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Implicative Negatively Partially Ordered Ternary Semigroups

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Abstract. In this paper, we introduce and examine the notion of implicative negatively partially ordered ternary semigroups, for short implicative n.p.o. ternary semigroup, which include an element that serves as both the greatest element and the multiplicative identity. We study the notion of implicative homomorphisms between these ternary semigroups, and have that any implicative homomorphism is a homomorphism. Let $\varphi : T_1 \longrightarrow T_2$ be an implicative homomorphism from a commutative implicative n.p.o. ternary semigroup T_1 onto T_2. We construct a quotient commutative implicative n.p.o. ternary semigroup $T_1/\rho_{\mathrm{Ker}\,\varphi}$, where $\rho_{\mathrm{Ker}\,\varphi}$ is a congruence relation defined by $\mathrm{Ker}\,\varphi$. We prove that there exists an implicative homomorphism ψ such that $\psi \circ \eta = \varphi$, where η is a canonical homomorphism from T_1 onto $T_1/\rho_{\mathrm{Ker}\varphi}$.

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1. Introduction

An *implicative semilattice* $(L, \leq, \wedge, *)$ consists of a non-empty set L, a partial order $\leq$, a greatest lower bound (with respect to $\leq$) $\wedge$, and a binary multiplication $*$ such that

$$z \leq x * y \Leftrightarrow z \wedge x \leq y$$

for any $x, y, z \in L$. The notion have been explored in the work of W. C. Nemitz in [13], the author investigated relationships between homomorphisms of implicative semilattices and their kernels. T. S. Blyth in [1] generalized some results of Nemitz by introducing the notion of Brouwerian semigroups. The results of Blyth [1] have been generalized further by M. F. Janowitz and C. S. Johnson Jr in [9]. In [10] Y. B. Jun introduced a special set in an implicative semigroup, from which the author derived an equivalent condition

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for a filter and proved that a filter can be represented by the union of such sets. In [14] D. A. Romano introduced the concept of an anti-filter within implicative semigroups and provided several equivalent conditions for when the special inhabited proper subset of an implicative semigroup qualifies as an ordered anti-filter.

In [6], a *partially ordered semigroup* $(S, \cdot, \leq)$ consists of a semigroup $(S, \cdot)$ together with a partial order $\leq$ on S that is compatible with the semigroup operation, that is for any $x, y, z \in S$, if $x \leq y$ then $xz \leq yz$ and $zx \leq zy$. A partially ordered semigroup $(S, \cdot, \leq)$ is called a *negatively partially ordered semigroup*, for short n.p.o. semigroup, if for any $x, y, z \in S$, $xy \leq x$ and $xy \leq y$. An n.p.o. semigroup $(S, \cdot, \leq)$ with an additional binary multiplication $*$ such that $z \leq x * y \Leftrightarrow zx \leq y$ for any $x, y, z \in S$ is called an *implicative n.p.o. semigroup.* Inspired by the works of Nemitz [13] and Blyth [1], In [2], M. W. Chan and K. P. Shum introduced the concept of implicative negatively partially ordered semigroups and explored the homomorphisms between these structures. Their work generalizes and expands upon some results by Nemitz regarding implicative semilattices.

A non-empty set T together with a ternary multiplication $[\ ]$ defined on the set T is called a *ternary semigroup* if the ternary multiplication $[\ ]$ satisfies the following associative law:

$$[[xyz]uv] = [x[yzu]v] = [xy[zuv]]$$

for all $x, y, z, u, v \in T$. This notion has been introduced and studied by S. Banach (cf. J. Los [12]) who is credited with an example of a ternary semigroup which does not reduce to a semigroup. D. H. Lehmer in [11] studied a system called triplexes which turns out to be a commutative ternary group. J. Los in [12] proved that any ternary semigroup can be embedded in a semigroup. F. M. Sioson in [17] considered ideals and radicals of ternary semigroups; various concepts such as primality, semiprimality, and regularity were introduced. A. Chronowski in [3] investigated ternary semigroups of mappings of sets; these algebraic structures are used for constructing the natural examples of ternary algebras. V. N. Dixit and S. Dewan in [5] studied quasi-ideals and bi-ideals of ternary semigroups; the authors proved that every quasi-ideals is a bi-ideal and gave several examples in different contexts to prove that the converse is not true in general. M. L. Santiago and S. Sri Bala in [15] investigated regular ternary semigroups and studied several properties.

A ternary semigroup $(T, [\ \ ])$ is called an *ordered ternary semigroup* if there is a partial order $\leq$ such that for any $a, b, x, y \in T$, if $a \leq b$ then $[axy] \leq [bxy], [xay] \leq [xby], [xya] \leq [xyb]$. A. Iampan in ([7], [8]) discussed ordered ternary semigroups and characterized the minimality and maximality of ordered lateral ideals in ordered ternary semigroups; the author also considered ideal extensions. V. R. Daddi and Y. S. Pawar in [4] introduced and studied quasi-ideals and bi-ideals in ordered ternary semigroups.

