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Jordan φ-Centralizers on Semiprime and Involution Rings

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Abstract. The intention of the current investigation is to demonstrate that if an additive mapping $\mathcal{H} : R \to R$ fulfills certain identities, then $\mathcal{H}$ is a φ-centralizer on R, where R is any suitable, torsion-free semiprime ring and p is a fixed integer greater than or equal to 1. As a result of the primary theorems, involution I_v related observations are also provided. We will also consider criticism and discussion alongside the proofs of theorems. Suitable examples are given in favor of justification.

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1. Introduction

In order to effectively comprehend our concept, we must first recall a few fundamental ideas. Throughout, R shall stand for an associative ring with unity e. A ring R is termed as p-torsion free, where $p > 1$, if $pr = 0$ entails $r = 0$ for every $r \in R$. A ring R is recognised as a prime if $rRt = \{0\}$ implies that either $r = 0$ or $t = 0$, and is termed as a semiprime if $rRr = \{0\}$ yields $r = 0$. The study of Helgosen [5], who introduced the idea of centralizers on Banach algebras, which is also known as multipliers, is supportive of our current understanding.

A possible idea of centralizers on commutative Banach algebra put out by Wang [14]. Further study on centralizers for topological algebras and the continuity of centralizers on Banach algebras is done by Johnson [9]. Further, Johnson studiedthe behaviour of centralizers on algebra of compact operators on Banach space over itself in [7]. Since, every centralizer on a commutative faithful Banach algebra is continuous. Simultaneously, he studied another class of Banach algebras that includes the group of algebras of locally

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compact topological groups in [8].

Additionally, Husain [6] has studied centralizers on topological algebras, specifically focusing on topological algebras with orthogonal bases and full metrizable locally convex algebras. Authors have since examined centralizers and double centralizers on certain topological algebras in [10, 12]. In [2], authors investigated the idea of Jordan $*$-derivations on standard operator algebras. The study of Hopf algebras, representation theory of Banach algebras, the study of Banach modules, the theory of singular integrals, interpolation theory, stochastic processes, the theory of semigroups of operators, partial differential equations, and the study of approximation problems are just a few of the fields in which centralizers have also been used (for more information, see Larsen [11]).

An additive mapping $\mathcal{H} : R \to R$ is said to be a left (right) centralizer if it holds $\mathcal{H}(rt) = \mathcal{H}(r)t$ $\big(respectively,\ \mathcal{H}(rt) = r\mathcal{H}(t)\big)$ for all $r, t \in R$ and it is known as a Jordan right (Jordan left) centralizer if $\mathcal{H}(r^2) = r\mathcal{H}(r)$ $\big(respectively,\ \mathcal{H}(r^2) = \mathcal{H}(r)r\big)$ for all $r \in R$. In recognizing that this mapping $\mathcal{H}$ is both a right centralizer and a left centralizer, we refer to it as a centralizer. As a result of Albas [1], $\mathcal{H} : R \to R$ is called as a left (right) φ-centralizer if $\mathcal{H}(rt) = \mathcal{H}(r)\varphi(t)$ $\big(\mathcal{H}(rt) = \varphi(r)\mathcal{H}(t)\big)$ and it is additive, for all $r, t \in R$ and is known as a Jordan right (Jordan left) φ-centralizer if $\mathcal{H}(r^2) = \varphi(r)\mathcal{H}(r)$ $\big(\mathcal{H}(r^2) = \mathcal{H}(r)\varphi(r)\big)$ for all $r \in R$, where φ is an endomorphism on R. An additive mapping $\mathcal{H}$ is recognised as a φ-centralizer, if $\mathcal{H}$ is both left as well as right φ-centralizer. Every Jordan φ-centralizer is a φ-centralizer. However, this isn't usually the case. Under appropriate torsion restrictions, the converse of this statement is also true for a semiprime ring provided in [1].

Motivated by previous literature review, in the present paper, authors presented an extension of this mathematical statement. Specifically, $\mathcal{H} : R \to R$ is a φ-centralizer, if $\mathcal{H}$ fulfills any one of the following $3\mathcal{H}(r^{3p}) = \mathcal{H}(r^p)\varphi(r^{2p}) + \varphi(r^p)\mathcal{H}(r^p)\varphi(r^p) + \varphi(r^{2p})\mathcal{H}(r^p)$, $2\mathcal{H}(r^{2p}) = \mathcal{H}(r^p)\varphi(r^p) + \varphi(r^p)\mathcal{H}(r^p)$ and $\mathcal{H}(r^{3p}) = \varphi(r^p)\mathcal{H}(r^p)\varphi(r^p)$ for every r in a semiprime ring R that is specifically torsion restricted.

