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Middle Graph of the Identity Graph of Finite Cyclic and Dihedral Groups

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Abstract. Given a group $\mathbb{G}$ with e as the identity element, the identity graph $\Gamma_{\mathbb{G}}$ having the vertex-set $\mathbb{G}$ and the edge-set E satisfies two conditions: (i) for every $x, y \in \mathbb{G}$ where $x \neq y$, x and y are adjacent in $\Gamma_{\mathbb{G}}$ if and only if $xy = e$; (ii) for each $x \in \mathbb{G}$, x and e are adjacent in $\Gamma_{\mathbb{G}}$. The middle graph of G denoted by $M(G)$ is the graph with vertex set $V(G) \cup E(G)$ where two vertices will be adjacent if and only if they are either adjacent edges of G or one is a vertex and the other is an edge incident to it. It can be obtained by inserting a new vertex into every edge of $\mathbb{G}$ and connecting the new obtained vertices if they are adjacent edges in $\mathbb{G}$. In this paper, we constructed the middle graph of the identity graph particular for finite cyclic and dihedral groups. Some parameters of a graph such as the size, order, graph measurements, independence number, domination number, vertex chromatic number and edge chromatic number were also investigated.

2020 Mathematics Subject Classifications: 05

Key Words and Phrases: Middle Graph, Identity Graph, Cyclic Groups, Dihedral Groups, Graph Properties

1. Introduction

The linking of group theory to graph theory was started in 2009 in the book of [1] by creating a new kind of graph from the concepts of group theory. They represented the finite groups in terms of graphs which they called identity graphs or identity graphs since the identity element of the group is the main role in order to create a graph.

A.D. Godase [2] in 2015 gave some examples of the identity graphs of some finite groups particular in finite cyclic and dihedral groups which he discovered that the graph formed were consists of lines and triangles. In the papers [3] and [4], the authors further

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studied the said graph by investigating its properties and characteristics.

Another focus of studies in graph theory is the construction new graphs from another graph. Given a graph, operation will be defined that yields to another form of graph. One particular example is the paper of Akiyama, Hamada, and Yoshimura [5] which introduced the concept of the middle graph and established some characterizations in particular for some common classes of graphs. The middle graph of a graph is obtained by inserting a new vertex into every edge of the original graph and connecting the new obtained vertices if they are adjacent edges in the original graph.

The combination of two concepts motivates the authors to explore the middle graphs of the identity graph of the finite cyclic and dihedral groups. This is in parallel with the study of Murusegan and Nair [6] which discussed the $(1, 2)$-domination in middle and central graph of a star, cycle and path. They also established some upperbounds and lower bounds. Later on 2017, they investigated the power domination of middle graph of path, cycle and star. Alib et al. [7] presented the construction of the central graph of the identity graph of finite cyclic group and investigated some of its graph properties.

This paper presents the construction of the middle graphs of the identity graph of the finite cyclic and dihedral groups. The properties particular in graph measurements, independence number, domination number and graph coloring were also investigated.

2. Preliminaries

For the purpose of further understanding concepts, examples, and illustrations are given.

2.1. Group Theory

This section contains some basic concepts in group theory and its examples that will be needed in the discussion of the following chapters. Groups can be finite or infinite. In general, groups can be classified into two categories, these are the cyclic and noncyclic groups. In this paper, we focus in finite cyclic groups and the dihedral groups. Let us start by defininng a binary operation.

Definition 1. *Let S be a set. A **binary operation** $*$ on S is a function that assigns each ordered pair of elements of S an element of S.*

Consider the set of even integers $S = 2\mathbb{Z}$ using the addition as the operation. If we take $a = 2k, b = j \in S$, for some integers k and j, $a + b = 2k + 2j = 2(k + j)$ which is also an even integer. Thus, addition is a binary operation in S.

Definition 2. *A **group** is a non-empty set $\mathbb{G}$ with binary operation $*$ such that*

i. $a * (b * c) = (a * b) * c$ for all a, b, c in $\mathbb{G}$ (associativity), *ii. there is an element $e \in \mathbb{G}$ such that $a * e = e * a = a$ for all $a \in \mathbb{G}$ (existence of identity),*

iii. if $a \in \mathbb{G}$, then there is an element $a^{-1} \in \mathbb{G}$ such that $a * a^{-1} = a^{-1} * a = e$ (existence of inverse).

Example 1. *The set of integers is a group under the operation of ordinary addition. Note that the set of integers under the operation of addition is closed, associative, contains identity element 0, and for any integer a it has an $-a$.*

The set of integers under ordinary multiplication is not a group. The third property fails since there is no integer b such that $2b = 1$ where 1 is the identity element.

Now we will define a cyclic subgroup,

Definition 3. *If $\mathbb{G}$ is a group and $a \in \mathbb{G}$, then the* ***cyclic subgroup generated by*** *a is the set*

$$\langle a \rangle = \{a^n : n \in Z\}$$

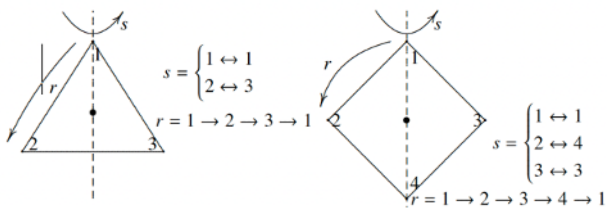
If, in $\mathbb{G}$, there exists an element a such that $\mathbb{G} = \langle a \rangle$, then we say that $\mathbb{G}$ is a ***cyclic group*** *and a is a* ***generator*** *of $\mathbb{G}$. We may also say that $\mathbb{G}$ is a* ***group generated*** *by a. If no such element exists in $\mathbb{G}$, then $\mathbb{G}$ is said to be a* ***noncyclic group****.*

Example 2. *Let $\mathbb{G} = \mathbb{Z}, +$. Then $\mathbb{G} = \langle 1 \rangle = \langle -1 \rangle$, so $\mathbb{G}$ is cyclic. On the other hand, $\mathbb{G} = \mathbb{Q}, +$ is noncyclic, since there is no rational number which generates all possible rational numbers. For the same reason, $\mathbb{G} = \mathbb{R}, +$ is noncyclic*

Another type of finite groups are dihedral groups which belongs to the classification of noncyclic.

Definition 4. *The group of symmetries of an n-sided regular polygon for $n \geq 1$ with rotations and reflections is termed* ***Dihedral group****, which is denoted as D_n. The order of the Dihedral group is $2n$.*

For $n \geq 3$, D_n is the group of symmetries of a regular polygon with n-sides. Number the vertices $1, ..., n$ in the counterclockwise direction. Let r be the rotation through $2\pi/n$ about the centre of polygon (so $i \mapsto i + 1$ mod n), and let s be the reflection in the line (= rotation about the line) through the vertex 1 and the centre of the polygon (so $i \mapsto n + 2 - i$ mod n). Here is an illustration.



2.2. Graph Theory

A **graph** G is an ordered pair $G = (V(G), E(G))$ where $V(G)$ is a nonempty set of elements called **vertices**, and $E(G)$ is a set of unordered pairs of vertices called **edges**. The number $|V(G)|$ is called the **order of G** and the number $|E(G)|$ is called the **size of G** .

Connected graphs are the graphs in which every two vertices is adjacent to each other. If $[u, v]$ is an edge of G, then u and v are **adjacent vertices**. If $[u, v]$ and $[v, w]$ are distinct edges in G, then $[u, v]$ and $[v, w]$ are **adjacent edges**. The vertex u and the edge $[u, v]$ are said to be **incident** with each other.