The purpose of this paper is to introduce and study the notion of implicative n.p.o. ternary semigroups. Main idea of this work is inspired by [2]. We apply concept of implicative n.p.o. semigroups to establish implicative n.p.o. ternary semigroups. In addition, we introduce and study implicative homomorphisms related to homomorphisms between implicative n.p.o. ternary semigroups. Moreover, we construct quotient commutative implicative n.p.o. ternary semigroups, and prove the homomorphism theorem.

2. Implicative ternary semigroups

We start this section with the definition of n.p.o. ternary semigroups.

Definition 1. *An n.p.o. ternary semigroup* $(T, [\ ], \leq)$ *consists of a non-empty set* T *together with a partial order* $\leq$ *and a ternary multiplication* $[\ ]$ *on* T *such that the following conditions are satisfied: for any* $x, y, z, u, v \in T$,

(1) $[[xyz]uv] = [x[yzu]v] = [xy[zuv]]$;

(2) if $x \leq y$, *then* $[xuv] \leq [yuv]$, $[uxv] \leq [uyv]$ *and* $[uvx] \leq [uvy]$;

(3) $[xyz] \leq x$, $[xyz] \leq y$ *and* $[xyz] \leq z$.

Definition 2. *An n.p.o. ternary semigroup* $(T, [\ ], \leq)$ *with an additional ternary multiplication* $[\ ]^*$ *on* T *such that*

$$u \leq [xyz]^* \Leftrightarrow [uxy] \leq z$$

for any $x, y, z, u \in T$ *is called an implicative n.p.o. ternary semigroup. The ternary multiplication* $[\ ]^*$ *is called a ternary implication.*

An element 1 of a ternary semigroup $(T, [\ ])$ is a multiplicative identity of T if $[11x] = [1x1] = [x11] = x$ for any $x \in T$. The following example shows that the greatest element of an implicative n.p.o. ternary semigroup need not be identity.

Example 1. *Let* $T = \{1, a, 0\}$. *Let us consider the implicative n.p.o. ternary semigroup* $(T, [\ ], \leq, [\ ]^*)$ *with a ternary multiplication, a ternary implication, and an order relation defined on* T *as follows:*

$[\ ]$	1	a	0
11	1	0	0
$1a$	0	0	0
10	0	0	0

$[\ ]$	1	a	0
aa	0	0	0
$a1$	0	0	0
$a0$	0	0	0

$[\ ]$	1	a	0
00	0	0	0
01	0	0	0
$0a$	0	0	0

$[\ ]^*$	1	a	0
11	1	1	1
$1a$	1	1	1
10	1	1	1

$[\ ]^*$	1	a	0
aa	1	1	1
$a1$	1	1	1
$a0$	1	1	1

$[\ ]^*$	1	a	0
00	1	1	1
01	1	1	1
$0a$	1	1	1

and

$$\leq = \{(0,0), (1,1), (a,a), (a,1), (0,a), (0,1)\}.$$

To express the calculation $[x_1x_2x_3]$ *using a multiplication table, we place* x_1x_2 *in the first column and* x_3 *in the first row. It is observed that* 1 *is the greatest element. However,* 1 *is not the identity since* $[11a] = 0 \neq a$.

The next example shows that not every n.p.o. ternary semigroup with identity admits the implicative structure.

Example 2. *Let $T = \{0, 1, a, b\}$. Let us consider the n.p.o. ternary semigroup $(T, [\ ], \leq)$ with a ternary multiplication and an order relation defined on T as follows:*

$[\ ]$	1	a	b	0
11	1	a	b	0
$1a$	a	0	0	0
$1b$	b	0	b	0
10	0	0	0	0

$[\ ]$	1	a	b	0
aa	0	0	0	0
$a1$	a	0	0	0
ab	0	0	0	0
$a0$	0	0	0	0

$[\ ]$	1	a	b	0
bb	b	0	b	0
$b1$	b	0	b	0
ba	0	0	0	0
$b0$	0	0	0	0

$[\ ]$	1	a	b	0
00	0	0	0	0
01	0	0	0	0
$0a$	0	0	0	0
$0b$	0	0	0	0

and

$$\leq = \{(0,0), (1,1), (a,a), (b,b), (0,a), (0,b), (0,1), (a,1), (b,1)\}.$$

Observed that 1 is the greatest element. Notice that T is not an implicative n.p.o. ternary semigroup. Indeed, if T is an implicative n.p.o. ternary semigroup with a ternary implication $[\ ]^*$, then $a \leq [1ab]^*$ and $b \leq [1ab]^*$ since $[a1a] = 0 \leq b$ and $[b1a] = 0 \leq b$, respectively. It follows that $[1ab]^* = 1$. As $1 \leq [1ab]^*$, we have $a = [11a] \leq b$. This is a contradiction.

An implicative n.p.o. ternary semigroup $(T, [\ ], \leq, [\ ]^*)$ is said to be *commutative* [16] if

$$[xyz] = [yzx] = [zxy] = [yxz] = [zyx] = [xzy]$$

for all elements x, y, z in T. The following example shows an infinite commutative implicative n.p.o. ternary semigroup with 1 as its greatest element.