2. On φ-centralizer

The following outcome is required to validate the basic theorems:

Lemma 1 ([4, Theorem 1.2])**.** *If φ is a surjective endomorphism on a semiprime ring R with 2 torsion free condition and $\mathcal{H} : R \to R$ is an additive mapping that satisfies $2\mathcal{H}(r^2) = \mathcal{H}(r)\varphi(r) + \varphi(r)\mathcal{H}(r)$ for all $r \in R$. Then $\mathcal{H}$ is a φ-centralizer on R.*

We start with the study considering the following problem:

Theorem 1. *If φ is a surjective endomorphism on a semiprime ring R with $(3p-1)!$ torsion free condition and $\mathcal{H} : R \to R$ is an additive mapping that satisfies*

$$3\mathcal{H}(r^{3p}) = \mathcal{H}(r^p)\varphi(r^{2p}) + \varphi(r^p)\mathcal{H}(r^p)\varphi(r^p) + \varphi(r^{2p})\mathcal{H}(r^p) \text{ for all } r \in R, \tag{1}$$

then $\mathcal{H}$ is a φ-centralizer on R, where p is a fixed integer greater than or equal to 1.

Proof. We commence with the equation (1) by replacing r by $r + \mathrm{k}t$, we get the following for k being a positive integer and $t \in R$

$$3\mathcal{H}(r^{3p}+\binom{3p}{1}(r^{3p-1})\mathrm{k}t+\binom{3p}{2}r^{3p-2}\mathrm{k}^2t^2+...+\mathrm{k}^{3p}t^{3p}) = \mathcal{H}(r^p+\binom{p}{1}r^{p-1}\mathrm{k}t+\binom{p}{2}r^{p-2}\mathrm{k}^2t^2+ ...+\mathrm{k}^pt^p)\cdot\varphi(r^{2p}+\binom{2p}{1}r^{2p-1}\mathrm{k}t+\binom{2p}{2}r^{2p-2}\mathrm{k}^2t^2+...+\mathrm{k}^{2p}t^{2p})+\varphi(r^p+\binom{p}{1}r^{p-1}\mathrm{k}t+\binom{p}{2}r^{p-2}\mathrm{k}^2t^2+ ...+\mathrm{k}^pt^p)\cdot\mathcal{H}(r^p+\binom{p}{1}r^{p-1}\mathrm{k}t+\binom{p}{2}r^{p-2}\mathrm{k}^2t^2+...+\mathrm{k}^pt^p)\cdot\varphi(r^p+\binom{p}{1}r^{p-1}\mathrm{k}t+\binom{p}{2}r^{p-2}\mathrm{k}^2t^2+...+ \mathrm{k}^pt^p)+\varphi(r^{2p}+\binom{2p}{1}r^{2p-1}\mathrm{k}t+\binom{2p}{2}r^{2p-2}\mathrm{k}^2t^2+...+\mathrm{k}^{2p}t^{2p})\cdot\mathcal{H}(r^p+\binom{p}{1}r^{p-1}\mathrm{k}t+\binom{p}{2}r^{p-2}\mathrm{k}^2t^2+ ...+\mathrm{k}^pt^p).$$

Restate the preceding expression using (1) as

$$\mathrm{k}\mathcal{A}_1(r,t) + \mathrm{k}^2\mathcal{A}_2(r,t) + ... + \mathrm{k}^{3p-1}\mathcal{A}_{3p-1}(r,t) = 0,$$

where the coefficients of k^i are delimited by $\mathcal{A}_i(r,t)$ for all $i = 1, 2, ..., (3p-1)$. A system of $(3p-1)$ homogeneous equations can be obtained if we substitute $1, 2, ..., (3p-1)$ for k one by one, it provides a Vandermonde matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2^2 & \cdots & 2^{3p-1} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ 3p-1 & (3p-1)^2 & \cdots & (3p-1)^{3p-1} \end{bmatrix}.$$

Which yields that $\mathcal{A}_i(r,t) = 0$ for all $r, t \in R$ and for $i = 1, 2, .., (3p-1)$. In particular, We have $\mathcal{A}_1(r,t) = 0$ implies that

$$\begin{aligned} 3\binom{3p}{1}\mathcal{H}(r^{3p-1}t) \;=\; & \binom{2p}{1}\mathcal{H}(r^p)\varphi(r^{2p-1}t) + \binom{p}{1}\mathcal{H}(r^{p-1}t)\varphi(r^{2p}) \\ & +\binom{p}{1}\varphi(r^{p-1}t)\mathcal{H}(r^p)\varphi(r^p) + \binom{p}{1}\varphi(r^p)\mathcal{H}(r^{p-1}t)\varphi(r^p) \\ & +\binom{p}{1}\varphi(r^p)\mathcal{H}(r^p)\varphi(r^{p-1}t) + \binom{2p}{1}\varphi(r^{2p-1}t)\mathcal{H}(r^p) \end{aligned}$$

for each $r, t \in R$. If we put e in place of r in above expression, then we find $9p\mathcal{H}(t) = 2p\mathcal{H}(e)t + 2pt\mathcal{H}(e) + 3p\mathcal{H}(t) + p\mathcal{H}(e)t + pt\mathcal{H}(e)$, for each $t \in R$. Making use of torsion restrictions on R to obtain

$$2\mathcal{H}(t) = \mathcal{H}(e)\varphi(t) + \varphi(t)\mathcal{H}(e), \text{ for all } t \in R. \tag{2}$$