A graph H is called a **subgraph** of a graph G, written as $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The **degree of a vertex** v in a graph G denoted by $deg_G(v)$ or simply by $deg(v)$ is the number of vertices in G that are adjacent to v. A vertex of degree 0 is referred to as an **isolated vertex** and a vertex of degree 1 is an **end-vertex** or a **leaf**. An edge incident with an end-vertex is called a **pendant edge**.

The distance $d(u, v)$ between $u, v \in V(G)$ is the length of a shortest $u - v$ path in the graph G. The eccentricity of a vertex $u \in V(G)$ is $e(u) = max\{d(u, v)|u \in V(G)\}$. The diameter of a graph G is $diam = max\{e(u)|u \in V(G)\}$. The radius of a graph G is $rad = min\{e(u)|u \in V(G)\}$. If $e(u) = rad(G)$, the vertex u is a central vertex. The set of all such vertices is the center of G. The girth of a graph G denoted by $gir(G)$ is the length of the shortest cycle (if any) in G.

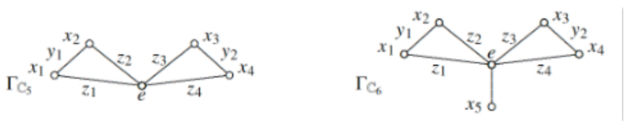
Definition 5. *Given a group $\mathbb{G}$ with e as the identity element, define the* ***identity graph*** *$\Gamma_{\mathbb{G}} = \Gamma_{(\mathbb{G},E)}$ to have the vertex-set $\mathbb{G}$ and the edge-set E satisfying two conditions:*
(i) For every $x, y \in \mathbb{G}$ where $x \neq y$, x and y are adjacent in $\Gamma_{\mathbb{G}}$ if and only if $xy = e$;
(ii) For each $x \in \mathbb{G}$, x and e are adjacent in $\Gamma_{\mathbb{G}}$.

Two important structures in identity graph are lines and triangles defines as follows:

Definition 6. *Given a group $\mathbb{G}$, a* ***line*** *in the identity graph $\Gamma_{\mathbb{G}}$ is an edge $[x, e]$ such that the degree of a vertex x is one. The number of lines in the identity graph $\Gamma_{\mathbb{G}}$ is denoted by $line(\mathbb{G})$.*

Definition 7. *A* ***triangle*** *in the identity graph $\Gamma_{\mathbb{G}}$ is a subgraph which is isomorphic to the cycle of length three. The number of the triangles in the identity graph is denoted by $tri(\mathbb{G})$.*

To give an example, consider the identity graphs of selected cyclic groups below.



To further understand the properties of groups and graphs, here are some propositions. Propositions are presented without proof and can be found in [5], [8], [1], [9] and [3].

Proposition 1. *For a cyclic groups $\mathbb{C}_n$ of order n, if n is odd, then $line(\mathbb{C}_n) = 0$ and $tri(\mathbb{C}_n) = \frac{n-1}{2}$. If n is even, then $line(\mathbb{C}_n) = 1$ and $tri(\mathbb{C}_n) = \frac{n-2}{2}$.*

Proposition 2. *If $\mathbb{C}_n$ is a cyclic group of order n, then we have*

$$|E(\Gamma_{\mathbb{C}_n})| = \begin{cases} \frac{3(n-1)}{2}, & \text{if } n \text{ is odd} \\ \frac{3(n-2)}{2} + 1, & \text{if } n \text{ is even.} \end{cases}$$

Proposition 3. *(Handshaking Lemma) If G is a graph of size m, then*

$$\sum_{v \in V(G)} deg(v) = 2m.$$

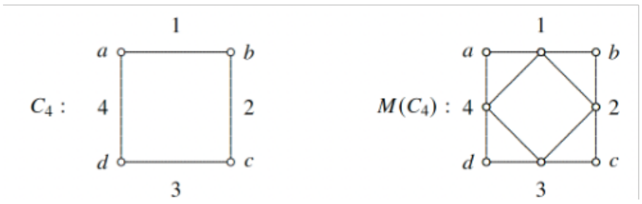
Proposition 4. *A nontrivial connected graph G is **Eulerian** if and only if every vertex of G has even degree.*

At this point, we will formally define the main focus of the study, the graph operation middle graph of a graph.

Definition 8. *Let $G = (V(G), E(G))$ be a simple graph. The **middle graph** of G denoted by $M(G)$ is the graph whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if (1) they are either adjacent edges of G or (2) one is a vertex and the other is an edge incident to it.*

For example,

Example 3. *Consider the graphs C_4 . The middle graph of C_4, $M(C_4)$, is given by*



The middle graph $M(C_4)$ of C_4 is a graph with $V(M(C_4)) = \{a, b, c, d, 1, 2, 3, 4\}$ and two vertices is adjacent if and only if they are adjacent edges of C_4 or one is a vertex and the other is an edge incident to it. For instance the vertices 4 and 1 in $M(C_4)$ are adjacent since they are adjacent edges in C_4. Also the vertices 3 and d in $M(C_4)$ are adjacent since 3 is an edge incident to a vertex d in C_4.

3. The Middle Graph of $\Gamma_{\mathbb{C}_n}$

Below is the structure of the middle graph of the identity graph of the finite cyclic groups. For easy reference, we refer to the middle graph of the identity graph as MIG. This will be used for the rest of this paper.

Definition 9. *Let $\mathbb{C}_n$ be a finite group of order n and $\Gamma_{\mathbb{C}_n}$ be the identity graph of $\mathbb{C}_n$. The **middle graph** of $\Gamma_{\mathbb{C}_n}$ denoted by $M(\Gamma_{\mathbb{C}_n})$ is the graph whose vertex set is $V(\Gamma_{\mathbb{C}_n}) \cup E(\Gamma_{\mathbb{C}_n})$ where two vertices are adjacent if (1) they are either adjacent edges of $\Gamma_{\mathbb{C}_n}$ or (2) one is a vertex and the other is an edge incident to it.*

In this study, the vertex-set and edge-set notations of $M(\Gamma_{\mathbb{C}_n})$ are fixed. Here are the following steps on how to construct the middle graph of the identity graph of finite cyclic groups.

Step 1. Draw the identity graph of finite cyclic group $\Gamma_{\mathbb{C}_n}$ for $n \geq 2$. Set the vertices $V(\Gamma_{\mathbb{C}_n}) = \{e\} \cup \{x_i | 1 \leq i \leq n-1\}$ where e is the identity element of the cyclic group $\mathbb{C}_n$. Set the edges $E(\Gamma_{\mathbb{C}_n}) = \{z_i | z_i = [x_i, e] \text{for all} 1 \leq i \leq n-1\}$

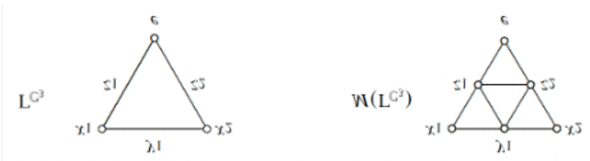
$$\bigcup \{y_p | y_p = [x_i, x_{i+1}] \text{and} p = \tfrac{i+1}{2}\} \begin{cases} \text{for all odd } 1 \leq i \leq n-2\}, & \text{if n is odd} \\ \text{for all odd } 1 \leq i \leq n-3\}, & \text{if n is even.} \end{cases}$$

Step 2. Subdivide the edges of the original graph. The additional new vertices will be obtained by subdividing the edges and the second condition of the middle graph will automatically be satisfied.

Step 3. Connect the new obtained vertices to each other if they are adjacent edges in the original graph.