Example 3. *Let $(\mathbb{Z}^+, [\ ])$ be the ternary semigroup of positive integers with the ternary multiplication induced by usual multiplication. For $a, b \in \mathbb{Z}^+$, an order relation $\leq$ on $\mathbb{Z}^+$ is defined by*

$$a \leq b \Leftrightarrow b \,|\, a.$$

Here, $b|a$ means b divides a. We have that $(\mathbb{Z}^+, [\ ], \leq)$ is a commutative n.p.o. ternary semigroup with 1 as its greatest element. Indeed: it is easy to see that $(\mathbb{Z}^+, [\ ])$ is a commutative ternary semigroup. Let $x, y, u, v \in \mathbb{Z}^+$ with $x \leq y$. Since $y \,|\, x$, there exists $q \in \mathbb{Z}^+$ such that $x = qy$. From $xuv = q(yuv)$, it follows that $yuv \,|\, xuv$. Hence, $[xuv] \leq [yuv]$. Similarly, we get $[uxv] \leq [uyv]$ and $[uvx] \leq [uvy]$. Let $x, y, z \in \mathbb{Z}^+$. Since $xyz = xyz$, $x \,|\, xyz$, and so $[xyz] \leq x$. Similarly, we get $[xyz] \leq y$ and $[xyz] \leq z$. Since $1 \,|\, x$ for all $x \in \mathbb{Z}^+$, $x \leq 1$ for all $x \in \mathbb{Z}^+$. Thus, 1 is the greatest element.

Moreover, we have the ternary implication on $\mathbb{Z}^+$ defined by $[xyz]^* = \frac{z}{\gcd(xy,z)}$ for all $x, y, z \in \mathbb{Z}^+$. To see this, let $x, y, z, u \in \mathbb{Z}^+$ with $gcd(xy, z) = d$. Assume that $[uxy] \leq z$;

then $\frac{z}{d} \mid u\frac{xy}{d}$. *Since* $gcd(\frac{xy}{d}, \frac{z}{d}) = 1$, $\frac{z}{d} \mid u$. *Therefore,* $u \leq \frac{z}{d}$. *Conversely, assume that* $u \leq \frac{z}{d}$. *Then there exists* $p \in \mathbb{Z}^+$ *such that* $u = p\frac{z}{d}$, *and then* $uxy = (p\frac{xy}{d})z$. *This implies that* $z \mid uxy$, *that is,* $[uxy] \leq z$.

The following theorem shows that an implicative n.p.o. ternary semigroup always contains the greatest element.

Theorem 1. *Let* $(T, [\ ], \leq, [\ ]^*)$ *be an implicative n.p.o. ternary semigroup. Then the following properties hold:*

(1) $x \leq [xxx]^*$;

(2) $[xxx]^* = [yyy]^*$;

(3) T *contains the greatest element, namely* $[xxx]^*$,

for any $x, y \in T$.

Proof. As $[xxx] \leq x$, $x \leq [xxx]^*$, so (1) is hold. Since $[[xxx]^*yy] \leq y$, then $[xxx]^* \leq [yyy]^*$. Similarly, $[yyy]^* \leq [xxx]^*$. Thus, (2) holds. By $y \leq [yyy]^* \leq [xxx]^*$, it follows that T contains greatest element, namely $[xxx]^*$. Hence, (3) holds.

Let 1 be the greatest element of an n.p.o. ternary semigroup $(T, [\ ], \leq, [\ ]^*)$ if exists. It is observed that if 1 is the multiplicative identity then it can be verified that $[xyz] = 1$ if and only if $x = y = z = 1$ for any $x, y, z \in T$. Indeed, if $x, y, z \in T$ such that $[xyz] = 1$ then $1 = [xyz] \leq x \leq 1$; so $x = 1$. In a similar argument we can deduce that $y = 1$ and $z = 1$. Clearly, if $x = y = z = 1$ then $[xyz] = 1$.

Throughout the rest of the paper, we deal with an implicative n.p.o. ternary semigroup with 1 which is both the greatest element and the multiplicative identity.

The following theorem collects several properties of elements of implicative n.p.o. ternary semigroups.

Theorem 2. *Let* $(T, [\ ], \leq, [\ ]^*)$ *be an implicative n.p.o. ternary semigroup. Then for any* $x, y, z, u, v \in T$, *the following conditions hold:*

(1) $x \leq 1$, $[xxx]^* = 1$, $x = [11x]^*$;

(2) $x \leq [yz[xyz]]^*$;

(3) $x \leq [xx[xxx]]^*$;

(4) $x \leq [yzx]^*$;

(5) *if* $x \leq y$, *then* $[yuv]^* \leq [xuv]^*$ *and* $[uvx]^* \leq [uvy]^*$;

(6) $x \leq y \Leftrightarrow [x1y]^* = 1 \Leftrightarrow [1xy]^* = 1$;

(7) $[xy[zuv]^*]^* = [[xyz]uv]^* = [x[yzu]v]^*$.

Proof. (1) It is clear that $x \leq 1$ and $[xxx]^* = 1$. As $[x11] = x \leq x$, we get that $x \leq [11x]^*$. Since $[11x]^* \leq [11x]^*$, we have $[11x]^* = [[11x]^*11] \leq x$.

(2) From $[xyz] \leq [xyz]$, we get $x \leq [yz[xyz]]^*$.

