$= $2t+1$,$b'_0$= $8t+1$ and $b'_1$= $8t+1$. So, $|v_0-v_1| = 0$ and $|e_0-e_1| = 1$. For the case $P_{4r+2}\odot L^4_{3,6}$, the labeling $[L_{4r}10;0_31_201_2, 0_31_201_2, 0_31_201_2, 0_31_201_2, ..., (r-times), 0_31_201_2, 0_31_201_2]$ is sufficient and thus $P_{4r+2}\odot L^4_{3,4t+2}$ is cordial.

**subcase 4.4.** $k \equiv 3(mod4)$

Let $k$= $4r+3$,$r \geq 0$ and $m$= $4t+2$, $t>1$. Then, one can select the labelling $[L_{4r}001;010_310\ 1_3M_{4t-6}, 010_3101_3M_{4t-6}, 010_310\ 1_3M_{4t-6}, 010_310\ 1_3M_{4t-6}, ..., (r-times), 010_310\ 1_3M_{4t-6}, 010_310\ 1_3M_{4t-6}, 010_310\ 1_3M_{4t-6}]$ for $P_{4r+3}\odot L^4_{3,4t+2}$. Therefore $x_0$= $2r+2$,$x_1$= $2r+1$,$a_0$=$a_1$= $2r+1$,$y_0$=$y_1$= $2t + 1$,$b_0$=$b_1$= $8t + 1$,$y'_0$=$y'_1$= $2t + 1$,$b'_0$= $8t + 1$ and $b'_1$= $8t + 1$. Hence, one can easily show that $|v_0 - v_1| = 1$ and $|e_0 - e_1| = 0$. For the case $P_{4r+3}\odot L^4_{3,6}$, the labeling $[L_{4r}001;0_3\ 1_20\ 1_2, 0_3\ 1_20\ 1_2, 0_3\ 1_20\ 1_2, 0_3\ 1_20\ 1_2, ..., (r-times), 0_3\ 1_20\ 1_2, 0_3\ 1_20\ 1_2, 0_3\ 1_20\ 1_2]$ is sufficient and thus $P_{4r+3}\odot L^4_{3,4t+2}$ is cordial.

**Case 5.** At $m \equiv 3(mod4)$, we consider the following subcases.

**subcase 5.1.** $k$ even

Let $k$= $4r, r \geq 1$ and $m$= $4t+3$, $t \geq 1$. Then, the labelling $[M_{2r};010\ 1_2L'_{4t}010_2, 010\ 1_2L'_{4t}010_2, 1010_2S'_{4t}10\ 1_2, 1010_2S'_{4t}10\ 1_2, ..., (r-times)]$ for $P_{2r}\odot L^2_{3,4t+3}$ can be applied. Therefore $x_0$=$x_1$=$r$, $a_0$= $0$,$a_1$= $2r-1$,$y_0$= $2t+3$,$y_1$= $2t+2$,$b_0$=$b_1$= $8t+3$,$y'_0$= $2t+2$,$y'_1$= $2t+3$,$b'_0$= $8t+3$ and $b'_1$= $8t+4$. Hence, $|v_0 - v_1| = 0$ and $|e_0 - e_1| = 1$. Thus $P_{2r}\odot L^2_{3,4t+3}$ is cordial.

**subcase 5.2.** $k$ odd

Let $k$= $4r, r \geq 1$ and $m$= $4t+3$, $t \geq 1$. Then, the labelling $[M_{2r+1};010\ 1_2L'_{4t}010_2, 010\ 1_2L'_{4t}010_2, 1010_2S'_{4t}10\ 1_2, 1010_2S'_{4t}10\ 1_2, ..., (r-times), 1010_2S'_{4t}10\ 1_2]$ for $P_{2r+1}\odot L^2_{3,4t+3}$ can be applied. Therefore $x_0$=$r+1$,$x_1$=$r$, $a_0$= $0$,$a_1$= $2r$, $y_0$= $2t+3$,$y_1$= $2t+2$,$b_0$=$b_1$= $8t+3$,$y'_0$=$y''_0$= $2t+2$,$y'_1$=$y_1$"= $2t+3$,$b'_0$=$b''_0$= $8t+3$ and $b'_1$=$b''_1$= $8t+4$. Hence, $|v_0-v_1| = 1$ and $|e_0-e_1| = 0$. Thus $P_{2r}\odot L^2_{3,4t+3}$ is cordial.

**Lemma 2.** *$P_k\odot L^4_{n,m}$ is cordial for all $k \geq 1$ and $m > 6$ except at $m = n = 7$.*

**Proof.** *Let $k = 4r + i'$ ( $i' = 0, 1, 2, 3$ and $r \geq 0$) , $n$= $4s + i$ and $m = 4t + j$ ($i, j = 0, 1, 2, 3$ and $s, t \geq 2$), then, we may use the labeling $A_{i'}$ or $A_{j'}$ for $P_k$ as given in Table 1. For a given value of $j$ with $0 \leq i, j \leq 3$, we may use one of the labeling in the set $\{B_{ij}, B'_{ij}\}$ for $L^4_{n,m}$, where $B_{ij}$ and $B^{'}_{ij}$ are the labeling of $L^4_{n,m}$ which are connected to the vertices labeled 0 in $P_k$, while $B_{ij}$ and $B'_{ij}$ are the labeling of $L^4_{n,m}$ which are connected to the vertices labeled 1 in $P_k$ as given in Table 3.2. Using Table 3.3 and the formulas $v_0 - v_1 = (x_0 - x_1) + x_0.(y_0 - y_1) + x_1.(y'_0 - y'_1)$ and $e_0 - e_1 = (a_0 - a_1) + x_0.(b_0 - b_1) + x_1.(b'_0 - b'_1) + x_0.(y_0 - y_1) - x_1.(y'_0 - y'_1)$, we can compute the values shown in the last two columns of Table 3.3. We see that $P_k\odot L^4_{n.m}$ is isomorphic to $P_k\odot L^4_{m,n}$. Since all of these values are 1 or 0, the lemma follows.*

**Table** 3.1.Labeling of $P_k$

| $K = 4r + i'$, $i' = 0, 1, 2, 3$ | labeling of $P_k$ | $x_0$ | $x_1$ | $a_0$ | $a_1$ |
|---|---|---|---|---|---|
| $i' = 0$ | $A_0 = L_{4r}$ | $2r$ | $2r$ | $2r$ | $2r-1$ |
| | $A'_0 = M_{4r}$ | $2r$ | $2r$ | $0$ | $4r-1$ |
| $i = 1$ | $A_1 = L_{4r}0$ | $2r+1$ | $2r$ | $2r$ | $2r$ |
| | $A'_1 = M_{4r+1}$ | $2r+1$ | $2r$ | $0$ | $4r$ |
| $i' = 2$ | $A_2 = L_{4r}01$ | $2r+1$ | $2r+1$ | $2r$ | $2r+1$ |
| | $A'_2 = M_{4r+2}$ | $2r+1$ | $2r+1$ | $0$ | $4r+1$ |
| | $A''_2 = L_{4r}10$ | $2r+1$ | $2r+1$ | $2r+1$ | $2r$ |
| $i' = 3$ | $A_3 = L_{4r}001$ | $2r+2$ | $2r+1$ | $2r+1$ | $2r+1$ |
| | $A'_3 = M_{4r+3}$ | $2r+2$ | $2r+1$ | $0$ | $4r+2$ |
| | $A''_3 = S_{4r}100$ | $2r+2$ | $2r+1$ | $2r+1$ | $2r+1$ |