Consider the following example,

Example 4. *Let $\Gamma_{\mathbb{C}_3}$ be the identity graph of $\mathbb{C}_3$ with vertices $V(\mathbb{C}_3) = \{e, x_1, x_2\}$ and $E(\mathbb{C}_3) = \{z_1, z_2, y_1\}$ shown in the figure below and its corresponding MIG.*



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3.1. The middle graph of $\Gamma_{\mathbb{C}_n}$ where n is odd

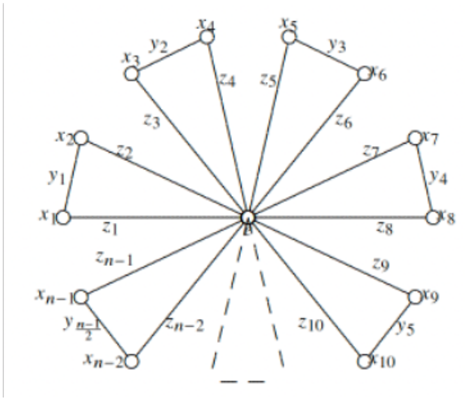
For the general structure of the middle graph of $\Gamma_{\mathbb{C}_n}$ where n is odd, set first the vertices of the identity graph of $\Gamma_{\mathbb{C}_n}$ as

$$V(\Gamma_{\mathbb{C}_n}) = \{e\} \cup \{x_i | 1 \leq i \leq n-1\}$$

and

$$E(\Gamma_{\mathbb{C}_n}) = \{z_i | 1 \leq i \leq n-1\} \cup \left\{ y_p | p = \frac{i+1}{2} \text{for all odd } 1 \leq i \leq n-2 \right\}$$

Here is the pictorial representation,



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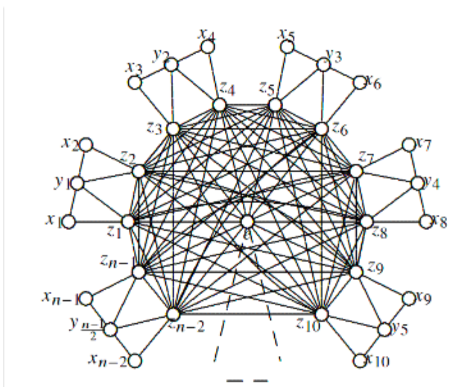
To generalize the vertex set and edge set of $M(\Gamma_{\mathbb{C}_n})$ for n is odd, we now have;

$$V(M(\Gamma_{\mathbb{C}_n})) = \{e\} \bigcup \{x_i | 1 \leq i \leq n-1\} \bigcup \{z_i | 1 \leq i \leq n-1\}$$
$$\bigcup \left\{ y_p | p = \tfrac{i+1}{2} \text{for all odd } 1 \leq i \leq n-2 \right\}$$

and

$$\begin{aligned} E(M(\Gamma_{\mathbb{C}_n})) = \{&[e, z_i] | 1 \leq i \leq n-1\} \\ &\bigcup \{[z_i, z_j] | 1 \leq i \leq n-1, 1 \leq j \leq n-1, i \neq j\} \\ &\bigcup \{[x_i, z_i] | 1 \leq i \leq n-1, 1 \leq j \leq n-1\} \\ &\bigcup \left\{ [y_p, z_i] | p = \tfrac{i+1}{2} \text{for all odd} 1 \leq i \leq n-2 \right\} \\ &\bigcup \left\{ [y_p, z_{i+1}] | p = \tfrac{i+1}{2} \text{for all odd} 1 \leq i \leq n-2 \right\} \\ &\bigcup \left\{ [y_p, x_i] | p = \tfrac{i+1}{2} \text{for all odd} 1 \leq i \leq n-2 \right\} \\ &\bigcup \left\{ [y_p, x_{i+1}] | p = \tfrac{i+1}{2} \text{for all odd} 1 \leq i \leq n-2 \right\} \end{aligned}$$

Here is the pictorial representation,



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The degree of the vertices of $M(\Gamma_{\mathbb{C}_n})$ where n is odd is summarized below:

a. $deg(e) = n - 1$

b. $deg(x_i) = 2,\ 1 \leq i \leq n - 1$

c. $deg(y_p) = 4,\ 1 \leq p \leq \frac{i+1}{2}$ for all odd $1 \leq i \leq n - 2$

d. $deg(z_i) = n + 1,\ 1 \leq i \leq n - 1.$

For the summation of all of the degrees of the vertices of $M(\Gamma_{\mathbb{C}_n})$ for n is odd, we have

$$\begin{aligned}\sum\nolimits_{v \in V(M(\Gamma_{\mathbb{C}_n}))} deg(v) &= (n-1) + 2(n-1) + 4(\tfrac{n-1}{2}) + (n+1)(n-1) \\ &= n - 1 + 2n - 2 + 2n - 2 + n^2 - 1 \\ &= n^2 + 5n - 6.\end{aligned}$$

3.2. The Middle Graph of $\Gamma_{\mathbb{C}_n}$, where n is even

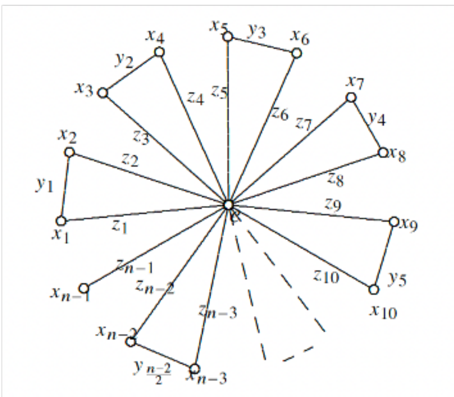
For the general structure of the middle graph of $\Gamma_{\mathbb{C}_n}$ where n is even, set first the vertices of the identity graph of $\Gamma_{\mathbb{C}_n}$ as

$$V(\Gamma_{\mathbb{C}_n}) = \{e\} \cup \{x_i | 1 \leq i \leq n - 1\}$$

and

$$E(\Gamma_{\mathbb{C}_n}) = \{z_i | 1 \leq i \leq n - 1\} \cup \left\{y_p | p = \tfrac{i+1}{2} \text{for all odd } 1 \leq i \leq n - 3\right\}$$

Here is the pictorial representation,



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To generalize the vertex set of $M(\Gamma_{\mathbb{C}_n})$ where n is odd, we have;

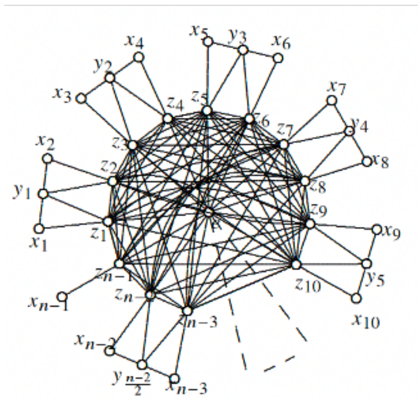
$$\begin{aligned}V(M(\Gamma_{\mathbb{C}_n})) = \{e\} \bigcup \{x_i | 1 \leq i \leq n - 1\} \bigcup \{z_i | 1 \leq i \leq n - 1\} \\ \bigcup \left\{y_p | p = \tfrac{i+1}{2} \text{for all odd } 1 \leq i \leq n - 3\right\}\end{aligned}$$

and

$$E(M(\Gamma_{\mathbb{C}_n})) = \{[e, z_i] | 1 \leq i \leq n - 1\}$$

$$\bigcup\{[z_i, z_j]|1 \le i \le n-1, 1 \le j \le n-1, i \ne j\}$$
$$\bigcup\{[x_i, z_i]|1 \le i \le n-1, 1 \le j \le n-1\}$$
$$\bigcup\left\{[y_p, z_i]|p = \tfrac{i+1}{2}\text{for all odd}1 \le i \le n-3\right\}$$
$$\bigcup\left\{[y_p, z_{i+1}]|p = \tfrac{i+1}{2}\text{for all odd}1 \le i \le n-3\right\}$$
$$\bigcup\left\{[y_p, x_i]|p = \tfrac{i+1}{2}\text{for all odd}1 \le i \le n-3\right\}$$
$$\bigcup\left\{[y_p, x_{i+1}]|p = \tfrac{i+1}{2}\text{for all odd}1 \le i \le n-3\right\}$$