(3) The assertion follows by (2).

(4) This is clear because $[xyz] \leq x$.

(5) Assume that $x \leq y$. Since $[yuv]^* \leq [yuv]^*$, $[[yuv]^*yu] \leq v$. By assumption, it follows that

$$[[yuv]^*xu] \leq [[yuv^*]yu] \leq v.$$

Then $[yuv]^* \leq [xuv]^*$. Similarly, if $x \leq y$, then $[uvx]^* \leq [uvy]^*$.

(6) If $x \leq y$, then $[1x1] \leq [11y]$. Hence,

$$1 \leq [x1[11y]]^* = [x1y]^* \leq 1.$$

Thus, $[x1y]^* = 1$. Similarly, if $[x1y]^* = 1$, then $[1xy]^* = 1$. Conversely, if $[1xy]^* = 1$, then $1 \leq [1xy]^*$, and so $x = [11x] \leq y$.

(7) Let $s = [xy[zuv]^*]^*$ and $t = [[xyz]uv]^*$. We have $[sxy] \leq [zuv]^*]$, and thus, $[s[xyz]u] = [[sxy]zu] \leq v$. Hence, $s \leq [[xyz]uv]^* = t$. By $[[txy]zu] = [t[xyz]u] \leq v$, it follows that $[txy] \leq [zuv]^*$. Then $t \leq [xy[zuv]^*]^* = s$. Hence, $s = t$.

Now, let $s = [xy[zuv]^*]^*$ and $w = [x[yzu]v]^*$. Then $[sxy] \leq [zuv]^*]$, and thus, $[sx[yzu]] = [[sxy]zu] \leq v$, so $s \leq [x[yzu]v]^* = w$. As $[[wxy]zu] = [wx[yzu]] \leq v$, we have $[wxy] \leq [zuv]^*$. Then $w \leq [xy[zuv]^*]^* = s$. Thus, $s = w$.

3. Implicative homomorphisms

We begin this section with the definition of implicative homomorphisms between implicative n.p.o. ternary semigroups.

Definition 3. *Let $(T_1, [\ ]_1, \leq_1, [\ ]_1^*)$ and $(T_2, [\ ]_2, \leq_2, [\ ]_2^*)$ be implicative n.p.o. ternary semigroups. A mapping $\varphi : T_1 \longrightarrow T_2$ from T_1 onto T_2 such that*

$$\varphi([xyz]_1^*) = [\varphi(x)\varphi(y)\varphi(z)]_2^*$$

for all $x, y, z \in T_1$ is called an implicative homomorphism from T_1 onto T_2.

To study the notion of quotient structures of implicative n.p.o. ternary semigroups, we need the concept of filters.

Definition 4. *Let $(T, [\ ], \leq)$ be an n.p.o. ternary semigroup. A non-empty subset F of T is called a filter of T if the following coditions hold:*

(1) $[xyz] \in F$ for any $x, y, z \in F$, that is F is a ternary subsemigroup of T;

(2) for any $x, y \in T$, if $x \leq y$ and $x \in F$, then $y \in F$.

Example 4. *Let $T = \{1, a, b, c, d\}$. Let us consider the n.p.o. ternary semigroup $(T, [\ ], \leq)$ with a ternary multiplication $[\ ]$ and an order relation $\leq$ defined on T as follows:*

[ ]	1	a	b	c	d
11	1	a	b	c	d
$1a$	a	a	d	c	d
$1b$	b	d	b	d	d
$1c$	c	c	d	c	d
$1d$	d	d	d	d	d

[ ]	1	a	b	c	d
$a1$	a	a	d	c	d
aa	a	a	d	c	d
ab	d	d	d	d	d
ac	c	c	d	c	d
ad	d	d	d	d	d

[ ]	1	a	b	c	d
$b1$	b	d	b	d	d
ba	d	d	d	d	d
bb	b	d	b	d	d
bc	d	d	d	d	d
bd	d	d	d	d	d

[ ]	1	a	b	c	d
$c1$	c	c	d	c	d
ca	c	c	d	c	d
cb	d	d	d	d	d
cc	c	c	d	c	d
cd	d	d	d	d	d

[ ]	1	a	b	c	d
$d1$	d	d	d	d	d
da	d	d	d	d	d
db	d	d	d	d	d
dc	d	d	d	d	d
dd	d	d	d	d	d

and

$$\begin{aligned}\leq\, = \{&(1,1),(a,a),(b,b),(c,c),(d,d),(b,1),(a,1),\\ &(c,a),(c,1),(d,c),(d,a),(d,b),(d,1)\}.\end{aligned}$$

Observe that 1 *is the greatest element. It is easy to verify that* $F_1 = \{1\}$, $F_2 = \{1, a\}$, $F_3 = \{1, b\}$, $F_4 = \{1, a, c\}$, $F_5 = T$ *are filters, but* $\{1, a, b\}$ *is not a filter.*

Now, we investigate some properties of implicative homomorphisms.