Next, $\mathcal{A}_2(r,t) = 0$ implies that

$$\begin{aligned} 3\binom{3p}{2}\mathcal{H}(r^{3p-2}t^2) \;=\; & \binom{2p}{2}\mathcal{H}(r^p)\varphi(r^{2p-2}t^2) + \binom{p}{1}\binom{2p}{1}\mathcal{H}(r^{p-1}t)\varphi(r^{2p-1}t) \\ & +\binom{p}{2}\mathcal{H}(r^{p-2}t^2)\varphi(r^{2p}) + \binom{p}{2}\varphi(r^p)\mathcal{H}(r^p)\varphi(r^{p-2}t^2) \\ & +\binom{p}{1}\binom{p}{1}\varphi(r^{p-1}t)\mathcal{H}(r^p)\varphi(r^{p-1}t) + \binom{p}{1}\binom{p}{1}\varphi(r^p)\mathcal{H}(r^{p-1}t)\varphi(r^{p-1}t) \\ & +\binom{p}{2}\varphi(r^p)\mathcal{H}(r^{p-2}t^2)\varphi(r^p) + \binom{p}{1}\binom{p}{1}\varphi(r^{p-1}t)\mathcal{H}(r^{p-1}t)\varphi(r^p) \\ & +\binom{p}{2}\varphi(r^{p-2}t^2)\mathcal{H}(r^p)\varphi(r^p) + \binom{p}{2}\varphi(r^{2p})\mathcal{H}(r^{p-2}t^2) \\ & +\binom{2p}{1}\binom{p}{1}\varphi(r^{2p-1}t)\mathcal{H}(r^{p-1}t) + \binom{2p}{2}\varphi(r^{2p-2}t^2)\mathcal{H}(r^p). \end{aligned}$$

Reword the above expression by putting e in place of r, we have

$$\begin{aligned} 3\tfrac{3p(3p-1)}{2}\mathcal{H}(t^2) &= \tfrac{2p(2p-1)}{2}\mathcal{H}(e)\varphi(t^2) + 2p^2\mathcal{H}(t)\varphi(t) + \tfrac{p(p-1)}{2}\mathcal{H}(t^2) \\ &+\tfrac{p(p-1)}{2}\mathcal{H}(e)\varphi(t^2) + p^2\varphi(t)\mathcal{H}(e)\varphi(t) + \tfrac{p(p-1)}{2}\mathcal{H}(t^2) \\ &+p^2\varphi(t)\mathcal{H}(e)\varphi(t) + p^2\varphi(t)\mathcal{H}(t) + \tfrac{p(p-1)}{2}\varphi(t^2)\mathcal{H}(e) \\ &+\tfrac{p(p-1)}{2}\mathcal{H}(t^2) + 2p^2\varphi(t)\mathcal{H}(t) + \tfrac{2p(2p-1)}{2}\varphi(t^2)\mathcal{H}(e) \end{aligned}$$

Simplify the above expression to find

$$\begin{aligned} 9p(3p-1)\mathcal{H}(t^2) &= 2p(2p-1)\mathcal{H}(e)\varphi(t^2) + 4p^2\mathcal{H}(t)\varphi(t) + p(p-1)\mathcal{H}(t^2) \\ &+p(p-1)\mathcal{H}(e)\varphi(t^2) + 2p^2\mathcal{H}(t)\varphi(t) + p(p-1)\mathcal{H}(t^2) \\ &+2p^2\varphi(t)\mathcal{H}(e)\varphi(t) + 2p^2\varphi(t)\mathcal{H}(t) + p(p-1)\varphi(t^2)\mathcal{H}(e) \\ &+p(p-1)\mathcal{H}(t^2) + 4p^2\varphi(t)\mathcal{H}(t) + 2p(2p-1)\varphi(t^2)\mathcal{H}(e) \end{aligned}$$

Collect the like terms in above expression and make them more comprehensible as

$$\begin{aligned} (24p^2-6p)\mathcal{H}(t^2) &= (5p^2-3p)2\mathcal{H}(t^2) + 6p^2(\mathcal{H}(t)\varphi(t) \\ &+\varphi(t)\mathcal{H}(t)) + 2p^2\varphi(t)\mathcal{H}(e)\varphi(t), \end{aligned}$$

for all $t \in R$. Again comparing the like terms both side and using equation (2), we come up with

$$14p^2\mathcal{H}(t^2) = 6p^2(\mathcal{H}(t)\varphi(t) + \varphi(t)\mathcal{H}(t)) + 2p^2\varphi(t)\mathcal{H}(e)\varphi(t),$$

for all $t \in R$. Hence, we get by applying the torsion freeness of R

$$14\mathcal{H}(t^2) = 6\big(\mathcal{H}(t)\varphi(t) + \varphi(t)\mathcal{H}(t)\big) + 2\varphi(t)\mathcal{H}(e)\varphi(t), \text{ for every } t \in R. \tag{3}$$