Here is the pictorial representation,



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The degree of the vertices of $M(\Gamma_{\mathbb{C}_n})$ where n is even is summarized below:

a. $deg(e) = n-1$

b. $deg(x_i) = 2,\ 1 \le i \le n-2$

c. $deg(x_{n-1}) = 1$

d. $deg(y_p) = 4,\ 1 \le p \le \frac{n-2}{2}$

e. $deg(z_i) = n+1,\ 1 \le i \le n-2$

f. $deg(z_{n-1}) = n$

To sum up all the degrees of the vertices of $M(\Gamma_{C_n})$ where n is even we have:

$$\begin{aligned}\sum\nolimits_{v \in V(M(\Gamma_{\mathbb{C}_n}))} deg(v) &= (n-1) + 1 + n + 2(n-2) + 4(\tfrac{n-2}{2}) + (n+1)(n-2)\\ &= 2n + 2n - 4 + 2n - 4 + n^2 - n - 2\\ &= n^2 + 5n - 10\end{aligned}$$

Theorem 1. *Let $\mathbb{C}_n$ be a cyclic group of order n and $M(\Gamma_{\mathbb{C}_n})$ be the MIG of $\mathbb{C}_n$ for $n \geq 2$. The order of $M(\Gamma_{\mathbb{C}_n})$ is*

$$|V(M(\Gamma_{\mathbb{C}_n}))| = \begin{cases} \frac{5n-3}{2}, & \text{if } n \text{ is odd} \\ \frac{5n-4}{2}, & \text{if } n \text{ is even.} \end{cases}$$

Proof.

i. For n is odd,

$$\begin{aligned} |V(M(\Gamma_{\mathbb{C}_n})| &= |V(\Gamma_{\mathbb{C}_n})| + |E(\Gamma_{\mathbb{C}_n})| \\ &= n + \frac{3(n-1)}{2} \\ &= \frac{2n+3n-3}{2} \\ &= \frac{5n-3}{2} \end{aligned}$$

ii. For n is even,

$$\begin{aligned} |V(M(\Gamma_{\mathbb{C}_n})| &= |V(\Gamma_{\mathbb{C}_n})| + |E(\Gamma_{\mathbb{C}_n})| \\ &= n + [\frac{3(n-2)}{2} + 1] \\ &= \frac{(2n+3n-6)+2}{2} \\ &= \frac{5n-4}{2} \end{aligned}$$

Now, for the size of the middle graph of identity graph of a cyclic group, refer to the theorem below.

Theorem 2. *The MIG of a cyclic group $\mathbb{C}_n$ of order n has size*

$$|E(M(\Gamma_{\mathbb{C}_n}))| = \begin{cases} \frac{n^2+5n-6}{2}, & \text{if } n \text{ is odd} \\ \frac{n^2+5n-10}{2}, & \text{if } n \text{ is even.} \end{cases}$$

Proof. To prove this, we need to consider two cases.

i. First we will consider if n is odd. From the summation of all of the degrees of the vertices where n is odd, $\sum_{v \in V(M(\Gamma_{\mathbb{C}_n}))} deg(v) = n^2 + 5n - 6$. By Theorem 3, for a graph of size m, $\sum_{v \in V(M(\Gamma_{\mathbb{C}_n}))} deg(v) = 2m$. By substitution, we have $n^2 + 5n - 6 = 2m$. Thus $m = \frac{n^2+5n-6}{2}$.

ii. For n is even, using the summary of the degree of the vertices where n is even, $\sum_{v \in V(M(\Gamma_{\mathbb{C}_n}))} deg(v) = n^2 + 5n - 10$. Also by Theorem 3, the sum of all of its vertices is $\sum_{v \in V(M(\Gamma_{\mathbb{C}_n}))} deg(v) = 2m$ where m is the size of the graph. By substitution, we have $n^2 + 5n - 10 = 2m$. Thus $m = \frac{n^2+5n-10}{2}$.

4. Properties of $M(\Gamma_{\mathbb{C}_n})$ on some parameters

In this section, we will explore the graphical properties of MIGs to further understand its structure.

4.1. Distance between two vertcices

Theorem 3. *Let $M(\Gamma_{\mathbb{C}_n})$ be the middle graph of $\Gamma_{\mathbb{C}_n}$ for $n \geq 4$. The distance*

i. $d(x_i, a) \leq 3$ *for all* $a \in V(M(\Gamma_{\mathbb{C}_n})) \setminus x_i$ *where* $1 \leq i \leq n-1$,

ii. $d(y_p, a) \leq 3$ *for all* $a \in V(M(\Gamma_{\mathbb{C}_n})) \setminus y_p$ *for all* $1 \leq p \leq \frac{n-1}{2}$ *if n is odd and* $1 \leq p \leq \frac{n-2}{2}$ *if n is even,*

iii. $d(z_i, a) \leq 2$ *for* $a \in V(M(\Gamma_{\mathbb{C}_n})) \setminus z_i$ *where* $1 \leq i \leq n-1$,

iv. $d(e, a) \leq 2 for$ $a \in V(M(\Gamma_{\mathbb{C}_n})) \setminus e$.

Proof. We divided it into four cases:

i. Let $a \in V(M(\Gamma_{\mathbb{C}_n})) \setminus x_i$. If $a = x_{i+1}$, then the distance $d(x_i, x_{i+1}) = 2$ since both $[x_i, y_{i+1}]$ and $[x_{i+1}, y_{i+1}] \in E(M(\Gamma_{\mathbb{C}_n}))$ and $[x_i, x_{i+1}] \notin E(M(\Gamma_{\mathbb{C}_n}))$. If $a = x_j$ such that $j \neq i$ or $i+1$, then $[x_j, z_j] \in E(M(\Gamma_{\mathbb{C}_n}))$. Also note that $[x_i, z_i] \in E(M(\Gamma_{\mathbb{C}_n}))$ and $[z_i, z_j] \in E(M(\Gamma_{\mathbb{C}_n}))$. Now since neither $[x_i, z_j]$ nor $[x_j, z_i] \in E(M(\Gamma_{\mathbb{C}_n}))$, then the shortest path from x_i to x_j is the path x_i, z_i, z_j, x_j of length 3. Thus the distance $d(x_i, x_j) = 3$. Similar argument if $a = y_p$. Hence the distance $d(x_i, y_p) = 3$. If $a = z_i$, then $d(x_i, z_i) = 1$ since $[x_i, z_i]] \in E(M(\Gamma_{\mathbb{C}_n}))$. If $a = z_j$ such that $i \neq j$, then $[z_i, z_j] \in E(M(\Gamma_{\mathbb{C}_n}))$. Also since $[x_i, z_i] \in E(M(\Gamma_{\mathbb{C}_n}))$, then we have a path x_i, z_i, z_j from x_i to z_j and this is the shortest since $[x_i, z_j] \notin E(M(\Gamma_{\mathbb{C}_n}))$. Thus the distance $d(x_i, z_j) = 2$. Lastly if $a = e$, the argument is similar to $a = z_j$. Hence the distance $d_(x_i, a) \leq 3$ for all $a \in V(M(\Gamma_{\mathbb{C}_n})) \setminus x_i$.