**Table** 3.2.Labeling of $L^4_{n,m}$

| $n = 4s + i$, $m = 4t + j$, $i, j = 0, 1, 2, 3$ | labeling of $L^4_{n,m}$ | $y_0$ | $y_1$ | $b_0$ | $b_1$ |
|---|---|---|---|---|---|
| $i, j = 0$ | $B_{00} = S'_{4s}1_2M'_{4t-6}0_3$ | $2s+2t$ | $2s+2t-1$ | $8s+8t-9$ | $8s+8t-9$ |
| $i, j = 0$ | $B'_{00} = L'_{4s}0_2M_{4t-6}1_3$ | $2s+2t-1$ | $2s+2t$ | $8s+8t-9$ | $8s+8t-9$ |
| $i = 0, j = 1$ | $B_{01} = 1_3M_{4s-6}0_3L_{4t-4}01_3$ | $2s+2t-1$ | $2s+2t+1$ | $8s+8t-6$ | $8s+8t-8$ |
| $i = 0, j = 1$ | $B_{01} = 0_3M'_{4s-6}1_3S_{4t-4}10_3$ | $2s+2t+1$ | $2s+2t-1$ | $8s+8t-6$ | $8s+8t-8$ |
| $i = 0, j = 2$ | $B_{02} = S'_{4s}01_3M_{4t-4}0_3$ | $2s+2t+1$ | $2s+2t$ | $8s+8t-5$ | $8s+8t-5$ |
| $i = 0, j = 2$ | $B'_{02} = L'_{4s}10_3M'_{4t-4}1_3$ | $2s+2t$ | $2s+2t+1$ | $8s+8t-5$ | $8s+8t-5$ |
| $i = 0, j = 3$ | $B_{03} = 1_3M_{4s-6}0_3L'_{4t-4}1_3010$ | $2s+2t$ | $2s+2t+2$ | $8s+8t-2$ | $8s+8t-4$ |
| $i = 0, j = 3$ | $B'_{03} = 0_3M'_{4s-6}1_3S'_{4t-4}0_3101$ | $2s+2t+2$ | $2s+2t$ | $8s+8t-2$ | $8s+8t-4$ |
| $i, j = 1$ | $B_{11} = L_{4s}0_2L'_{4t-4}1_3$ | $2s+2t$ | $2s+2t-1$ | $8s+8t-5$ | $8s+8t-5$ |
| $i, j = 1$ | $B'_{11} = S_{4s}1_2S'_{4t-4}0_3$ | $2s+2t-1$ | $2s+2t$ | $8s+8t-5$ | $8s+8t-5$ |
| $i = 1, j = 2$ | $B_{12} = 1_3L'_{4s-4}0_4101_3M_{4t-6}$ | $2s+2t$ | $2s+2t+2$ | $8s+8t-2$ | $8s+8t-4$ |
| $i = 1, j = 2$ | $B'_{12} = 0_3S'_{4s-4}1_4010_3M'_{4t-6}$ | $2s+2t+2$ | $2s+2t$ | $8s+8t-2$ | $8s+8t-4$ |
| $i = 1, j = 3$ | $B_{13} = 1_3L'_{4s-4}0_210L'_{4t-4}$ | $2s+2t+1$ | $2s+2t+2$ | $8s+8t-1$ | $8s+8t-1$ |
| $i = 1, j = 3$ | $B'_{13} = 0_3S_{4s-4}1_201S_{4t-4}$ | $2s+2t+2$ | $2s+2t+1$ | $8s+8t-1$ | $8s+8t-1$ |
| $i, j = 2$ | $B_{22} = 0_31_3L'_{4s-4}0_2101_3M_{4t-6}$ | $2s+2t+1$ | $2s+2t+2$ | $8s+8t-1$ | $8s+8t-1$ |
| $i, j = 2$ | $B'_{22} = 1_30_3S'_{4s-4}1_2010_3M'_{4t-6}$ | $2s+2t+2$ | $2s+2t+1$ | $8s+8t-1$ | $8s+8t-1$ |
| $i = 2, j = 3$ | $B_{33} = 0_3101_3M_{4s-6}0L'_{4s-4}1$ | $2s+2t+2$ | $2s+2t+2$ | $8s+8t+1$ | $8s+8t+1$ |
| $i = 2, j = 3$ | $B_{33} = 1_3010_3M'_{4s-6}1S'_{4s-4}0$ | $2s+2t+2$ | $2s+2t+2$ | $8s+8t+1$ | $8s+8t+1$ |
| $i, j = 3$ | $B'_{33} = 0_2M'_{4s-2}1_3S'_{4t-4}0_3101$ | $2s+2t+3$ | $2s+2t+2$ | $8s+8t+3$ | $8s+8t+3$ |
| $i, j = 3$ | $B_{33} = 1_2M_{4s-2}0_3L'_{4t-4}1_3010$ | $2s+2t+2$ | $2s+2t+3$ | $8s+8t+3$ | $8s+8t+3$ |

***Table*** 3.3.Labeling of $P_k \bigodot L^4_{n,m}$

| $i$ | $ij$ | $P_k$ | $L^4_{n,m}$ | $\|v_0 - v_1\|$ | $\|e_0 - e_1\|$ |
|---|---|---|---|---|---|
| 0 | 00 | $A'_0$ | $B_{00}, B'_{00}, B_{00}, B'_{00}$ | 0 | 1 |
| 1 | 00 | $A'_1$ | $B_{00}, B'_{00}, B_{00}, B'_{00}, ..., B'_{00}$ | 0 | 1 |
| 2 | 00 | $A'_2$ | $B_{00}, B'_{00}, B_{00}, B'_{00}, ..., B_{00}, B'_{00}$ | 0 | 1 |
| 3 | 00 | $A'_3$ | $B_{00}, B'_{00}, B_{00}, B'_{00}, ..., B_{00}, B'_{00}, B'_{00}$ | 0 | 1 |
| 0 | 01 | $A_0$ | $B_{01}, B_{01}, B'_{01}, B'_{01}$ | 0 | 1 |
| 1 | 01 | $A_1$ | $B_{01}, B_{01}, B'_{01}, B'_{01}, ..., B_{01}$ | 1 | 0 |
| 2 | 01 | $A_2$ | $B_{01}, B_{01}, B'_{01}, B'_{01}, ..., B_{01}, B'_{01}$ | 0 | 1 |
| 3 | 01 | $A_3$ | $B_{01}, B_{01}, B'_{01}, B'_{01}, ..., B_{01}, B_{01}, B'_{01}$ | 1 | 0 |
| 0 | 02 | $A'_0$ | $B_{02}, B'_{02}, B_{02}, B'_{02}$ | 0 | 1 |
| 1 | 02 | $A'_1$ | $B_{02}, B'_{02}, B_{02}, B'_{02}, ..., B'_{02}$ | 0 | 1 |
| 2 | 02 | $A'_2$ | $B_{02}, B'_{02}, B_{02}, B'_{02}, ..., B_{02}, B'_{02}$ | 0 | 1 |
| 3 | 02 | $A'_3$ | $B_{02}, B'_{02}, B_{02}, B'_{02}, ..., B_{02}, B'_{02}, B'_{02}$ | 0 | 1 |
| 0 | 03 | $A_0$ | $B_{03}, B_{03}, B'_{03}, B'_{03}$ | 0 | 1 |
| 1 | 03 | $A_1$ | $B_{03}, B_{03}, B'_{03}, B'_{03}, ..., B_{03}$ | 1 | 0 |
| 2 | 03 | $A''_2$ | $B_{03}, B_{03}, B'_{03}, B'_{03}, ..., B'_{03}, B_{03}$ | 0 | 1 |
| 3 | 03 | $A''_3$ | $B_{03}, B_{03}, B'_{03}, B'_{03}, ..., B'_{03}, B_{03}, B_{03}$ | 1 | 0 |
| 0 | 11 | $A'_0$ | $B_{11}, B'_{11}, B_{11}, B'_{11}$ | 0 | 1 |
| 1 | 11 | $A'_1$ | $B_{11}, B'_{11}, B_{11}, B'_{11}, ..., B'_{11}$ | 0 | 1 |
| 2 | 11 | $A'_2$ | $B_{11}, B'_{11}, B_{11}, B'_{11}, ..., B_{11}, B'_{11}$ | 0 | 1 |
| 3 | 11 | $A'_3$ | $B_{11}, B'_{11}, B_{11}, B'_{11}, ..., B_{11}, B'_{11}, B'_{11}$ | 0 | 1 |
| 0 | 12 | $A_0$ | $B_{12}, B_{12}, B'_{12}, B'_{12}$ | 0 | 1 |
| 1 | 12 | $A_1$ | $B_{12}, B_{12}, B'_{12}, B'_{12}, ..., B_{12}$ | 1 | 0 |
| 2 | 12 | $A_2$ | $B_{12}, B_{12}, B'_{12}, B'_{12}, ..., B_{12}, B'_{12}$ | 0 | 1 |
| 3 | 12 | $A_3$ | $B_{12}, B_{12}, B'_{12}, B'_{12}, ..., B_{12}, B_{12}, B'_{12}$ | 1 | 0 |
| 0 | 13 | $A'_0$ | $B_{13}, B'_{13}, B_{13}, B'_{13}$ | 0 | 1 |
| 1 | 13 | $A'_1$ | $B_{13}, B'_{13}, B_{13}, B'_{13}, ..., B'_{13}$ | 0 | 1 |
| 2 | 13 | $A'_2$ | $B_{13}, B'_{13}, B_{13}, B'_{13}, ..., B_{13}, B'_{13}$ | 0 | 1 |
| 3 | 13 | $A'_3$ | $B_{13}, B'_{13}, B_{13}, B'_{13}, ..., B_{13}, B'_{13}, B'_{13}$ | 0 | 1 |