Multiplying from right side by $\varphi(r)$ to (2), we obtain

$$2\mathcal{H}(r)\varphi(r) = \mathcal{H}(e)\varphi(r^2) + \varphi(r)\mathcal{H}(e)\varphi(r) \text{ for all } r \in R.$$

Multiply from left side by $\varphi(r)$ to (2) to get

$$2\varphi(r)\mathcal{H}(r) = \varphi(r)\mathcal{H}(e)\varphi(r) + \varphi(r^2)\mathcal{H}(e) \text{ for all } r \in R.$$

Adding these equations, we find

$$2(\mathcal{H}(r)\varphi(r) + \varphi(r)\mathcal{H}(r)) = 2\varphi(r)\mathcal{H}(e)\varphi(r) + 2\mathcal{H}(r^2),$$

which implies that

$$2\varphi(r)\mathcal{H}(e)\varphi(r) = 2(\mathcal{H}(r)\varphi(r) + \varphi(r)\mathcal{H}(r)) - 2\mathcal{H}(r^2) \text{ for all } r \in R.$$

Using this equation in (3), we have

$$14\mathcal{H}(r^2) = 6\big(\mathcal{H}(r)\varphi(r) + \varphi(r)\mathcal{H}(r)\big) + 2\big(\mathcal{H}(r)\varphi(r) + \varphi(r)\mathcal{H}(r)\big) - 2\mathcal{H}(r^2) \text{ for all } r \in R.$$

Using torsion restrictions on R, we get

$$2\mathcal{H}(r^2) = \mathcal{H}(r)\varphi(r) + \varphi(r)\mathcal{H}(r) \text{ for all } r \in R.$$

Using Lemma 1, we obtained the required outcome.

Theorem 2. *If φ is a surjective endomorphism on a semiprime ring R with $(2p-1)!$ torsion free condition and $\mathcal{H} : R \to R$ is an additive mapping that satisfies*

$$2\mathcal{H}(r^{2p}) = \mathcal{H}(r^p)\varphi(r^p) + \varphi(r^p)\mathcal{H}(r^p) \text{ for all } r \in R, \tag{4}$$

then $\mathcal{H}$ is a φ-centralizer on R, where p is a fixed integer greater than or equal to 1.

Proof. We proceed with (4) and replace r by $r + \mathrm{k}t$ for $\mathrm{k} \geq 1$ and $t \in R$, to get

$$2\mathcal{H}(r^{2p} + \binom{2p}{1}(r^{2p-1})\mathrm{k}t + \binom{2p}{2}r^{2p-2}\mathrm{k}^2t^2 + ... + \mathrm{k}^{2p}t^{2p}) = \mathcal{H}(r^p + \binom{p}{1}r^{p-1}\mathrm{k}t + \binom{p}{2}r^{p-2}\mathrm{k}^2t^2 + ... + \mathrm{k}^pt^p) \cdot \varphi(r^p + \binom{p}{1}r^{p-1}\mathrm{k}t + \binom{p}{2}r^{p-2}\mathrm{k}^2t^2 + ... + \mathrm{k}^pt^p) + \varphi(r^p + \binom{p}{1}r^{p-1}\mathrm{k}t + \binom{p}{2}r^{p-2}\mathrm{k}^2t^2 + ... + \mathrm{k}^pt^p) \cdot \mathcal{H}(r^p + \binom{p}{1}r^{p-1}\mathrm{k}t + \binom{p}{2}r^{p-2}\mathrm{k}^2t^2 + ... + \mathrm{k}^pt^p) \text{ for all } r, t \in R \text{ and } \mathrm{k} \geq 1.$$

Rewrite the above expression by using (4) as

$$\mathrm{k}\mathcal{Q}_1(r,t) + \mathrm{k}^2\mathcal{Q}_2(r,t) + ... + \mathrm{k}^{2p-1}\mathcal{Q}_{2p-1}(r,t) = 0.$$

A system of $(2p-1)$ homogeneous equations can be identified if k is substituted by $1, 2, ..., (2p-1)$ in turn. It provides a Vandermonde matrix of type $(2p-1) \times (2p-1)$, then for $i = 1, 2, .., (2p-1)$, $\mathcal{Q}_i(r,t) = 0$ for all $r, t \in R$. Particularly, $\mathcal{Q}_1(r,t) = 0$ suggests

$$\begin{aligned} 2\binom{2p}{1}\mathcal{H}(r^{2p-1}\mathrm{k}t) \quad = \quad & \binom{p}{1}\mathcal{H}(r^p)\varphi(r^{p-1}t) + \binom{p}{1}\mathcal{H}(r^{p-1})\varphi(tr^p) \\ & + \binom{p}{1}\varphi(r^p)\mathcal{H}(r^{p-1}t) + \binom{p}{1}\varphi(r^{p-1}t)\mathcal{H}(r^p) \end{aligned} \tag{5}$$