ii. The proof for the distance $d(y_p, a) \leq 3$ is analogous to case i.

iii. Let $a \in V(M(\Gamma_{\mathbb{C}_n})) \setminus z_i$. If $a = z_j$ such that $i \neq j$, then $d(z_i, z_j) = 1$ since $[z_i, z_j] \in E(M(\Gamma_{\mathbb{C}_n}))$. Similar argument if $a = e$. If $a = x_i$, then clearly $d(x_i, z_i) = 1$. Now suppose $a = x_j$ where $i \neq j$, note that $[x_j, z_j] \in E(M(\Gamma_{\mathbb{C}_n}))$ and also $[z_i, z_j] \in E(M(\Gamma_{\mathbb{C}_n}))$, it follows that z_i, z_j, x_j is a shortest path from z_i to x_j since $[z_i, x_j] \notin E(M(\Gamma_{\mathbb{C}_n}))$. Thus the distance $d(z_i, x_j) = 2$. Lastly, if $a = y_p$ where $p = \frac{i+1}{2}$, then clearly $d(z_i, y_{\frac{i+1}{2}}) = 1$. For $p = \frac{j+1}{2}$ where $i \neq j$, it follows that $[y_{\frac{j+1}{2}}, z_j] \in E(M(\Gamma_{\mathbb{C}_n}))$ and we know that $[z_i, z_j] \in E(M(\Gamma_{\mathbb{C}_n}))$, thus a shortest path from z_i to $y_{\frac{i+1}{2}}$ is $z_i, z_j, y_{\frac{i+1}{2}}$ since $[y_{\frac{i+1}{2}}, z_i] \notin E(M(\Gamma_{\mathbb{C}_n}))$. Hence the distance $d(z_i, y_p) \leq 2$. Therefore the distance $d(z_i, a) \leq 2$ for all $a \in V(M(\Gamma_{\mathbb{C}_n})) \setminus z_i$.

iv. Let $M(\Gamma_{\mathbb{C}_n})$ be the middle graph of $\Gamma_{\mathbb{C}_n}$ and let $a \in V(M(\Gamma_{\mathbb{C}_n})) \setminus e$. If $a = z_i$, then $d(e, z_i) = 1$ since $[e, z_i] \in E(M(\Gamma_{\mathbb{C}_n}))$. Now if $a = x_i$ for $1 \leq i \leq n-1$, then

$[x_i, z_i] \in E(M(\Gamma_{\mathbb{C}_n}))$ also since $[e, z_i] \in E(M(\Gamma_{\mathbb{C}_n}))$ then we have a path e, z_i, x_i of length 2 from e to x_i. And also since $[e, x_i] \notin E(M(\Gamma_{\mathbb{C}_n}))$, thus this is the shortest path from e to x_i. Hence $d(e, x_i) = 2$. Finally if $a = y_p$ for $p = \frac{i+1}{2}$ for all odd $1 \leq i \leq n-2$, then $[y_p, z_i] \in E(M(\Gamma_{\mathbb{C}_n}))$. The other arguments are similar for $a = x_i$. Thus $d(e, y_p) = 2$. Therefore the distance $d(e, a) \leq 2$ for all $a \in V(M(\Gamma_{\mathbb{C}_n})) \setminus e$.

4.2. Eccentricity of the vertcices

Theorem 4. *Let $M(\Gamma_{\mathbb{C}_n})$ be the middle graph of $\Gamma_{\mathbb{C}_n}$. The eccentricity of the vertices*

i. $e(x_i) = \begin{cases} 2, & \text{if } n=2 \text{ or } 3 \\ 3, & \text{if } n \geq 4 \end{cases}$

for all $1 \leq i \leq n-1$,

ii. $e(y_p) = \begin{cases} 2, & \text{if } n=3 \\ 3, & \text{for } n \geq 4 \end{cases}$

for all $1 \leq p \leq \frac{n-2}{2}$,

iii. $e(z_i) = \begin{cases} 1, & \text{if } n=2 \text{ or } 3 \\ 2, & \text{if } n \geq 4 \end{cases}$

for all $1 \leq i \leq n-1$,

iv. $e(e) = 2$

Proof. Let $M(\Gamma_{\mathbb{C}_n})$ be the middle graph of $\Gamma_{\mathbb{C}_n}$.

i. For $n = 2$, it is very obvious since the middle graph $M(\Gamma_{\mathbb{C}_2})$ is isomorphic to a path P_3 with the vertex set $V(M(\Gamma_{\mathbb{C}_2}) = \{e, x_1, z_1\}$ and edges $E(M(\Gamma_{\mathbb{C}_2}) = \{[e, z_1], [z_1, x_1]\}$.

ii. For $n = 3$, note that $\Gamma_{\mathbb{C}_3}$ is isomorphic to a cycle C_3 with the vertex set $V(\Gamma_{\mathbb{C}_3}) = \{e, x_1, x_2\}$ and edge set $E(\Gamma_{\mathbb{C}_3}) = \{z_1 = [e, x_1], z_2 = [e, x_2], y_1 = [x_1 x_2]\}$. Now for $M(\Gamma_{\mathbb{C}_3})$, the vertex set $V(M(\Gamma_{\mathbb{C}_3})) = \{e, x_1, x_2, z_1, z_2, y_1\}$ where the vertices x_1, x_2 and e are the corner vertices and the vertices z_1, z_2 and y_1 are the inner vertices. Note that $[x_1, z_1]$ and $[z_1, z_2] \in E(\Gamma_{\mathbb{C}_3})$ but $[x_1, x_2] \notin E(\Gamma_{\mathbb{C}_3})$ thus the distance from x_1 to z_2 is 2 same with x_2 to x_1 and e to y_1. Also the distance of each corner vertex to each other is 2. Now for the distance of all inner vertices to each other is 1 since they are adjacent edges in $\Gamma_{\mathbb{C}_3}$. Thus the maximum distance or eccentricity $e(x_1) = e(x_2) = e(e) = e(z_1) = e(z_2) = e(y_1) = 2$.

iii. For $n \geq 4$, the proof follows from Theorem 3.

4.3. Radius

Theorem 5. *If $M(\Gamma_{\mathbb{C}_n})$ be the middle graph of $\Gamma_{\mathbb{C}_n}$, then*
$rad(M(\Gamma_{\mathbb{C}_n})) = \begin{cases} 1, & \text{if } n=2 \\ 2, & \text{for } n \geq 3 \end{cases}$ and $diam(M(\Gamma_{\mathbb{C}_n})) = \begin{cases} 2, & \text{if } n=2 \text{ or } 3 \\ 3, & \text{for } n \geq 4. \end{cases}$

Proof. Let $M(\Gamma_{\mathbb{C}_n})$ be the middle graph of $\Gamma_{\mathbb{C}_n}$. We divided it into three cases:

i. For $n = 2$, since $M(\Gamma_{\mathbb{C}_2})$ is isomorphic to a path of length 3, clearly $rad(M(\Gamma_{\mathbb{C}_2})) = 1$ and the diameter $diam(M(\Gamma_{\mathbb{C}_2})) = 2$.

ii. For $n = 3$, by Theorem 4, the eccentricity of all the vertices is equal to 2, thus the radius and diameter $rad(M(\Gamma_{\mathbb{C}_3})) = 2$ and $diam(M(\Gamma_{\mathbb{C}_3})) = 2$ respectively.

iii. For $n \geq 4$, from Theorem 4, $e(z_i) = 2$ for all $1 \leq i \leq n-1$ and $e(e) = 2$ and this is the minimum eccentricity, thus the radius $rad(M(\Gamma_{\mathbb{C}_n})) = 2$. Also using the same reference, the vertices with the maximum eccentricities are the vertices x_i and y_i with $e(x_i) = 3$ for all $1 \leq i \leq n-1$ and $e(y_p) = 3$ for all $1 \leq p \leq \frac{n-1}{2}$ if n is odd and $1 \leq p \leq \frac{n-2}{2}$ if n is even. Hence the diameter $diam(M(\Gamma_{\mathbb{C}_n})) = 3$.