| $i$ | $ij$ | $P_k$ | $L^4_{n,m}$ | $\|v_0 - v_1\|$ | $\|e_0 - e_1\|$ |
|---|---|---|---|---|---|
| 0 | 22 | $A'_0$ | $B_{22}, B'_{22}, B_{22}, B'_{22}$ | 0 | 1 |
| 1 | 22 | $A'_1$ | $B_{22}, B'_{22}, B_{22}, B'_{22}, ..., B'_{22}$ | 0 | 1 |
| 2 | 22 | $A'_2$ | $B_{22}, B'_{22}, B_{22}, B'_{22}, ..., B_{22}, B'_{22}$ | 0 | 1 |
| 3 | 22 | $A'_3$ | $B_{22}, B'_{22}, B_{22}, B'_{22}, ..., B_{22}, B'_{22}, B'_{22}$ | 0 | 1 |
| 0 | 23 | $A_0$ | $B_{23}, B_{23}, B'_{23}, B'_{23}$ | 0 | 1 |
| 1 | 23 | $A_1$ | $B_{23}, B_{23}, B'_{23}, B'_{23}, ..., B_{23}$ | 1 | 0 |
| 2 | 23 | $A_2$ | $B_{23}, B_{23}, B'_{23}, B'_{23}, ..., B_{23}, B'_{23}$ | 0 | 1 |
| 3 | 23 | $A_3$ | $B_{23}, B_{23}, B'_{23}, B'_{23}, ..., B_{23}, B_{23}, B'_{23}$ | 1 | 0 |
| 0 | 33 | $A'_0$ | $B_{33}, B'_{33}, B_{33}, B'_{33}$ | 0 | 1 |
| 1 | 33 | $A'_1$ | $B_{33}, B'_{33}, B_{33}, B'_{33}, ..., B'_{33}$ | 0 | 1 |
| 2 | 33 | $A'_2$ | $B_{33}, B'_{33}, B_{33}, B'_{33}, ..., B_{33}, B'_{33}$ | 0 | 1 |
| 3 | 33 | $A'_3$ | $B_{33}, B'_{33}, B_{33}, B'_{33}, ..., B_{33}, B'_{33}, B'_{33}$ | 0 | 1 |

**Lemma 3.** *$P_k \odot L^4_{4,m}$ is cordial for all $k \geq 1$ and $m > 3$ .*

**Proof.** *We need to examine the following cases :*

**Case 1.** At $m \equiv 0(mod4)$, we consider the following sub subcases.

**subcase 1.1.**$k \equiv 0(mod4)$

Let $k= 4r, r \geq 1$ and $m= 4t$, $t> 1$. Then, the labelling $[L_{4r};0\ 1_5M_{4t-6}0_3, 0\ 1_5M_{4t-6}0_3, 10_5M'_{4t-6}\ 1_3, 10_5M'_{4t-6}\ 1_3..., (r-times)]$ for $P_{4r} \odot L^4_{4,4t}$ is applied. Therefore $x_0=x_1= 2r, a_0= 2r, a_1= 2r-1, y_0= 2t+1, y_1= 2t+2, b_0= 8t-1, b_1= 8t-2, y'_0= 2t+2, y'_1= 2t+1, b'_0= 8t-1$ and $b'_1= 8t-2$. So, $|v_0-v_1| = 0$ and $|e_0-e_1| = 1$. For the case $P_{4r} \odot L^4_{4,4}$, the labeling $[M_{4r};01010_3, 1010\ 1_3, 01010_3, 1010\ 1_3, ..., (r-times)]$ is sufficient and thus $P_{4r} \odot L^4_{4,4t}$ is cordial.

**subcase 1.2.**$k \equiv 1(mod4)$

Let $k= 4r+1, r \geq 0$ and $m= 4t$, $t> 1$. Then, the labelling $[L_{4r}0;01_5M_{4t-6}0_3, 01_5M_{4t-6}0_3, 10_5M'_{4t-6}\ 1_3\ , 10_5M'_{4t-6}\ 1_3..., (r-times), 0\ 1_5M_{4t-6}0_3]$ for $P_{4r+1} \odot L^4_{4,4t}$ is applied. Therefore $x_0= 2r+1, x_1= 2r, a_0=a_1= 2r, y_0= 2t+1, y_1= 2t+2, b_0= 8t-1, b_1= 8t-2, y'_0= 2t+2, y'_1= 2t+1, b'_0= 8t-1$ and $b'_1= 8t-2$. So, $|v_0-v_1| = 0$ and $|e_0-e_1| = 0$. For the case $P_{4r+1} \odot L^4_{4,4}$, the labeling $[M_{4r+1};01010_3, 1010\ 1_3, 01010_3, 1010\ 1_3, ..., (r-times), 1010\ 1_3]$ is sufficient and thus $P_{4r+1} \odot L^4_{4,4t}$ is cordial.