Theorem 3. *Let* $(T_1, [\ ]_1, \leq_1, [\ ]_1^*)$ *and* $(T_2, [\ ]_2, \leq_2, [\ ]_2^*)$ *be implicative n.p.o. ternary semigroups. Let* $\varphi : T_1 \longrightarrow T_2$ *be an implicative homomorphism from* T_1 *onto* T_2*. Then the following conditions hold:*

(1) $\varphi(1) = 1'$*, where* 1 *and* $1'$ *are the identities as well as the greatest elements of* T_1 *and of* T_2*, respectively;*

(2) φ *is isotonic, that is for any* $x, y \in T_1$*, if* $x \leq_1 y$ *then* $\varphi(x) \leq_2 \varphi(y)$*;*

(3) φ *is a (ternary semigroup) homomorphism (i.e., for any* $x, y, z \in T_1$, $\varphi[xyz]_1 = [\varphi(x)\varphi(y)\varphi(z)]_2$*);*

(4) $\varphi^{-1}(1')$ *is a filter of* T_1*, when* $\varphi^{-1}(1') = \{x \in T_1 : \varphi(x) = 1'\}$*;*

(5) φ *is an (ternary semigroup) isomorphism if and only if* $\varphi^{-1}(1') = \{1\}$*.*

Proof. (1) By Theorem 2 (1), we have $1 = [111]_1^*$. Consequently,

$$\varphi(1) = \varphi([111]_1^*) = [\varphi(1)\varphi(1)\varphi(1)]_2^* = 1'.$$

(2) If $x, y \in T_1$ such that $x \leq_1 y$, then by Theorem 2 (6) we get that $[x1y]_1^* = 1$. Hence,

$$[\varphi(x)1'\varphi(y)]_2^* = [\varphi(x)\varphi(1)\varphi(y)]_2^* = \varphi([x1y]_1^*) = \varphi(1) = 1'.$$

By Theorem 2 (6), we have $\varphi(x) \leq_2 \varphi(y)$.

(3) Let $x, y, z \in T_1$; then $\varphi(u) = [\varphi(x)\varphi(y)\varphi(z)]_2$ for some $u \in T_1$. Since φ is an implicative homomorphism, we have

$$\begin{aligned}
[\varphi([xyz]_1)\varphi(1)\varphi(u)]_2^* &= \varphi([[xyz]_1 1u]_1^*) \\
&= \varphi([xy[z1u]_1]_1^*) \\
&= [\varphi(x)\varphi(y)\varphi([z1u]_1)]_2^* \\
&= [\varphi(x)\varphi(y)[\varphi(z)\varphi(1)\varphi(u)]_2]_2^* \\
&= [[\varphi(x)\varphi(y)\varphi(z)]_2\varphi(1)\varphi(u)]_2^* \\
&= [\varphi(u)\varphi(1)\varphi(u)]_2^* \\
&= 1'.
\end{aligned}$$

Then, by Theorem 2 (6), $\varphi([xyz]_1) \leq_2 [\varphi(x)\varphi(y)\varphi(z)]_2$. From $[xyz]_1 \leq_1 [xyz]_1$, it follows that $x \leq_1 [yz[xyz]_1]_1^*$. By (2),

$$\varphi(x) \leq_2 \varphi([yz[xyz]_1]_1^*) = [\varphi(y)\varphi(z)\varphi([xyz]_1)]_2^*.$$

That is, $[\varphi(x)\varphi(y)\varphi(z)]_2 \leq_2 \varphi([xyz]_1)$. Hence, $\varphi([xyz]_1) = [\varphi(x)\varphi(y)\varphi(z)]_2$.

(4) Let $x, y, z \in \varphi^{-1}(1')$, that is, $\varphi(x) = \varphi(y) = \varphi(z) = 1'$. By (3),

$$\varphi([xyz]_1) = [\varphi(x)\varphi(y)\varphi(z)]_2 = [1'1'1']_2 = 1'.$$

Thus, $[xyz] \in \varphi^{-1}(1')$. Assume that $x, y \in T_1$ such that $x \leq_1 y$ and $x \in \varphi^{-1}(1')$. Then

$$1' = \varphi(1) = \varphi([x1y]_1^*) = [\varphi(x)\varphi(1)\varphi(y)]_2^* = [1'1'\varphi(y)]_2^* = \varphi(y),$$

so $y \in \varphi^{-1}(1')$. Hence, $\varphi^{-1}(1')$ is a filter.

(5) Assume that $\varphi^{-1}(1') = \{1\}$. Let $x, y \in T_1$ be such that $\varphi(x) = \varphi(y)$. Thus,

$$\varphi([x1y]_1^*) = [\varphi(x)\varphi(1)\varphi(y)]_2^* = [\varphi(x)1'\varphi(x)]_2^* = 1'.$$

This means that $[x1y]_1^* \in \varphi^{-1}(1')$, that is, $[x1y]_1^* = 1$. Then by Theorem 2 (6), we get that $x \leq_1 y$. Similarly, we have $[y1x]_1^* = 1$, then $y \leq_1 x$. Hence, $x = y$. On the other hand, assume that φ is an isomorphism. If $x \in \varphi^{-1}(1')$, then $\varphi(x) = 1' = \varphi(1)$, so by assumption we have $x = 1$. Hence, $\varphi^{-1}(1') = \{1\}$.