Reinstate the above equation by putting e in place of r to have $4p\mathcal{H}(t) = p\mathcal{H}(e)\varphi(t) + p\mathcal{H}(t) + p\mathcal{H}(t) + p\varphi(t)\mathcal{H}(e)$. On simplifying the last relation we can obtain $2p\mathcal{H}(t) = p\mathcal{H}(e)\varphi(t) + p\varphi(t)\mathcal{H}(e)$ for all $t \in R$. A torsion restriction given in the hypothesis enable us to write

$$2\mathcal{H}(t) = \mathcal{H}(e)\varphi(t) + \varphi(t)\mathcal{H}(e), \quad \text{for all } t \in R. \tag{6}$$

Now consider the following $\mathcal{Q}_2(r,t) = 0$, we have

$$\begin{aligned} 2\binom{2p}{2}\mathcal{H}(r^{2p-2}t^2) \quad = \quad & \binom{p}{2}\mathcal{H}(r^p)\varphi(r^{p-2}t^2) + \binom{p}{1}\binom{p}{1}\mathcal{H}(r^{p-1}t)\varphi(r^{p-1}t) \\ & + \binom{p}{2}\mathcal{H}(r^{p-2}t^2)\varphi(r^p) + \binom{p}{2}\varphi(r^p)\mathcal{H}(r^{p-2}t^2) \\ & + \binom{p}{1}\binom{p}{1}\varphi(r^{p-1}t)\mathcal{H}(r^{p-1}t) + \binom{p}{2}\varphi(r^{p-2}t^2)\mathcal{H}(r^p) \end{aligned}$$

Substitute e for r in above expression to obtain

$$\begin{aligned} 2\frac{2p(2p-1)}{2}\mathcal{H}(t^2) \quad = \quad & \frac{p(p-1)}{2}\mathcal{H}(e)\varphi(t^2) + p^2\mathcal{H}(t)\varphi(t) + \frac{p(p-1)}{2}\mathcal{H}(t^2) \\ & + \frac{p(p-1)}{2}\mathcal{H}(t^2) + p^2\varphi(t)\mathcal{H}(t) + \frac{p(p-1)}{2}\varphi(t^2)\mathcal{H}(e) \end{aligned}$$

A simple manipulation yields that $2p^2\mathcal{H}(t^2) = p^2(\mathcal{H}(t)\varphi(t) + \varphi(t)\mathcal{H}(t))$. By utilizing the torsion-freeness of R, we achieved $2\mathcal{H}(t^2) = \mathcal{H}(t)\varphi(t) + \varphi(t)\mathcal{H}(t)$ for all $t \in R$. Hence $\mathcal{H}$ is carry oneself like φ-centralizer, as desired.

Theorem 3. *If φ is a surjective endomorphism on a semiprime ring R with $(3p-1)!$ torsion free condition and $\mathcal{H} : R \to R$ is an additive mapping that satisfies*

$$\mathcal{H}(r^{3p}) = \varphi(r^p)\mathcal{H}\varphi((r^p))r^p \text{ for all } r \in R. \tag{7}$$

Then $\mathcal{H}$ is a φ-centralizer on R, where p is a fixed integer greater than or equal to 1.

Proof. Replacing r by $r + \mathrm{k}t$ in (7), we obtain

$\mathcal{H}(r^{3p}+\binom{3p}{1}(r^{3p-1})\mathrm{k}t+\binom{3p}{2}r^{3p-2}\mathrm{k}^2t^2+...+\mathrm{k}^{3p}t^{3p}) = \varphi(r^p+\binom{p}{1}r^{p-1}\mathrm{k}t+\binom{p}{2}r^{p-2}\mathrm{k}^2t^2+...+\mathrm{k}^pt^p)\cdot\mathcal{H}(r^p+\binom{p}{1}r^{p-1}\mathrm{k}t+\binom{p}{2}r^{p-2}\mathrm{k}^2t^2+...+\mathrm{k}^pt^p)\varphi(r^p+\binom{p}{1}r^{p-1}\mathrm{k}t+\binom{p}{2}r^{p-2}\mathrm{k}^2t^2+...+\mathrm{k}^pt^p)$, where k is a positive integer.