**subcase 1.3.**$k \equiv 2(mod4)$

Let $k= 4r+2, r \geq 0$ and $m= 4t$, $t> 1$. Then, the labelling $[L_{4r}10;0\ 1_5M_{4t-6}0_3, 0\ 1_5M_{4t-6}0_3, 10_5M'_{4t-6}\ 1_3, 10_5M'_{4t-6}\ 1_3..., (r-times), 10_5M'_{4t-6}\ 1_3, 0\ 1_5M_{4t-6}0_3]$ for $P_{4r+2} \odot L^4_{4,4t}$ is applied. Therefore $x_0=x_1= 2r+1, a_0= 2r+1, a_1= 2r, y_0= 2t+1, y_1= 2t+2, b_0= 8t-1, b_1= 8t-2, y'_0= 2t+2, y'_1= 2t+1, b'_0= 8t-1$ and $b'_1= 8t-2$. So, $|v_0-v_1| = 0$ and $|e_0-e_1| = 1$. For the case $P_{4r+2} \odot L^4_{4,4}$, the labeling $[M_{4r+2};01010_3, 1010\ 1_3, 01010_3, 1010\ 1_3, ..., (r-times), 1010\ 1_3,\ 01010_3]$ is sufficient and thus $P_{4r+2} \odot L^4_{4,4t}$ is cordial.

**subcase 1.4.**$k \equiv 3(mod4)$

Let $k= 4r+3, r \geq 0$ and $m= 4t$, $t> 1$. Then, the labelling $[L_{4r}001;0\ 1_5M_{4t-6}0_3, 0\ 1_5M_{4t-6}0_3, 10_5M'_{4t-6}\ 1_3, 10_5M'_{4t-6}\ 1_3..., (r-times), 0\ 1_5M_{4t-6}0_3, 0\ 1_5M_{4t-6}0_3, 10_5M'_{4t-6}\ 1_3]$ for $P_{4r+3} \odot L^4_{4,4t}$ is applied. Therefore $x_0= 2r+2, x_1= 2r+1, a_0=a_1= 2r+1, y_0= 2t+1, y_1= 2t+2, b_0= 8t-1, b_1= 8t-2, y'_0= 2t+2, y'_1= 2t+1, b'_0= 8t-1$ and $b'_1= 8t-2$. So, $|v_0-v_1| = 0$

and $|e_0 - e_1| = 0$. For the case $P_{4r+3} \odot L^4_{4,4}$, the labeling $[M_{4r+3}; 01010_3, 1010\ 1_3, 01010_3, 1010\ 1_3, ..., (r - times), 01010_3, 1010\ 1_3, 1010\ 1_3]$ is sufficient and thus $P_{4r+3} \odot L^4_{4,4t}$ is cordial.

**Case 2.** At $m \equiv 1(mod4)$, we consider the following subcases.

**subcase 2.1.**$k \equiv 0(mod4)$

Let $k = 4r, r \geq 1$ and $m = 4t+1$, $t > 1$. Then, the labelling $[M_{4r}; 010\ 1_2S'_{4t-4}0_3, 1010_2L'_{4t}\ 1_3, 010\ 1_2S'_{4t-4}0_3, 1010_2L'_{4t}\ 1_3, ..., (r-times)]$ for $P_{4r} \odot L^4_{4,4t+1}$ is applied. Therefore $x_0 = x_1 = 2r$, $a_0 = 0, a_1 = 4r - 1, y_0 = 2t + 3, y_1 = 2t + 1, b_0 = 8t, b_1 = 8t + 1, y'_0 = 2t + 1, y'_1 = 2t + 3, b'_0 = 8t$ and $b'_1 = 8t + 1$. So, $|v_0 - v_1| = 0$ and $|e_0 - e_1| = 1$. For the case $P_{4r} \odot L^4_{4,5}$, the labeling $[M_{4r}; 0_3\ 1_20_21, 0_3\ 1_20_21,\ 1_30_2\ 1_20,\ 1_30_2\ 1_20, ..., (r - times)]$ is sufficient and thus $P_{4r} \odot L^4_{4,4t+1}$ is cordial.

**subcase 2.2.**$k \equiv 1(mod4)$

Let $k = 4r+1, r \geq 0$ and $m = 4t+1$, $t > 1$. Then, the labelling $[M_{4r+1}; 010\ 1_2S'_{4t-4}0_3, 1010_2L'_{4t}\ 1_3, 010\ 1_2S'_{4t-4}0_3, 1010_2L'_{4t}\ 1_3, ..., (r - times), 010\ 1_3L'_{4t-4}0_3]$ for $P_{4r+1} \odot L^4_{4,4t+1}$ is applied. Therefore $x_0 = 2r+1, x_1 = 2r, a_0 = 0, a_1 = 4r, y_0 = 2t+3, y_1 = 2t+1, b_0 = 8t, b_1 = 8t+1, y'_0 = 2t+1, y'_1 = 2t+3, b'_0 = 8t$ , $b'_1 = 8t+1, y''_0 = 2t+2, y''_1 = 2t+2, b''_0 = 8t$ and $b''_1 = 8t+1$. So, $|v_0 - v_1| = 1$ and $|e_0 - e_1| = 1$. For the case $P_{4r+1} \odot L^4_{4,5}$, the labeling $[M_{4r}; 0_3\ 1_20_21, 0_3\ 1_20_21,\ 1_30_2\ 1_20, 1_30_2\ 1_20, ..., (r - times), 010\ 1_40]$ is sufficient and thus $P_{4r+1} \odot L^4_{4,4t+1}$ is cordial.

**subcase 2.3.**$k \equiv 2(mod4)$

Let $k = 4r+2, r \geq 0$ and $m = 4t$, $t > 1$. Then, the labelling $[M_{4r+1}; 010\ 1_2S'_{4t-4}0_3, 1010_2L'_{4t}\ 1_3, 010\ 1_2S'_{4t-4}0_3, 1010_2L'_{4t}\ 1_3, ..., (r - times), 010\ 1_2S'_{4t-4}0_3, 1010_2L'_{4t}\ 1_3]$ for $P_{4r+2} \odot L^4_{4,4t}$ is applied. Therefore $x_0 = x_1 = 2r + 1, a_0 = 0, a_1 = 4r + 1, y_0 = 2t + 3, y_1 = 2t + 1, b_0 = 8t, b_1 = 8t + 1, y'_0 = 2t + 1, y'_1 = 2t + 3, b'_0 = 8t$ and $b'_1 = 8t + 1$. So, $|v_0 - v_1| = 0$ and $|e_0 - e_1| = 1$. For the case $P_{4r+2} \odot L^4_{4,5}$, the labeling $[L_{4r}01; 0_3\ 1_20_21, 0_3\ 1_20_21,\ 1_30_2\ 1_20,\ 1_30_2\ 1_20, ..., (r-times), 0_3\ 1_20_21, 0_3\ 1_20_21]$ is sufficient and thus $P_{4r+2} \odot L^4_{4,4t+1}$ is cordial.

**subcase 2.4.**$k \equiv 3(mod4)$

Let $k = 4r+3, r \geq 0$ and $m = 4t+1$, $t > 1$. Then, the labelling $[M_{4r+1}; 010\ 1_2S'_{4t-4}0_3, 1010_2L'_{4t}\ 1_3, 010\ 1_2S'_{4t-4}0_3, 1010_2L'_{4t}\ 1_3, ..., (r - times), 010\ 1_3L'_{4t-4}0_3]$ for $P_{4r+3} \odot L^4_{4,4t+1}$ is applied. Therefore $x_0 = 2r + 2, x_1 = 2r + 1, a_0 = 0, a_1 = 4r + 2, y_0 = 2t + 3, y_1 = 2t + 1, b_0 = 8t, b_1 = 8t + 1, y'_0 = 2t+1, y'_1 = 2t+3, b'_0 = 8t$ , $b'_1 = 8t+1, y''_0 = 2t+2, y''_1 = 2t+2, b''_0 = 8t$ and $b''_1 = 8t+1$. So, $|v_0 - v_1| = 1$ and $|e_0 - e_1| = 1$. For the case $P_{4r+3} \odot L^4_{4,5}$, the labeling $[M_{4r}; 0_3\ 1_20_21,\ 1_30_2\ 1_20, 0_3\ 1_20_21,\ 1_30_2\ 1_20, ..., (r-times), 0_3\ 1_20_21,\ 1_30_2\ 1_20, 010\ 1_40]$is sufficient and thus $P_{4r+3} \odot L^4_{4,4t+1}$ is cordial.