4. Commutative implicative n.p.o. ternary semigroups

Hereafter, we deal with commutative implicative n.p.o. ternary semigroups.

Proposition 1. *If F is a filter of a commutative implicative n.p.o. ternary semigroup $(T, [\ ], \leq, [\ ]^*)$, then $1 \in F$.*

Proof. The assertion follows by 1 is the greatest element of T.

Let $(T, [\ ])$ be a ternary semigroup. An equivalence relation α on T is called a *congruence* if for any $x, y, s, t \in T$, if $(x, y) \in \alpha$, then $([stx], [sty]), ([xst], [yst]), ([sxt], [syt]) \in \alpha$.

Definition 5. *Let $(T, [\ ], \leq, [\ ]^*)$ be a commutative implicative n.p.o. ternary semigroup, and let F be a filter of T. For any $x, y \in T$, the relation ρ_F defined on T as follows:*

$$x\rho_F y \Leftrightarrow \text{ there exist } a, b \in F \text{ such that } [abx] \leq y \text{ and } [aby] \leq x.$$

Lemma 1. *Let $(T, [\ ], \leq, [\ ]^*)$ be a commutative implicative n.p.o. ternary semigroup, and let F be a filter of T. Then the relation ρ_F is a congruence defined on T.*

Proof. Let $x, y, z \in T$. As $1 \in F$ and $[11x] \leq x$, we have $x\rho_F x$. If $x\rho_F y$, then there exist $a, b \in F$ such that $[abx] \leq y$ and $[aby] \leq x$, hence, $y\rho_F x$. Assume that $x\rho_F y$ and $y\rho_F z$. Then there exist $a, b, c, d \in F$ such that $[abx] \leq y$ and $[aby] \leq x$, $[cdy] \leq z$ and $[cdz] \leq y$. We have

$$[[abc]dx] = [[cab]dx] = [c[abd]x] = [c[dab]x] = [cd[abx]] \leq [cdy] \leq z$$

and

$$[[abc]dz] = [ab[cdz]] \leq [aby] \leq x.$$

Thus, $x\rho_F z$. Therefore, ρ_F is an equivalence relation on T. Suppose that $x\rho_F y$. Then there exist $a, b \in F$ such that $[abx] \leq y$ and $[aby] \leq x$. Let $s, t \in T$. We have $[ab[xst]] = [[abx]st] \leq [yst]$ and $[ab[yst]] = [[aby]st] \leq [xst]$; so $[xst]\rho_F[yst]$. Similarly, $[stx]\rho_F[sty]$ and $[sxt]\rho_F[syt]$. Hence, ρ_F is a congruence relation on T.

Let $(T, [\ ], \leq, [\ ]^*)$ be a commutative implicative n.p.o. ternary semigroup, and let F be a filter of T. As usual, for each $x \in T$ the corresponding element in T/ρ_F, denoted by $[x]_{\rho_F}$, is the equivalence class $[x]_{\rho_F} = \{y \in T : y\rho_F x\}$, that is $T/\rho_F = \{[x]_{\rho_F} : x \in T\}$. Define the ternary multiplication $[\![\ ]\!] : T/\rho_F \times T/\rho_F \times T/\rho_F \longrightarrow T/\rho_F$ by

$$[\![[x]_{\rho_F}[y]_{\rho_F}[z]_{\rho_F}]\!] = [[xyz]]_{\rho_F}$$

for all $[x]_{\rho_F}, [y]_{\rho_F}, [z]_{\rho_F} \in T/\rho_F$. Then $(T/\rho_F, [\![\ ]\!])$ is a commutative ternary semigroup. Indeed, let $[x]_{\rho_F}[y]_{\rho_F}[z]_{\rho_F} \in T/\rho_F$. We have

$$[\![[x]_{\rho_F}[y]_{\rho_F}[z]_{\rho_F}]\!] = [[xyz]]_{\rho_F} = [[yzx]]_{\rho_F} = [\![[y]_{\rho_F}[z]_{\rho_F}[x]_{\rho_F}]\!].$$

Similarly, we get $[\![[x]_{\rho_F}[y]_{\rho_F}[z]_{\rho_F}]\!] = [\![[z]_{\rho_F}[x]_{\rho_F}[y]_{\rho_F}]\!]$, $[\![[x]_{\rho_F}[y]_{\rho_F}[z]_{\rho_F}]\!] = [\![[y]_{\rho_F}[x]_{\rho_F}[z]_{\rho_F}]\!]$, $[\![[x]_{\rho_F}[y]_{\rho_F}[z]_{\rho_F}]\!] = [\![[z]_{\rho_F}[y]_{\rho_F}[x]_{\rho_F}]\!]$, $[\![[x]_{\rho_F}[y]_{\rho_F}[z]_{\rho_F}]\!] = [\![[x]_{\rho_F}[z]_{\rho_F}[y]_{\rho_F}]\!]$. The order relation $\preceq$ on T/ρ_F is induced by the relation $\leq$ as follows: for any $[x]_{\rho_F}$, $[y]_{\rho_F} \in T/\rho_F$, defined $\preceq$ by

$$[x]_{\rho_F} \preceq [y]_{\rho_F} \text{ if for any } a \in [x]_{\rho_F}, b \in [y]_{\rho_F}, \text{ there exist } c, d \in F \text{ such that } [cda] \leq b.$$