4.4. Central Vertices

Theorem 6. *For the middle graph $M(\Gamma_{\mathbb{C}_n})$ for $n \geq 4$, the central vertices are the vertices $z_i \in V(M(\Gamma_{\mathbb{C}_n}))$ and e.*

Proof. From Theorem 4, the eccentricity $e(z_i) = 2$ for all i, $1 \leq i \leq n-1$ and $e(e) = 2$. Now by Theorem 5, the radius $rad(M(\Gamma_{\mathbb{C}_n})) = e(z_i) = e(e) = 2$ for all $1 \leq i \leq n-1$. The ramaining vertices $x_{i's}$ and $y_{p's}$ has the eccentricity of 3. Thus the central vertices are all the $z_i \in V(M(\Gamma_{\mathbb{C}_n}))$ and e.

4.5. Center

Theorem 7. *The set of vertices $Cen(M(\Gamma_{\mathbb{C}_n})) = \{\{z_i | 1 \leq i \leq n-1\} \bigcup \{e\}\}$ is the center of $M(\Gamma_{\mathbb{C}_n})$ for $n \geq 4$.*

Proof. The proof of this theorem follows from Theorem 6.

4.6. Complete Subgraph

Theorem 8. *Let H be a subgraph induced by $Cen(M(\Gamma_{\mathbb{C}_n}))$ for $n \geq 4$, then H is a complete subgraph of order n .*

Proof.

Let $M(\Gamma_{\mathbb{C}_n})$ be the middle graph of $\Gamma_{\mathbb{C}_n}$ for $n \geq 4$. Suppose H is a subgraph induced by $Cen(M(\Gamma_{\mathbb{C}_n}))$, then $V(H) = \{\{z_i | 1 \leq i \leq n-1\} \bigcup \{e\}\} = Z \cup \{e\}$ and clearly $|V(H)| = n$. Note that for every vertex z_i and z_j element of Z where $i \neq j$, $[z_i, z_j] \in E(M(\Gamma_{\mathbb{C}_n}))$ and also $[z_i, e] \in E(M(\Gamma_{\mathbb{C}_n}))$ thus $deg_H(z_i) = deg_H(e) = n-1$. Consequently, the size $|E(H)| = |E(K_n)| = \frac{n(n-1)}{2}$. Hence H is a complete graph of order n.

4.7. Circumference

Theorem 9. *If $M(\Gamma_{\mathbb{C}_n})$ be the middle graph of $\Gamma_{\mathbb{C}_n}$, then the circumference*

$$c(M(\Gamma_{\mathbb{C}_n})) = \begin{cases} \frac{5n-3}{2}, & \text{if } n \text{ is odd,} \\ \frac{5n-6}{2}, & \text{if } n \text{ is even.} \end{cases}$$

Proof. Let $M(\Gamma_{\mathbb{C}_n})$ be the middle graph of $\Gamma_{\mathbb{C}_n}$. We divided it into two cases.

i. For n is odd, we will find a cycle $C \subseteq M(\Gamma_{\mathbb{C}_n})$ with the largest size. From the structure of $M(\Gamma_{\mathbb{C}_n})$ where n is odd, since $[e, z_i] \in E(M(\Gamma_{\mathbb{C}_n}))$ for $1 \leq i \leq n-1$, take e as our initial vertex, followed by z_1. Note that $[z_i, x_i] \in E(M(\Gamma_{\mathbb{C}_n}))$, then we have a path e, z_1, x_1. Now also $[x_i, y_{\frac{i+1}{2}}]$ and $[x_{i+1}, y_{\frac{i+1}{2}}] \in E(M(\Gamma_{\mathbb{C}_n}))$, then we can extend our path to $e, z_1, x_1, y_1, x_2, z_2,$. Also since $[z_i, z_i] \in E(M(\Gamma_{\mathbb{C}_n}))$, we can have $e, z_1, x_1, y_1, x_2, z_2, z_3$ then repeat the process. By continuing, we now have $e, z_1, x_1, y_1, x_2, z_2, z_3, x_3, y_2, x_4,$
$z_4, ...z_i, x_i, y_{\frac{i+1}{2}}, x_{i+1}, z_{i+1}, ...z_{n-2}, x_{n-2}, y_{\frac{n-1}{2}}, x_{n-1},$
z_{n-1}. Now we can connect z_{n-1} to e since $[e, z_i] \in E(M(\Gamma_{\mathbb{C}_n}))$ for $1 \leq i \leq n-1$. Clearly, $C : e, z_1, x_1, y_1, x_2, z_2, z_3, x_3, y_2, x_4, z_4, ...z_i, x_i, y_{\frac{i+1}{2}}, x_{i+1}, z_{i+1}, ...z_{n-2},$
$x_{n-2}, y_{\frac{n-1}{2}}, x_{n-1}, z_{n-1}, e$ is a cycle since no vertex is repeated except for the first and the last. To compute the length, we have to compute its order since the length of a cycle is equal to its order. So $|e| = 1$, $|z_{i's}| = n-1$, $|x_{i's}| = n-1$, $|y_{i's}| = \frac{n-2}{2}$. Thus $|V(C)| = 1 + (n-1) + (n-1) + (\frac{n-1}{2}) = \frac{5n-3}{2}$. Note that $C \subseteq M(\Gamma_{\mathbb{C}_n})$, thus $|V(C)| \leq |V(M(\Gamma_{\mathbb{C}_n}))|$. And from Theorem 1 the order $|V(M(\Gamma_{\mathbb{C}_n}))| = \frac{5n-3}{2}$ if n is odd . It implies that $|V(C)| = |V(M(\Gamma_{\mathbb{C}_n}))| = \frac{5n-3}{2}$, hence the length of the maximum cycle in $M(\Gamma_{\mathbb{C}_n}) = \frac{5n-3}{2}$.