Rewrite the above expression by using (7) as

$$\mathrm{k}\mathcal{R}_1(r,t) + \mathrm{k}^2\mathcal{R}_2(r,t) + ... + \mathrm{k}^{3p-1}\mathcal{R}_{3p-1}(r,t) = 0,$$

where $\mathcal{R}_i(r,t)$ signifies for the coefficients of power of k up to $(3p-1)$. Replacing k by $1, 2, ..., (3p-1)$ in turn, we obtain a system of $(3p-1)$ homogeneous equations that provides a Vander Monde matrix of $(3p-1)$ by $(3p-1)$ that implies that for all $r, t \in R$, $\mathcal{R}_i(r,t) = 0$, where $i = 1, 2, .., (3p-1)$. In particular, We have $\binom{3p}{1}\mathcal{H}(r^{3p-1}t) = \binom{p}{1}\varphi(r^p)\mathcal{H}(r^{p-1}t)\varphi(r^p) + \binom{p}{1}\varphi(r^p)\mathcal{H}(r^p)\varphi(r^{p-1}t) - \binom{p}{1}\varphi(r^{p-1}t)\mathcal{H}(r^p)\varphi(r^p)$ for all $r, t \in R$. Replacing r by e and making use of torsion restriction on R to get

$$2\mathcal{H}(t) = \mathcal{H}(e)\varphi(t) + \varphi(t)\mathcal{H}(e), \text{ for all } t \in R. \tag{8}$$

Further, $\mathcal{R}_2(r,t) = 0$ implies that

$$\begin{aligned}\binom{3p}{2}\mathcal{H}(r^{3p-2}t^2) \;=\;& \binom{p}{2}\varphi(r^p)\mathcal{H}(r^p)\varphi(r^{p-2}t^2) + \binom{p}{1}\binom{p}{1}\varphi(r^{p-1}t)\mathcal{H}(r^p)\varphi(r^{p-1}t)\\ &+\binom{p}{1}\binom{p}{1}\varphi(r^p)\mathcal{H}(r^{p-1}t)\varphi(r^{p-1}t) + \binom{p}{2}\varphi(r^p)\mathcal{H}(r^{p-2}t^2)\varphi(r^p)\\ &+\binom{p}{1}\binom{p}{1}\varphi(r^{p-1}t)\mathcal{H}(r^{p-1}t)\varphi(r^p) + \binom{p}{2}\varphi(r^{p-2}t^2)\mathcal{H}(r^p)\varphi(r^p).\end{aligned}$$

Reword the above expression by putting e in place of r, we have

$$\begin{aligned}\frac{3p(3p-1)}{2}\mathcal{H}(t^2) \;=\;& \binom{p}{2}\mathcal{H}(e)\varphi(t^2) + \binom{p}{1}\binom{p}{1}\varphi(t)\mathcal{H}(e)\varphi(t)\\ &+\binom{p}{1}\binom{p}{1}\mathcal{H}(t)\varphi(t) + \binom{p}{2}\mathcal{H}(t^2)\\ &+\binom{p}{1}\binom{p}{1}\varphi(t)\mathcal{H}(t) + \binom{p}{2}\varphi(t^2)\mathcal{H}(e).\end{aligned}$$

Simplify the above expression using the same steps as we did in last theorems to find $3p^2\mathcal{H}(t^2) = p^2(\mathcal{H}(t)\varphi(t) + \varphi(t)\mathcal{H}(t)) + p^2\varphi(t)\mathcal{H}(e)\varphi(t)$, for all $t \in R$. Hence, we get by applying the torsion freeness of R

$$3\mathcal{H}(t^2) = (\mathcal{H}(t)\varphi(t) + \varphi(t)\mathcal{H}(t)) + \varphi(t)\mathcal{H}(e)\varphi(t), \text{ for all } t \in R. \tag{9}$$

Multiplying from right side by $\varphi(r)$ to (8), we obtain

$$2\mathcal{H}(r)\varphi(r) = \mathcal{H}(e)\varphi(r^2) + \varphi(r)\mathcal{H}(e)\varphi(r) \text{ for every } r \in R.$$

Multiply from left side by $\varphi(r)$ to (8) to get

$$2\varphi(r)\mathcal{H}(r) = \varphi(r)\mathcal{H}(e)\varphi(r) + \varphi(r^2)\mathcal{H}(e) \text{ for every } r \in R.$$

Adding these equations, we find

$$2(\mathcal{H}(r)\varphi(r) + \varphi(r)\mathcal{H}(r)) = 2\varphi(r)\mathcal{H}(e)\varphi(r) + 2\mathcal{H}(r^2),$$

which implies that

$$\varphi(r)\mathcal{H}(e)\varphi(r) = \mathcal{H}(r)\varphi(r) + \varphi(r)\mathcal{H}(r) - \mathcal{H}(r^2) \text{ for every } r \in R.$$

Using this equation in (9), we have

$$4\mathcal{H}(r^2) = 2(\mathcal{H}(r)\varphi(r) + \varphi(r)\mathcal{H}(r)) \text{ for every } r \in R.$$

Using torsion restrictions on R, get $2\mathcal{H}(r^2) = \mathcal{H}(r)\varphi(r) + \varphi(r)\mathcal{H}(r)$ for every $r \in R$. Using Lemma 1, $\mathcal{H}$ is a φ-centralizer on R.