**Case 3.** At $m \equiv 2(mod4)$, we consider the following sub subcases.

**subcase 3.1.**$k \equiv 0(mod4)$

Let $k = 4r, r \geq 1$ and $m = 4t+2$, $t > 1$. Then, the labelling $[L_{4r}; 0\ 1_5M_{4t-6}0_310, 0\ 1_5M_{4t-6}0_310, 10_5M'_{4t-6}\ 1_301, 10_5M'_{4t-6}\ 1_301, ..., (r-times)]$ for $P_{4r} \odot L^4_{4,4t+2}$ is applied. Therefore $x_0 = x_1 = 2r$, $a_0 = 2r, a_1 = 2r-1, y_0 = 2t+2, y_1 = 2t+3, b_0 = 8t+3, b_1 = 8t+2, y'_0 = 2t+3, y'_1 = 2t+2, b'_0 = 8t+3$ and $b'_1 = 8t + 2$. So, $|v_0 - v_1| = 0$ and $|e_0 - e_1| = 1$. For the case $P_{4r} \odot L^4_{4,6}$, the labeling $[L_{4r}; 0_3\ 1_40_2, 0_3\ 1_40_2,\ 1_30_4\ 1_2,\ 1_30_4\ 1_2, ..., (r-times)]$ is sufficient and thus $P_{4r} \odot L^4_{4,4t+2}$ is cordial.

**subcase 3.2.**$k \equiv 1(mod4)$

Let $k$= $4r$+1,$r \geq 0$ and $m$= $4t$+2, $t$> 1. Then, the labelling $[L_{4r}0;01_5M_{4t-6}0_310, 01_5\ M_{4t-6}0_310, 10_5M'_{4t-6}\ 1_301, 10_5M'_{4t-6}\ 1_301, ..., (r-times), 0\ 1_5M_{4t-6}0_310]$ for $P_{4r+1}\odot L^4_{4,4t+2}$ is applied. Therefore $x_0$= $2r$+1,$x_1$= $2r$, $a_0$=$a_1$= $2r$, $y_0$= $2t$+2,$y_1$= $2t$+3,$b_0$= $8t$+3,$b_1$= $8t$+2,$y'_0$= $2t$+ 3,$y'_1$= $2t+2$,$b'_0$= $8t+3$ and $b'_1$= $8t+2$. So, $|v_0 - v_1| = 0$ and $|e_0 - e_1| = 0$. For the case $P_{4r+1}\odot L^4_{4,6}$, the labeling $[L_{4r}0;0_3\ 1_40_2, 0_3\ 1_40_2,\ 1_30_4\ 1_2,\ 1_30_4\ 1_2, ..., (r-times), 0_3\ 1_40_2]$ is sufficient and thus $P_{4r+1}\odot L^4_{4,4t+2}$ is cordial.

**subcase 3.3.**$k \equiv 2(mod4)$

Let $k$= $4r$+2,$r \geq 0$ and $m$= $4t$+2, $t$> 1. Then, the labelling $[L_{4r}10;01_5M_{4t-6}0_310,\ 01_5M_{4t-6}0_310, 10_5M'_{4t-6}\ 1_301, 10_5M'_{4t-6}\ 1_301, ..., (r-times), 10_5M'_{4t-6}\ 1_301, 0\ 1_5M_{4t-6}0_310]$ for $P_{4r+2}\odot L^4_{4,4t+2}$ is applied. Therefore $x_0$=$x_1$= $2r+1$,$a_0$= $2r+1$,$a_1$= $2r$, $y_0$= $2t+2$,$y_1$= $2t+3$,$b_0$= $8t$+ 3,$b_1$= $8t$+2,$y'_0$= $2t$+3,$y'_1$= $2t$+2,$b'_0$= $8t$+3 and $b'_1$= $8t$+2. So, $|v_0-v_1| = 0$ and $|e_0-e_1| = 1$. For the case $P_{4r+2}\odot L^4_{4,6}$, the labeling $[L_{4r}01;0_3\ 1_40_2, 0_3\ 1_40_2,\ 1_30_4\ 1_2,\ 1_30_4\ 1_2, ..., (r-times), 0_3\ 1_40_2,\ 1_30_4\ 1_2]$ is sufficient and thus $P_{4r+2}\odot L^4_{4,4t+2}$ is cordial.

**subcase 3.4.**$k \equiv 3(mod4)$

Let $k$= $4r+3$,$r \geq 0$ and $m$= $4t+2$, $t$> 1. Then, the labelling $[L_{4r}0_21;0\ 1_5M_{4t-6}0_310, 01_5M_{4t-6}0_310, 10_5M'_{4t-6}\ 1_301, 10_5M'_{4t-6}\ 1_301, ..., (r-times), 0\ 1_5M_{4t-6}0_310, 0\ 1_5M_{4t-6}0_310, 10_5M'_{4t-6}\ 1_301]$ for $P_{4r+3}\odot\ L^4_{4,4t+2}$ is applied. Therefore $x_0$= $2r$+2,$x_1$= $2r$+1,$a_0$=$a_1$= $2r$+ 1,$y_0$= $2t$+2,$y_1$= $2t$+3,$b_0$= $8t$+3,$b_1$= $8t$+2,$y'_0$= $2t$+3,$y'_1$= $2t$+2,$b'_0$= $8t$+3 and $b'_1$= $8t$+2. So, $|v_0-v_1| = 1$ and $|e_0-e_1| = 0$. For the case $P_{4r+2}\odot L^4_{4,6}$, the labeling $[L_{4r}001;\ 1_30_4\ 1_2,\ 1_30_4\ 1_2, 0_3\ 1_40_2, 0_3\ 1_40_2, ..., (r-times),\ 1_30_4\ 1_2,\ 1_30_4\ 1_2, 0_3\ 1_40_2]$ is sufficient and thus $P_{4r+3}\odot L^4_{4,4t+2}$ is cordial.

**Case 4.** At $m \equiv 3(mod4)$, we consider the following sub subcases.

**subcase 4.1.**$k$ is even.

Let $k$= $2r$, $r \geq 1$ and $m$= $4t$+3, $t \geq 1$. Then, the labelling $[M_{2r};10_3\ 1_2L_{4t-4}010_2, 1010M_{4t}\ 1_2, ..., (r-times)]$ for $P_{2r}\odot\ L^4_{4,4t+3}$ is applied. Therefore $x_0$=$x_1$=$r$, $a_0$= 0,$a_1$= $2r$−1,$y_0$= $2t$+ 4,$y_1$= $2t+2$,$b_0$= $8t+5$,$b_1$= $8t+4$,$y'_0$= $2t+2$,$y'_1$= $2t+4$,$b'_0$= $8t+3$ and $b'_1$= $8t+6$. Hence, $|v_0 - v_1| = 0$ and $|e_0 - e_1| = 1$ and thus $P_{2r}\odot L^4_{4,4t+3}$ is cordial.