2. For n is even, we will find a cycle $C \subseteq M(\Gamma_{\mathbb{C}_n})$ of maximum length. From the structure of $M(\Gamma_{\mathbb{C}_n})$ where n is even, since $[e, z_i] \in E(M(\Gamma_{\mathbb{C}_n}))$ for $1 \leq i \leq n-1$, then choose e as our initial vertex followed by z_{n-1}, then z_1 since $[z_i, z_j] \in E(M(\Gamma_{\mathbb{C}_n}))$ so that we have a path e, z_{n-1}, z_1. also since $[z_i, x_i] \in E(M(\Gamma_{\mathbb{C}_n}))$, then we can extend it to $e, z_{n-1}, z_1, x_1,$. Similar to Case 1, $[x_i, y_{\frac{i+1}{2}}]$ and $[x_{i+1}, y_{\frac{i+1}{2}}] \in E(M(\Gamma_{\mathbb{C}_n}))$ for all odd $1 \leq i \leq n-3$, thus we have $e, z_{n-1}, z_1, x_1, y-1, x_2$. Now choose z_2 as our next vertex so that we have $e, z_{n-1}, z_1, x_1, y-1, x_2, z_2$. By continuing the process,we now have a cycle
$C : e, z_{n-1}, z_1, x_1, y-1, x_2, z_2, z_3, x_3, y_2, x_4, z_4, ...z_i, x_i, y_{\frac{i+1}{2}}, x_{i+1}, z_{i+1}, ...z_{n-3}, x_{n-3},$
$y_{\frac{n-2}{2}}, x_{n-2}, z_{n-2}, e$. To compute the order of C, we have

$V(C) = \{e\} \bigcup \{z_i | 1 \leq i \leq n-1\} \bigcup \{x_i | 1 \leq i \leq n-2\} \bigcup \{y_p | 1 \leq p \leq \frac{n-2}{2}\}$. Implies that

$$\begin{aligned}|V(C)| &= 1 + (n-1) + (n-2) + (\frac{n-2}{2}) \\ &= 2n - 2 + \frac{n-2}{2} \\ &= \frac{4n-4+n-2}{2} \\ &= \frac{5n-6}{2}.\end{aligned}$$

By Theorem 13, there exist exactly one vertex $v \in V(M(\Gamma_{\mathbb{C}_n}))$ of degree 1. Thus clearly $v \notin V(C)$. It follows that

$$\begin{aligned}|V(C)| &\leq |V(M(\Gamma_{\mathbb{C}_n}))| - 1 \\ &\leq \frac{5n-4}{2} - 1 \\ &\leq \frac{5n-4-2}{2} \\ &\leq \frac{5n-6}{2}.\end{aligned}$$

Thus $|V(C_k)| = |V(M(\Gamma_{\mathbb{C}_n}))| - 1 = \frac{5n-6}{2}$. Hence $\frac{5n-6}{2}$ is the maximum length of a cycle contained in $(M(\Gamma_{\mathbb{C}_n})$. Therefore the circumference $c(M(\Gamma_{\mathbb{C}_n})) = \frac{5n-6}{2}$, if n is even.

4.8. Girth

Theorem 10. *If $M(\Gamma_{\mathbb{C}_n})$ be a middle graph of a $\Gamma_{\mathbb{C}_n}$, then the $gir(M(\Gamma_{\mathbb{C}_n})) = 3$ for $n \geq 3$.*

Proof. Let e be the vertex representing the identity element in $\Gamma_{\mathbb{C}_n}$. Now for $\Gamma_{\mathbb{C}_n}$, there are $n-1$ edges incident to e. Pick any edges namely z_1, z_2. By the Definition 8 of MIG, z_1 and z_2 are vertices in $M(\Gamma_{\mathbb{C}_n})$. Also by the condition (1), $[z_1, z_2]$ is an edge in $M(\Gamma_{\mathbb{C}_n})$. Now by the condition (2), $[z_1, e]$ and $[z_2, e]$ are edges in $M(\Gamma_{\mathbb{C}_n})$. Thus z_1, z_2, e, z_1 is a cycle of length 3 in $M(\Gamma_{\mathbb{C}_n})$. Therefore the girth $gir(M(\Gamma_{\mathbb{C}_n})) = 3$ for $n \geq 3$.

4.9. Clique Number

Theorem 11. *If $M(\Gamma_{\mathbb{C}_n})$ be a middle graph of a $\Gamma_{\mathbb{C}_n}$, then the clique number $\omega[M(\Gamma_{\mathbb{C}_n})] = n$ for $n \geq 3$.*

Proof. Let $M(\Gamma_{\mathbb{C}_n})$ be the middle graph of $\Gamma_{\mathbb{C}_n}$. We will show that the clique number $\omega[M(\Gamma_{\mathbb{C}_n})] = n$. Note that from Theorem 8, the subgraph induced by the center $Cen(M(\Gamma_{\mathbb{C}_n})) = \{z_i | 1 \leq i \leq n-1\} \bigcup \{e\}$ is a complete graph of order n. Suppose there is another complete subgraph K_m of order m where $m \geq n$. But from the summary of the degree of the vertices, the degree $deg(x_i) = 2$ for $1 \leq i \leq n-1$ and $deg(y_p) = 4$ for

$1 \leq p \leq \frac{n-1}{2}$ if n is odd and $1 \leq p \leq \frac{n-2}{2}$ if n is even. Thus clearly $x_{i's}$ and $y_{p's}$ are not elements of $V(K_m)$. Now we are left with the vertices z_i for $1 \leq i \leq n-1$ and e since $V(M(\Gamma_{\mathbb{C}_n})) \setminus \{x_i \cup y_p\} = \{z_i\} \cup \{e\}$ for all $1 \leq i \leq n-1$ and $1 \leq p \leq \frac{n-1}{2}$ if n is odd and $1 \leq p \leq \frac{n-2}{2}$ if n is even which is clearly the center $Cen(M(\Gamma_{\mathbb{C}_n}))$. Hence it is not possible to have a complete subgraph K_m of order m where $m \geq n$. Therefore the clique number $\omega[M(\Gamma_{\mathbb{C}_n})] = n$.

4.10. Independence Number

Theorem 12. *The indepedendence number $\alpha(M(\Gamma_{\mathbb{C}_n})) = n$.*

Proof. To start with, we know that from Theorem 11, the largest complete graph K contained in $(M(\Gamma_{\mathbb{C}_n})$ has order n so that the clique number $\omega(M(\Gamma_{\mathbb{C}_n})) = n$. Note that the vertices $V(K) = \{z_i | 1 \leq i \leq n-1\} \bigcup \{e\}$. Let S be an independent set that has a maximum number of elements. It follows that exactly one vertex $V(K)$ must be in S. Thus it is either e only or one of the $z_{i's}$ only. For n is odd, we have two cases.

1. Let z_j be a fixed element in S, then it follows that x_j and y_q are not in S for any fixed x_j and y_q such that $y_q = [x_j, xj+1] \in V(\Gamma_{\mathbb{C}_n})$ but $x_{j+1} \in S$ since z_j is not adjacent to x_{j+1}. Now we are left with the set of vertices $X = \{x_i | 1 \leq i \leq n-1\} \setminus \{x_j, x_{j+1}\}$ for all odd $1 \leq j \leq n-2$ and the set $Y = y_p | 1 \leq p \leq \frac{n-1}{2} \setminus y_q$. But note that for every y_p, there are exactly 2 $x_{i's}$ adjacent to it. Thus for every $y_p \in S$ there are exactly 2 $x_i \notin S$ so that $S = \{z_j, x_{i+1}\} \cup Y$. Hence $|S| = |\{z_j, x_{i+1}\}| \cup |Y| = 2 + \frac{n-1}{2} - 1 = \frac{n+1}{2}$. Now if we choose $x_i \in S$, it follows that y_p is not in S for $y_p = [x_i, x_{i+1}$. But since x_i is not adjacent to x_{i+1}, then x_{i+1} must be in S so that $S = \{z_j, x_{j+1}\} \cup X$. Thus $|S| = |\{z_j, x_{j+1}\}| \cup |X| = 2 + n - 3 = n - 1$.