The subsequent illustration supports our theorems:

Example 1. *Let* $R = \left\{ \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} \mid z_1, z_2 \in 2\mathbb{Z}_8 \right\}$*, whereby the meaning of* $\mathbb{Z}_8$ *is as usual. Define mappings* $\mathcal{H}, \varphi : R \to R$ *by* $\mathcal{H}(r) = \begin{pmatrix} 0 & 0 \\ 0 & z_2 \end{pmatrix}$ *and* $\varphi(r) = \begin{pmatrix} z_2 & 0 \\ 0 & z_1 \end{pmatrix}$ *for all* $r \in R$*. Clearly,* R *is not a semiprime ring but* $\mathcal{H}$ *satisfy the algebraic identity of main theorems ((ii)* $p > 1$*) of this section. It is easy to see that* $\mathcal{H}$ *is not a* φ*-centralizer. Therefore semiprimeness hypothesis is crucial for the above theorems.*

3. On involution ring

Next, an additive mapping $I_v : R \to R$ is said to be an involution if it satisfies $I_v(rt) = I_v(t)I_v(r)$ and $I_v(I_v(r)) = r$ for all $r, t \in R$. A ring with involution is a ring that possesses an involution I_v. An additive mapping $\mathcal{H} : R \to R$ is termed as a right (resp. left) I_v-centralizer if $\mathcal{H}(rt) = I_v(r)\mathcal{H}(t)$ (resp. $\mathcal{H}(rt) = \mathcal{H}(r)I_v(t)$) holds for all $r, t \in R$ and $\mathcal{H}$ is termed as a right (resp. left) Jordan I_v-centralizer if for all $r \in R$, $\mathcal{H}(r^2) = I_v(r)\mathcal{H}(r)$ (resp. $\mathcal{H}(r^2) = \mathcal{H}(r)I_v(r)$). If $\mathcal{H}$ is both right and left (Jordan) I_v-centralizer then, recognised as a (Jordan) I_v-centralizer of R. Let φ be an endomorphism on R. An additive mapping

$\mathcal{H} : R \to R$ is known as a right (resp. left) φ-I_v-centralizer if $\mathcal{H}(rt) = \varphi(I_v(r))\mathcal{H}(t)$ (resp. $\mathcal{H}(rt) = \mathcal{H}(r)\varphi(I_v(t)))$ holds for all $r, t \in R$ and it is φ-I_v-centralizer if it left as well right φ-I_v-centralizer. An additive mapping $\mathcal{H}$ is a right (resp. left) Jordan φ-I_v-centralizer if for all $r \in R$, $\mathcal{H}(r^2) = \varphi(I_v(r))\mathcal{H}(r)$ (resp. $\mathcal{H}(r^2) = \mathcal{H}(r)\varphi(I_v(r)))$. If it is both, then it is Jordan φ-I_v-centralizer of R.

We investigate about the interesting extension of the results presented in the last section in context of involution ring. The study of some identities on involution ring is the main focus of this section. However, evidence suggests that an additive mappings $\mathcal{H}$ on a suitable torsion free restricted semiprime ring R satisfying $2\mathcal{H}(r^{2p}) = \mathcal{H}(r^p)\varphi(I_v(r))^p + \varphi(I_v(r))^p\mathcal{H}(r^p)$, $3\mathcal{H}(r^{3p}) = \mathcal{H}(r^p)\varphi(I_v(r))^{2p} + \varphi(I_v(r))^p\mathcal{H}(r^p)\varphi(I_v(r))^p + \varphi(I_v(r))^{2p}\mathcal{H}(r^p)$ and $\mathcal{H}(r^{3p}) = \varphi(I_v(r))^p\mathcal{H}(r^p)\varphi(I_v(r))^p$ for all $r \in R$, will be a φ-I_v-centralizer of R.

We require the following lemma that supports our primary findings.

Lemma 2 ([3, Corollary 2.1]). *If φ is a surjective endomorphism on a 2 torsion free semiprime ring R with involution I_v and $\mathcal{H} : R \to R$ is an additive mapping that satisfies $2\mathcal{H}(r^2) = \mathcal{H}(r)\varphi(I_v(r)) + \varphi(I_v(r))\mathcal{H}(r)$ for all $r \in R$, then $\mathcal{H}$ is a φ-I_v-centralizer on R.*

Next, start main result of this part.