**subcase 4.2.**$k$ is odd.

Let $k$= $2r$+1,$r \geq 0$ and $m$= $4t$+3, $t \geq 1$. Then, the labelling $[M_{2r+1};10_3\ 1_2L_{4t-4}010_2, 1010M_{4t}\ 1_2, ..., (r-times), 10_3M_{4t}\ 1_2]$ for $P_{2r+1}\odot L^4_{4,4t+3}$ is applied. Therefore $x_0$=$r$ + 1,$x_1$=$r$, $a_0$= 0,$a_1$= $2r$, $y_0$= $2t+4$,$y_1$= $2t+2$,$b_0$= $8t+5$,$b_1$= $8t+4$,$y'_0$= $2t+2$,$y'_1$= $2t$ + 4,$b'_0$= $8t$+3,$b'_1$= $8t$+6,$y''_0$= $2t$+3,$y''_1$= $2t$+3,$b''_0$= $8t$+4 and $b''_1$= $8t$+5. Hence, $|v_0 - v_1| = 1$ and $|e_0 - e_1| = 1$. Thus $P_{2r+1}\odot L^4_{4,4t+3}$ is cordial and the lemma follows.■

**Lemma 4.** $P_k\odot L^4_{5,m}$ *is cordial for all* $k \geq 1$ *and* $m \geq 3$ *.*

**Proof .** *We need to examine the following two cases :*

**Case 1.** At $m \equiv 0(mod4)$, i.e $m$= $4t$,$t \geq 1$ .

We see that $P_k\odot L^4_{5,4}$ and $P_k\odot L^4_{5,4t}$,$t \geq 1$ are cordial. This is clear since these graphs are isomorphic to $P_k\odot L^4_{4,5}$ and $P_k\odot L^4_{4t,5}$ respectively. So, by lemma 3, we conclude that $P_k\odot L^4_{5,4}$ and $P_k\odot L^4_{5,4t}$ are cordial.

**Case 2.** At $m \equiv 1(mod4)$, we consider the following subcases.

**subcase 2.1.**$k \equiv 0(mod4)$

Let $k$= $4r$, $r \geq 1$ and $m$= $4t$+1, $t$> 1. Then, the labelling $[L_{4r};01_4M_{4t-4}0_31, 0\ 1_4M_{4t-4}0_31,$

$10_4M'_{4t-4}\ 1_30, 10_4M'_{4t-4}\ 1_30, ..., (r-times)]$ for $P_{4r}\odot\ L^4_{5,4t+1}$ is applied. Therefore $x_0$=$x_1$= $2r$, $a_0$= $2r$, $a_1$= $2r-1$,$y_0$= $2t+2$,$y_1$= $2t+3$,$b_0$= $8t-3$,$b_1$= $8t+2$,$y'_0$= $2t+3$,$y'_1$= $2t+2$,$b'_0$= $8t+3$ and $b'_1$= $8t+2$. So, $|v_0 - v_1| = 0$ and $|e_0 - e_1| = 1$. For the case $P_{4r}\odot L^4_{5,5}$, the labeling $[M_{4r};10_5\ 1_3, 0\ 1_50_310_5,\ 1_3, 0\ 1_50_3, ..., (r-times)]$ is sufficient and thus $P_{4r}\odot L^4_{5,4t+1}$ is cordial.

**subcase 2.2.**$k\equiv 1(mod4)$

Let $k$= $4r$+1,$r\geq 0$ and $m$= $4t$+1, $t$> 1. Then, the labelling $[L_{4r}0;0\ 1_4M_{4t-4}0_31, 0\ 1_4M_{4t-4}0_31, 10_4M'_{4t-4}\ 1_30, 10_4M'_{4t-4}\ 1_30, ..., (r-times), 0\ 1_4M_{4t-4}0_31]$ for $P_{4r+1}\odot\ L^4_{5,4t+1}$ is applied. Therefore $x_0$= $2r$+1,$x_1$= $2r$, $a_0$=$a_1$= $2r$, $y_0$= $2t$+2,$y_1$= $2t$+3,$b_0$= $8t-3$,$b_1$= $8t$+2,$y'_0$= $2t$+3,$y'_1$= $2t+2$,$b'_0$= $8t+3$ and $b'_1$= $8t+2$. So, $|v_0 - v_1| = 0$ and $|e_0 - e_1| = 0$. For the case $P_{4r+1}\odot L^4_{5,5}$, the labeling $[M_{4r+1};10_5\ 1_3, 0\ 1_50_3, 10_5\ 1_3, 0\ 1_50_3, ..., (r-times), 0\ 1_50_3]$ is sufficient and thus $P_{4r+1}\odot L^4_{5,4t+1}$ is cordial.

**subcase 2.3.**$k\equiv 2(mod4)$

Let $k$= $4r$+2,$r\geq 0$ and $m$= $4t$+1, $t$> 1. Then, the labelling $[L_{4r}01;0\ 1_4M_{4t-4}0_31, 0\ 1_4M_{4t-4}0_31, 10_4M'_{4t-4}\ 1_30, 10_4M'_{4t-4}\ 1_30, ..., (r-times), 0\ 1_4M_{4t-4}0_31, 10_4M'_{4t-4}\ 1_30]$ for $P_{4r+2}\odot\ L^4_{5,4t+1}$ is applied. Therefore $x_0$=$x_1$= $2r+1$,$a_0$= $2r$, $a_1$= $2r+1$,$y_0$= $2t+2$,$y_1$= $2t+3$,$b_0$= $8t-3$,$b_1$= $8t+2$,$y'_0$= $2t+3$,$y'_1$= $2t+2$,$b'_0$= $8t+3$ and $b'_1$= $8t+2$. So, $|v_0 - v_1| = 0$ and $|e_0 - e_1| = 1$. For the case $P_{4r+2}\odot L^4_{5,5}$, the labeling $[M_{4r+2};10_5\ 1_3, 0\ 1_50_3, 10_5\ 1_3, 0\ 1_50_3, ..., (r-times), 10_5\ 1_3, 0\ 1_50_3]$ is sufficient and thus $P_{4r+2}\odot L^4_{5,4t+1}$ is cordial.

**subcase 2.4.**$k\equiv 3(mod4)$ Let $k$= $4r$+3,$r\geq 0$ and $m$= $4t$+1, $t$> 1. Then, the labelling $[L_{4r}001;0\ 1_4M_{4t-4}0_31, 0\ 1_4M_{4t-4}0_31, 10_4M'_{4t-4}\ 1_30, 10_4M'_{4t-4}\ 1_30, ..., (r-times), 01_4M_{4t-4}0_31, 0\ 1_4M_{4t-4}0_31, 10_4M'_{4t-4}\ 1_30]$ for $P_{4r+3}\odot L^4_{5,4t+1}$ is applied. Therefore $x_0$= $2r+2$,$x_1$= $2r+1$,$a_0$=$a_1$= $2r+1$,$y_0$= $2t+2$,$y_1$= $2t+3$,$b_0$= $8t-3$,$b_1$= $8t+2$,$y'_0$= $2t+3$,$y'_1$= $2t+2$,$b'_0$= $8t+3$ and $b'_1$= $8t+2$. So, $|v_0 - v_1| = 0$ and $|e_0 - e_1| = 0$. For the case $P_{4r+3}\odot L^4_{5,5}$, the labeling $[M_{4r+3};10_5\ 1_3, 0\ 1_50_3,\ 10_5\ 1_3, 0\ 1_50_3, ..., (r-times), 0\ 1_50_3, 10_5\ 1_3, 0\ 1_50_3]$ is sufficient and thus $P_{4r+3}\odot L^4_{5,4t+1}$ is cordial.