2. Suppose $e \in S$ then clearly each of the z_i for all $1 \leq i \leq n-1$ is not in S. Now we are left with the set of vertices $X = \{x_i | 1 \leq i \leq n-1\}$ and the set $Y = y_p | 1 \leq p \leq \frac{n-1}{2}$. Similar to Case 1, for every y_p, there are exactly 2 $x_{i's}$ adjacent to it. Thus for every $y_p \in S$ there are exactly 2 $x_i \notin S$ so that $S = \{e\} \cup Y$. Hence $|S| = |\{e\}| \cup |Y| = 1 + \frac{n-1}{2} = \frac{n+1}{2}$. Now if we choose $x_i \in S$, it follows that y_p is not in S for $y_p = [x_i, x_{i+1}$. But since x_i is not adjacent to x_{i+1}, then x_{i+1} must be in S so that $S = \{e\} \cup X$. Thus $|S| = |\{e\}| \cup |X| = 1 + n - 1 = n$. Therefore the cardinality of the maximum independent set in $(M(\Gamma_{\mathbb{C}_n})$ is n for n is odd.

   The proof for n is even is analogous for n is odd.

4.11. Other Properties

Theorem 13. *If n is even, then the middle graph $M(\Gamma_{\mathbb{C}_n})$ contains exactly one vertex of degree 1.*

Proof. By Theorem 1, the line in the identity graph $\Gamma_{\mathbb{C}_n}$ is 1 if n is even. Now let x be the vertex of degree 1 in $\Gamma_{\mathbb{C}_n}$ where n is even and let a be the edge connecting x to another vertex say y. Now by the definition of the middle graph, a will become a vertex in $M(\Gamma_{\mathbb{C}_n})$. Also by condition (2) from the definition of MIG in Definition 8, $[x,a]$ is an edge. Suppose there exists another vertex w such that $[x,w]$ is an edge in $M(\Gamma_{\mathbb{C}_n})$. By (2) in the definition of the MIG, it follows that w is an edge in $\Gamma_{\mathbb{C}_n}$ that is incident to x which contradicts the fact that x has degree 1 in $\Gamma_{\mathbb{C}_n}$. Thus $M(\Gamma_{\mathbb{C}_n})$ contains a vertex of degree 1. Suppose there exists another vertex u where $u \neq x$ in $M(\Gamma_{\mathbb{C}_n})$ of degree 1, then u cannot be in $E(\Gamma_{\mathbb{C}_n})$ since there are two edges namely q and r incident to u and by (2) of Definition 8, u will become a vertex adjacent to q and r. Thus it must be $u \in V(\Gamma_{\mathbb{C}_n})$. Note that $\Gamma_{\mathbb{C}_n}$ contains only one line, it follows that u lies in some triangles of $\Gamma_{\mathbb{C}_n}$ which implies that atleast two edges say s and t are incident to u that will eventually become vertices in $M(\Gamma_{\mathbb{C}_n})$. By the (2) of Definition 8, s and t will be vertices adjacent to u which contradicts that u has of degree 1. Hence there is only one vertex in $M(\Gamma_{\mathbb{C}_n})$ of degree 1.

Theorem 14. *Every vertex of $M(\Gamma_{\mathbb{C}_n})$ has an even degree if n is odd.*

Proof. The summary of the degree of the vertices of $M(\Gamma_{\mathbb{C}_n})$ whwre n is odd is sufficient enough to prove this theorem.

Theorem 15. *The middle graph $M(\Gamma_{\mathbb{C}_n})$ is Eulerian if n is odd.*

Proof.

Let $M(\Gamma_{\mathbb{C}_n})$ be the middle graph of $\Gamma_{\mathbb{C}_n}$. Suppose n is odd, by Theorem 14, every vertex of $M(\Gamma_{\mathbb{C}_n})$ is of even degree. Thus by Theorem 4, $M(\Gamma_{\mathbb{C}_n})$ is Eulerian.

Theorem 16. *If n is odd, then the middle graph $M(\Gamma_{\mathbb{C}_n})$ is Hamiltonian.*

Proof. From Theorem 9, if n is odd, the order of the largest cycle $|V(C)|$ contained in $M(\Gamma_{\mathbb{C}_n})$ is $\frac{5n-3}{2} = |M(\Gamma_{\mathbb{C}_n})|$. Thus C is Hamiltonian cycle. Consequently, $M(\Gamma_{\mathbb{C}_n})$ where n is odd is a Hamiltonian graph.

5. Conclusion and Recommendations

This paper focuses on the middle graph of the identity graph of finite cyclic and dihedral groups denoted by $M(\Gamma_{\mathbb{C}_n})$ and $M(\Gamma_{D_n})$ respectively. These graphs are simple, finite, connected and undirected graphs. The concept of the identity graph and middle graph is introduced in this paper.

Using these concepts, the middle graphs $M(\Gamma_{\mathbb{C}_n})$ and $M(\Gamma_{D_n})$ were constructed and the labeling for the vertices and edges were discussed. In addition, some parameters such as the size and order were easily shown using the construction and other existing theorems and propositions. It is also found that the middle graph $M(\Gamma_{\mathbb{C}_n})$ is both Eulerian and Hamiltonian if n is odd. Other properties on some parameters is summarized in Table **??**.

Here is the table for the properties of $M(\Gamma_{\mathbb{C}_n})$ and $M(\Gamma_{D_n})$ on some parameters of a graph.

Parameters	Values
$gir(M(\Gamma_{\mathbb{C}_n}))$	3
$\omega[M(\Gamma_{\mathbb{C}_n})]$	n
$\alpha(M(\Gamma_{\mathbb{C}_n}))$	n
$\gamma(M(\Gamma_{\mathbb{C}_n}))$	$\frac{n+1}{2}$ if n is odd and $\frac{n}{2}$ if n is even
$\chi(M(\Gamma_{\mathbb{C}_n}))$	n
$\chi'(M(\Gamma_{\mathbb{C}_n}))$	$n+2$ if n is odd and $n+1$ if n is even
$gir(M(\Gamma_{D_n}))$	3
$\omega[M(\Gamma_{D_n})]$	$2n$
$\alpha(M(\Gamma_{D_n}))$	$2n$
$\gamma(M(\Gamma_{D_n}))$	$\frac{3n-1}{2}$ if n is odd and $\frac{3n}{2}$ if n is even.
$\chi(M(\Gamma_{D_n}))$	$2n$
$\chi'(M(\Gamma_{D_n}))$	$2n+1$

The problem on the identity graphs is still open. For instance, the middle graph of the identity graph of symmetric groups is also interesting to investigate.

References

[1] W. B. V. Kandasamy and F. Smarandache. *Groups as Graphs.* Editura CuArt, Slobozia, 2009.

[2] A. D. Godase. Unit graph of some finite group. *International Journal of Universal Science and Technology*, 1(1):12–18, 2015.

[3] N. F. Yalcin and Y. Kirgil. Identity graph of finite cyclic groups. *Journal of Balikesir University Institute of Science and Technology*, 21(2):614–624, 2019.

[4] J. U. Jeeshma. Coloring for the identity graphs of groups. *International Research Journal of Engineering and Technology*, 7(7):3459–3462, 2020.

[5] J. Akiyama, T. Hamada, and I. Yoshimora. On characterization of the middle graphs. https://www.researchgate.net/publication/269002499, 1975.

[6] N. Murugesan and D. S. Nair. Power domination of middle graph of central graph of path, cycle, and star. *International Journal of Pure and Applied Mathematics*, 115(6):121–126, 2017.

[7] C. Alib and D. Magpantay. On some parameters of the central graphs of the identity graphs of finite cyclic groups. *European Journal of Pure and Applied Mathematics*, 15(4):1888–1904, 2022.

[8] G. Chartrand and P. Zhang. *A First Course in Graph Theory.* Dover Publications, Mineola, NY, 2012.

[9] W. Somnuek. Counting lines and triangle in a unit graph. *Current Applied Science and Technology*, 18(2):148–155, 2018.