Lemma 2. *Let $(T, [\ ], \leq, [\ ]^*)$ be a commutative implicative n.p.o. ternary semigroup, and let F be a filter of T. The ordered relation $\preceq$ is a partial order on T/ρ_F.*

Proof. If $x' \in [x]_{\rho_F}$, then by $1 \in F$ we have $[11x'] \leq x'$; so $[x]_{\rho_F} \preceq [x]_{\rho_F}$. Assume $[x]_{\rho_F} \preceq [y]_{\rho_F}$ and $[y]_{\rho_F} \preceq [x]_{\rho_F}$. Since $x \in [x]_{\rho_F}$ and $y \in [x]_{\rho_F}$, there exist $c_1, d_1, c_2, d_2 \in F$ such that $[c_1d_1x] \leq y$ and $[c_2d_2y] \leq x$. We have

$$[[c_1d_1c_2]d_2x] = [c_1d_1[c_2d_2x]] = [c_1d_1[xc_2d_2]] = [[c_1d_1x]c_2d_2] \leq [yc_2d_2] \leq y.$$

Also,

$$[[c_1d_1c_2]d_2y] = [c_1d_1[c_2d_2y]] \leq [c_1d_1x] \leq x.$$

Thus, $x\rho_F y$, that is, $[x]_{\rho_F} = [y]_{\rho_F}$. Assume that $[x]_{\rho_F} \preceq [y]_{\rho_F}$ and $[y]_{\rho_F} \preceq [z]_{\rho_F}$. Let $x' \in [x]_{\rho_F}$, $y' \in [y]_{\rho_F}$ and $z' \in [z]_{\rho_F}$. Then there exist $c_3, d_3, c_4, d_4 \in F$ such that $[c_3d_3x'] \leq y'$ and $[c_4d_4y'] \leq z'$. From

$$[[c_3d_3c_4]d_4x'] = [c_3d_3[c_4d_4x']] = [c_3d_3[x'c_4d_4]] = [[c_3d_3x']c_4d_4] \leq [y'c_4d_4] = [c_4d_4y'] \leq z'$$

it follows that $[x]_{\rho_F} \preceq [z]_{\rho_F}$. These show that $\preceq$ is a partial order on T/ρ_F.

Lemma 3. *Let $(T, [\ ], \leq)$ be a commutative n.p.o. ternary semigroup, and let F be a filter of T. Then $(T/\rho_F, [\![\ ]\!], \preceq)$ is a commutative n.p.o. ternary semigroup.*

Proof. We have seen that $(T/\rho_F, [\![\ ]\!])$ is a commutative ternary semigroup. By Lemma 2, $\preceq$ is a partial order on T/ρ_F. Let $[x]_{\rho_F}, [y]_{\rho_F}, [u]_{\rho_F}, [v]_{\rho_F} \in T/\rho_F$ be such that $[x]_{\rho_F} \preceq [y]_{\rho_F}$. Let $a \in [[xuv]]_{\rho_F}$ and $b \in [[yuv]]_{\rho_F}$. Since $x \in [x]_{\rho_F}$, $y \in [y]_{\rho_F}$, and $[x]_{\rho_F} \preceq [y]_{\rho_F}$, there exist $c, d \in F$ such that $[cdx] \leq y$. From $a \in [[xuv]]_{\rho_F}$, there exist $c_1, d_1 \in F$ such that $[c_1d_1[xuv]] \leq a$ and $[c_1d_1a] \leq [xuv]$. Similarly, by $b \in [[yuv]]_{\rho_F}$, there exist $c_2, d_2 \in F$ such that $[c_2d_2[yuv]] \leq b$ and $[c_2d_2b] \leq [yuv]$. Consider:

$$\begin{aligned}
[[c_2d_2[cdc_1]]d_1a] &= [c_2d_2[[cdc_1]d_1a]] \\
&= [c_2d_2[cd[c_1d_1a]]] \\
&\leq [c_2d_2[cd[xuv]]] \\
&= [c_2d_2[[cdx]uv]] \\
&\leq [c_2d_2[yuv]] \\
&\leq b.
\end{aligned}$$

Then $[[c_2d_2[cdc_1]]d_1a] \leq b$, so $[[xuv]]_{\rho_F} \preceq [[yuv]]_{\rho_F}$. Hence,

$$[\![[x]_{\rho_F}[u]_{\rho_F}[v]_{\rho_F}]\!] = [[xuv]]_{\rho_F} \preceq [[yuv]]_{\rho_F} = [\![[y]_{\rho_F}[u]_{\rho_F}[v]_{\rho_F}]\!].$$

Similarly, $[\![[u]_{\rho_F}[x]_{\rho_F}[v]_{\rho_F}]\!] \preceq [\![[u]_{\rho_F}[y]_{\rho_F}[v]_{\rho_F}]\!]$, and $[\![[u]_{\rho_F}[v]_{\rho_F}[x]_{\rho_F}]\!] \preceq [\![[u]_{\rho_F}[v]_{\rho_F}[y]_{\rho_F}]\!]$.