**Case 3.** At $m\equiv 2(mod4)$, we consider the following sub subcases.

**subcase 3.1.**$k$ is even. Let $k$= $2r, r\geq 1$ and $m$= $4t+2$, $t$> 1. Then, the labelling $[M_{2r};0_3\ 1_4010_3M'_{4t-6},\ 1_30_40\ 1_3M_{4t-6}, ..., (r-times)]$ for $P_{2r}\odot\ L^4_{5,4t+2}$ is applied. Therefore $x_0$=$x_1$=$r$, $a_0$= 0,$a_1$= $2r-1$,$y_0$= $2t+4$,$y_1$= $2t+2$,$b_0$= $8t+4$,$b_1$= $8t+5$,$y'_0$= $2t+2$,$y'_1$= $2t+4$,$b'_0$= $8t+4$ and $b'_1$= $8t+5$. So, $|v_0 - v_1| = 0$ and $|e_0 - e_1| = 1$. For the case $P_{2r}\odot L^4_{5,6}$, the labeling $[M_{2r};0_3\ 1_30_4,\ 1_30_3\ 1_4, ..., (r-times)]$ is sufficient and thus $P_{2r}\odot L^4_{5,4t+2}$ is cordial.

**subcase 3.2.**$k$ is odd. Let $k$= $2r+1$,$r\geq 0$ and $m$= $4t+2$, $t$> 1. Then, the labelling $[M_{2r+1};0_3\ 1_4010_3M'_{4t-6},\ 1_30_40\ 1_3M_{4t-6}\ , ..., (r-times), 10_3L_{4t}\ 1_2]$ for $P_{2r+1}\odot L^4_{5,4t+2}$ is applied. Therefore $x_0$=$r+1$,$x_1$=$r$, $a_0$= 0,$a_1$= $2r$, $y_0$= $2t+4$,$y_1$= $2t+2$,$b_0$= $8t+4$,$b_1$= $8t+5$,$y'_0$= $2t+2$,$y'_1$= $2t+4$,$b'_0$= $8t+4$,$b'_1$= $8t+5$,$y''_0$= $2t+3$,$y''_1$= $2t+3$,$b''_0$= $8t+5$ and $b''_1$= $8t+4$. So, $|v_0 - v_1| = 1$ and $|e_0 - e_1| = 1$. For the case $P_{2r+1}\odot L^4_{5,6}$, the labeling $[M_{2r+1};0_3\ 1_30_4,\ 1_30_3\ 1_4, ..., (r-times), 10_5\ 1_4]$ is sufficient and thus $P_{2r+1}\odot L^4_{5,4t+2}$ is cordial.

**Case 4.** At $m\equiv 3(mod4)$, we consider the following sub subcases.

**subcase 4.1.**$k\equiv 0(mod4)$

Let $k$= $4r, r\geq 1$ and $m$= $4t+3$, $t\geq 1$. Then, the labelling $[L_{4r};0\ 1_4M'_{4t}0_2, 0\ 1_4$

$M'_{4t}0_2, 10_4M_{4t}\ 1_2, 10_4M_{4t}\ 1_2, ..., (r{-}times)]$ for $P_{4r}\odot L^4_{5,4t+3}$ is applied. Therefore $x_0{=}x_1{=}\ 2r$, $a_0{=}\ 2r, a_1{=}\ 2r-1, y_0{=}\ 2t+3, y_1{=}\ 2t+4, b_0{=}\ 8t+7, b_1{=}\ 8t+6, y'_0{=}\ 2t+4, y'_1{=}\ 2t+3, b'_0{=}\ 8t+7$ and $b'_1{=}\ 8t+6$. So, $|v_0 - v_1| = 0$ and $|e_0 - e_1| = 1$. Thus $P_{4r}\odot L^4_{5,4t+3}$ is cordial.

**subcase 4.2.**$k \equiv 1(mod4)$

Let $k{=}\ 4r{+}1, r \geq 0$ and $m{=}\ 4t{+}3$, $t \geq 1$. Then, the labelling $[L_{4r}0;0\ 1_4M'_{4t}0_2, 0\ 1_4M'_{4t}0_2, 10_4M_{4t}\ 1_2,\ 10_4M_{4t}\ 1_2, ..., (r-times), 0\ 1_4M'_{4t}0_2]$ for $P_{4r+1}\odot\ L^4_{5,4t+3}$ is applied. Therefore $x_0{=}\ 2r{+}1, x_1{=}\ 2r, a_0{=}a_1{=}2r, y_0{=}\ 2t{+}3, y_1{=}\ 2t{+}4, b_0{=}\ 8t{+}7, b_1{=}\ 8t{+}6, y'_0{=}\ 2t{+}4, y'_1{=}\ 2t{+}3, b'_0{=}\ 8t+7$ and $b'_1{=}\ 8t+6$. So, $|v_0 - v_1| = 0$ and $|e_0 - e_1| = 0$. Thus $P_{4r+1}\odot L^4_{5,4t+3}$ is cordial.

**subcase 4.3.**$k \equiv 2(mod4)$

Let $k{=}\ 4r{+}2, r \geq 0$ and $m{=}\ 4t{+}3$, $t \geq 1$. Then, the labelling $[L_{4r}01;0\ 1_4M'_{4t}0_2, 01_4M'_{4t}0_2, 10_4M_{4t}\ 1_2,\ 10_4M_{4t}\ 1_2, ..., (r-times), 0\ 1_4M'_{4t}0_2, 10_4M_{4t}\ 1_2]$ for $P_{4r+2}\odot L^4_{5,4t+3}$ is applied. Therefore $x_0{=}x_1{=}\ 2r+1, a_0{=}\ 2r, a_1{=}\ 2r+1, y_0{=}\ 2t+3, y_1{=}\ 2t+4, b_0{=}\ 8t+7, b_1{=}\ 8t+6, y'_0{=}\ 2t+4, y'_1{=}\ 2t+3, b'_0{=}\ 8t+7$ and $b'_1{=}\ 8t+6$. So, $|v_0 - v_1| = 0$ and $|e_0 - e_1| = 1$. Thus $P_{4r+2}\odot L^4_{5,4t+3}$ is cordial.

**subcase 4.4.**$k \equiv 3(mod4)$

Let $k{=}\ 4r{+}3, r \geq 0$ and $m{=}\ 4t{+}3$, $t \geq 1$. Then, the labelling $[L_{4r}001;0\ 1_4M'_{4t}0_2, 01_4M'_{4t}0_2, 10_4M_{4t}\ 1_2,\ 10_4M_{4t}\ 1_2, ..., (r-times), 0\ 1_4M'_{4t}0_2, 0\ 1_4M'_{4t}0_2, 10_4M_{4t}\ 1_2]$ for $P_{4r+3}\odot L^4_{5,4t+3}$ is applied. Therefore $x_0{=}x_1{=}\ 2r{+}1, a_0{=}a_1{=}\ 2r{+}1, y_0{=}\ 2t{+}3, y_1{=}\ 2t{+}4, b_0{=}\ 8t{+}7, b_1{=}\ 8t{+}6, y'_0{=}\ 2t+4, y'_1{=}\ 2t+3, b'_0{=}\ 8t+7$ and $b'_1{=}\ 8t+6$. So, $|v_0 - v_1| = 1$ and $|e_0 - e_1| = 0$. Thus $P_{4r+3}\odot L^4_{5,4t+3}$ is cordial and the lemma follows.■

**Lemma 5.** *$P_k\odot L^4_{6,m}$; is cordial for all $m, k$.*

**Proof.** *Let $n = 6$, then we need to examine the following cases :*

**Case 1.** At $m \equiv 0(mod4)$, i.e $m{=}\ 4t, t \geq 1$ .

We see that $P_k\odot L^4_{6,4}$ and $P_k\odot L^4_{6,4t}, t \geq 1$ are cordial. This is clear since these graphs are isomorphic to $P_k\odot L^4_{4,6}$ and $P_k\odot L^4_{4t,6}$ respectively. So, by lemma 3, we conclude that $P_k\odot L^4_{6,4}$ and $P_k\odot L^4_{6,4t}$ are cordial.