Theorem 4. *If φ is a surjective endomorphism on a $(3p-1)!$ torsion free semiprime ring R with involution I_v and $\mathcal{H} : R \to R$ is an additive mapping that satisfies any one of the following algebraic identities:*

(i) $3\mathcal{H}(r^{3p}) = \mathcal{H}(r^p)\varphi(I_v(r))^{2p} + \varphi(I_v(r))^p\mathcal{H}(r^p)\varphi(I_v(r))^p + \varphi(I_v(r))^{2p}\mathcal{H}(r^p)$

(ii) $2\mathcal{H}(r^{2p}) = \mathcal{H}(r^p)\varphi(I_v(r))^p + \varphi(I_v(r))^p\mathcal{H}(r^p)$

(iii) $\mathcal{H}(r^{3p}) = \varphi(I_v(r))^p\mathcal{H}(r^p)\varphi(I_v(r))^p$ *for all $r \in R$,*

then, $\mathcal{H}$ is a φ-I_v-centralizer on R, where p is a fixed integer greater than or equal to 1.

Proof. (*i*) Define a mapping $\mathcal{T} : R \to R$ such that $\mathcal{T}(r) = \mathcal{H}(I_v(r))$ for all $r \in R$. Then,

$$\begin{aligned} \mathcal{T}(r+s) &= \mathcal{H}(I_v(r+s)) \\ &= \mathcal{H}(I_v(r) + I_v(s)) \\ &= \mathcal{H}(I_v(r)) + \mathcal{H}(I_v(s)) \\ &= \mathcal{T}(r) + \mathcal{T}(s) \end{aligned}$$

for every $r, s \in R$. Therefore, $\mathcal{T}$ is an additive mapping on R. Now, consider

$$\begin{aligned} 3\mathcal{T}(r^{3p}) &= 3\mathcal{H}(I_v((r^{3p})) \\ &= \mathcal{H}((I_v(r))^p)\varphi((r)^{2p}) + \varphi((r)^p)\mathcal{H}((I_v(r))^p)\varphi((r)^p) + \varphi((r)^{2p})\mathcal{H}(I_v(r)) \\ &= \mathcal{T}(r^p)\varphi(r^{2p}) + \varphi(r^p)\mathcal{T}(r^p)\varphi(r^p) + \varphi(r^{2p})\mathcal{T}(r^p) \text{ for all } r, t \in R. \end{aligned}$$

Use first theorem of the last section to find that $\mathcal{T}$ is a φ-centralizer on R. Hence,

$$\begin{aligned}\mathcal{H}(r^2) &= \mathcal{H}(I_v(I_v(r^2))\\ &= \mathcal{T}(I_v(r^2)\\ &= \mathcal{T}(I_v(r)I_v(r))\\ &= \mathcal{T}(I_v(r))\varphi(I_v(r)), \quad \text{as } \mathcal{T} \text{ is a } \varphi\text{-centralizer on } R\\ &= \mathcal{H}(r)\varphi(I_v(r))\end{aligned}$$

This implies that $\mathcal{H}$ is a Jordan left φ-I_v-centralizer on R. Similarly, we can prove that $\mathcal{H}$ is a Jordan right φ-I_v-centralizer on R. Now, combine and use Lemma 2 to obtain the required conclusion.

Using same technique, we can get proof of *(ii)* and *(iii)* part.

Example 2. *Let* $R = \left\{ \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} \mid z_1, z_2 \in 2\mathbb{Z}_8 \right\}$ *is a ring with involution* $I_v : R \to R$ *by* $I_v(r) = \begin{pmatrix} z_2 & 0 \\ 0 & z_1 \end{pmatrix}$*, where* $\mathbb{Z}_8$ *signifies what it normally does. Define mappings* $\mathcal{H}, \varphi : R \to R$ *by* $\mathcal{H}(r) = \begin{pmatrix} 0 & 0 \\ 0 & z_2 \end{pmatrix}$ *and* $\phi(r) = \begin{pmatrix} 2z_2 & 0 \\ 0 & 2z_1 \end{pmatrix}$ *for all* $r \in R$*. It is clear that* $\mathcal{H}$ *satisfy the identities in Theorem 4 ((ii)* $p > 1$*) and* R *is neither a 2-torsion free semiprime ring nor* $\mathcal{H}$ *is a* φ-I_v*-centralizer on* R*, hence semiprimeness hypothesis is crucial for Theorem 4.*

4. Conclusion

We explore φ-centralizer on rings and φ-I_v-centralizer on rings with involution in depth, which is an intriguing topic. Obtaining continuity theorems on other algebraic structures, such as Banach algebra, semi-simple Banach algebra, Lie algebra, C^* algebra, etc., is the area of future study in the framework of the provided research. It would be fascinating to view our concept in the context of [13] with the aid of algebra of linear operators (transformations). It is also interesting that the reader can consider various functional identities involving specific sorts of derivations, such as generalized (α, β)-derivations on semiprime rings with involution and generalized (α, β)-higher derivations. The forms of additive maps applying to rings and their corresponding subsets have been described using just algebraic techniques.

Competing Interests

Regarding the publication of this work, the authors affirm that they have no conflicts of interest.

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