**Case 2.** At $m \equiv 1(mod4)$, i.e $m{=}\ 4t+1, t \geq 1$.

We see that $P_k\odot L^4_{6,5}$ and $P_k\odot L^4_{6,4t+1}, t \geq 1$ are cordial. This is clear since these graphs are isomorphic to $P_k\odot L^4_{5,6}$ and $P_k\odot L^4_{4t+1,6}$ respectively. So, by lemma 4, we conclude that $P_k\odot L^4_{6,5}$ and $P_k\odot L^4_{6,4t+1}$ are cordial.

**Case 3.** At $m \equiv 2(mod4)$, we need to examine the following two subcases:

**subcase 3.1.**$k$ even

Let $k{=}\ 2r, r \geq 1$ and $m{=}\ 4t{+}3$, $t \geq 1$. Then, the labelling $[M_{2r};\ 1_20_5M_{4t}\ 1_201,\ 1_20_5M_{4t}\ 1_201, 0_2\ 1_5M'_{4t}0_210, 0_2\ 1_5M'_{4t}0_210, ..., (r{-}times)]$ for $P_{2r}\odot L^4_{6,4t+2}$ is applied. Therefore $x_0{=}x_1{=}r$, $a_0{=}\ 0, a_1{=}\ 2r-1, y_0{=}\ 2t+4, y_1{=}\ 2t+3, b_0{=}b_1{=}\ 8t+7, y'_0{=}\ 2t+3, y'_1{=}\ 2t+4, b'_0{=}\ 8t+7$ and $b'_1{=}\ 8t+7$. So, $|v_0 - v_1| = 0$ and $|e_0 - e_1| = 1$. Thus $P_{2r}\odot L^4_{6,4t+2}$ is cordial.

**subcase 3.2.**$k$ odd

Let $k{=}\ 2r+1, r \geq 0$ and $m{=}\ 4t+3$, $t \geq 1$. Then, the labelling $[M_{2r+1};\ 1_20_5M_{4t}\ 1_201,\ 1_20_5M_{4t}\ 1_201,\ 0_2\ 1_5M'_{4t}0_210, 0_2\ 1_5M'_{4t}0_210, ..., (r{-}times), 0_2\ 1_5M'_{4t}0_210]$ for $P_{2r+1}\odot L^4_{5,4t+3}$ is applied. Therefore $x_0{=}r+1, x_1{=}r, a_0{=}\ 0, a_1{=}\ 2r, y_0{=}\ 2t+4, y_1{=}2t+3, b_0{=}b_1{=}\ 8t+7, y'_0{=}y''_0{=}\ 2t+4, y'_1{=}y''_1{=}\ 2t+3, b'_0{=}b''_0{=}\ 8t+7$ and $b'_1{=}b''_1{=}\ 8t+7$. So, $|v_0 - v_1| = 0$ and

$|e_0 - e_1| = 0$. Thus $P_{2r+1} \odot L^4_{6,4t+3}$ is cordial.

**Case 4.** At $m \equiv 3(mod4)$, we need to examine the following subcases:

**subcase 4.1.** At $k \equiv 0(mod4)$ .

Let $k= 4r$, $r \geq 1$. Then, the labeling $[L_{4r};S_40_3M_{4t}0, S_40_3M_{4t}0, S_40_3M_{4t}0, S_40_3M_{4t}0, ..., (r-time)]$ for $P_{4r} \odot L^4_{6,4t+3}$ is applied. Therefore $x_0$=$x_1$= $2r, a_0$= $2r, a_1$= $2r - 1$,$y_0$=$y_1$= $2t + 4$,$b_0$=$b_1$= $8t + 9$,$y'_0$=$y'_1$= $2t + 4$ and $b'_0$=$b'_1$= $8t + 9$. Consequently, it is easy to show that $|v_0 - v_1| = 0$ and $|e_0 - e_1| = 1$. Thus $P_{4r} \odot L^4_{6,4t+3}$, is cordial.

**subcase 4.2.** At $k \equiv 1(mod4)$ .

Let $k$= $4r$+1, $r \geq 0$. Then one can choose the labeling $[L_{4r}0;S_40_3M_{4t}0, S_40_3M_{4t}0, S_40_3M_{4t}0, S_40_3M_{4t}0, ..., (r-time), S_40_3M_{4t}0]$ for $P_{4r+1} \odot L^4_{6,4t+3}$. Therefore $x_0$= $2r$+1,$x_1$= $2r$, $a_0$=$a_1$= $2r$, $y_0$=$y_1$= $2t + 4$,$b_0$=$b_1$= $8t + 9$,$y'_0$=$y'_1$= $2t + 4$ and $b'_0$=$b'_1$= $8t + 9$. So, $|v_0 - v_1| = 1$ and $|e_0 - e_1| = 0$. Thus $P_{4r+1} \odot L^4_{6,4t+3}$, is cordial.

**subcase 4.3.** At $k \equiv 2(mod4)$ .

Let $k$= $4r$+2, $r \geq 0$. Then one can take the labeling $[L_{4r}10;S_40_3M_{4t}0, S_40_3M_{4t}0, S_40_3M_{4t}0, S_40_3M_{4t}0, ..., (r - time), S_40_3M_{4t}0, S_40_3M_{4t}0]$ for $P_{4r+2} \odot L^4_{6,4t+3}$. Therefore $x_0$=$x_1$= $2r + 1$,$a_0$= $2r + 1$,$a_1$= $2r$, $y_0$=$y_1$= $2t + 4$,$b_0$=$b_1$= $8t + 9$,$y'_0$=$y'_1$= $2t + 4$ and $b'_0$=$b'_1$= $8t + 9$. Hence, $|v_0 - v_1| = 0$ and $|e_0 - e_1| = 1$. Thus $P_{4r+2} \odot L^4_{6,4t+3}$, is cordial.

**subcase 4.4.** At $k \equiv 3(mod4)$ .

Let $k$= $4r$+3, $r \geq 0$. Then one can select the labeling $[L_{4r}001;S_40_3M_{4t}0, S_40_3M_{4t}0, S_40_3M_{4t}0, S_40_3M_{4t}0, ..., (r-time), S_40_3M_{4t}0, S_40_3M_{4t}0, S_40_3M_{4t}0]$ for $P_{4r+3} \odot L^4_{6,4t+3}$. Therefore $x_0$= $2r + 2$,$x_1$= $2r + 1$,$a_0$=$a_1$= $2r + 1$,$y_0$=$y_1$= $2t + 4$,$b_0$=$b_1$= $8t + 9$,$y'_0$=$y'_1$= $2t + 4$ and $b'_0$=$b'_1$= $8t + 9$. Consequently, it is easy to show that $|v_0 - v_1| = 1$ and $|e_0 - e_1| = 0$. Thus $P_{4r+3} \odot L^4_{6,4t+3}$, is cordial and the lemma follows.■

## 4. Conclusion

*We proved that the corona* $P_k \odot L^4_{n,m}$ *between paths* $P_k$ *and fourth power of lemniscate graphs* $L^4_{n,m}$ *is cordial for al* $k \geq 1, n, m \geq 3$. *In the future, we will apply cordial labeling to other types of graphs.*

## Acknowledgements

The Researchers would like to thank the Deanship of Graduate Studies and Scientific Research at Qassim University for financial support (QU-APC-2024-9/1)

## Data Availability Statement

All data generated or analyzed during this study are included in this published article.

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