Let $[x]_{\rho_F}, [y]_{\rho_F}, [z]_{\rho_F} \in T/\rho_F$. To show that $[\![[x]_{\rho_F}[y]_{\rho_F}[z]_{\rho_F}]\!] \preceq [x]_{\rho_F}$, let $a' \in [[xyz]]_{\rho_F}$, $b' \in [x]_{\rho_F}$. As $[xyz]\rho_F a'$, there exist $c_1, d_1 \in F$ such that $[c_1d_1[xyz]] \leq a'$ and $[c_1d_1a'] \leq [xyz]$. By $x\rho_F b'$, there exist $c_2, d_2 \in F$ such that $[c_2d_2x] \leq b'$ and $[c_2d_2b'] \leq x$. We have

$[[c_1d_1c_2]d_2a'] = [c_1d_1[c_2d_2a']] = [c_1d_1[a'c_2d_2]] = [[c_1d_1a']c_2d_2] \leq [[xyz]c_2d_2] = [c_2d_2[xyz]] = [[c_2d_2x]yz] \leq [b'yz] \leq b'$. Then $[[c_1d_1c_2]d_2a'] \leq b'$; hence, $[[xyz]]_{\rho_F} \preceq [x]_{\rho_F}$. That is $[\![x]_{\rho_F}[y]_{\rho_F}[z]_{\rho_F}]\!] \preceq [x]_{\rho_F}$. In the same manner, we can prove that $[\![x]_{\rho_F}[y]_{\rho_F}[z]_{\rho_F}]\!] \preceq [y]_{\rho_F}$ and $[\![x]_{\rho_F}[y]_{\rho_F}[z]_{\rho_F}]\!] \preceq [z]_{\rho_F}$.

Lemma 4. *Let $(T, [\ ], \leq, [\ ]^*)$ be a commutative implicative n.p.o. ternary semigroup, and let F be a filter of T. Then $(T/\rho_F, [\![\ ]\!], \preceq, [\![\ ]\!]^*)$ is a commutative implicative n.p.o. ternary semigroup with the ternary implication*

$$[\![x]_{\rho_F}[y]_{\rho_F}[z]_{\rho_F}]\!]^* = [[xyz]^*]_{\rho_F}$$

for all $[x]_{\rho_F}, [y]_{\rho_F}, [z]_{\rho_F} \in T/\rho_F$.

Proof. We first show that $[\![\ ]\!]^*$ is well-defined on T/ρ_F. Let $[x]_{\rho_F}$, $[y]_{\rho_F}$, $[z]_{\rho_F}$, $[x']_{\rho_F}$, $[y']_{\rho_F}$, $[z']_{\rho_F} \in T/\rho_F$ be such that $[x]_{\rho_F} = [x']_{\rho_F}$, $[y]_{\rho_F} = [y']_{\rho_F}$ and $[z]_{\rho_F} = [z']_{\rho_F}$. Then there exist $c_1, d_1, c_2, d_2, c_3, d_3 \in F$ such that $[c_1d_1x] \leq x'$, $[c_1d_1x'] \leq x$, $[c_2d_2y] \leq y'$, $[c_2d_2y'] \leq y$, $[c_3d_3z] \leq z'$ and $[c_3d_3z'] \leq z$. We denote $[xyz]^*$ by u and $[x'y'z']^*$ by t. Since $u \leq [xyz]^*$, $[uxy] \leq z$. Using the associativity and commutativity of T, we have $[[[[c_1d_1c_2]d_2c_3]d_3u]x'y'] \leq z'$. This shows that

$$[[[[c_1d_1c_2]d_2c_3]d_3[xyz]^*]] \leq [x'y'z']^*.$$

In the same manner, we have

$$[[[[c_1d_1c_2]d_2c_3]d_3[x'y'z']^*]] \leq [xyz]^*.$$

Thus we have shown that $[xyz]^*\rho_F[x'y'z']^*$ and hence $[[xyz]^*]_{\rho_F} = [[x'y'z']^*]_{\rho_F}$. Let $[x]_{\rho_F}, [y]_{\rho_F}, [z]_{\rho_F}, [u]_{\rho_F} \in T/\rho_F$ be such that $[\![u]_{\rho_F}[x]_{\rho_F}[y]_{\rho_F}]\!] \preceq [z]_{\rho_F}$; then $[[uxy]]_{\rho_F} \preceq [z]_{\rho_F}$. To show that $[u]_{\rho_F} \preceq [[xyz]^*]_{\rho_F}$, that is, $[u]_{\rho_F} \preceq [\![x]_{\rho_F}[y]_{\rho_F}[z]_{\rho_F}]\!]^*$. Let $a \in [u]_{\rho_F}$ and $b \in [[xyz]^*]_{\rho_F}$. Since $[[uxy]]_{\rho_F} \preceq [z]_{\rho_F}$, there exist $c, d \in F$ such that $[cd[uxy]] \leq z$. By $a \in [u]_{\rho_F}$, there exist $c_1, d_1 \in F$ such that $[c_1d_1u] \leq a$ and $[c_1d_1a] \leq u$. Similarly, by $b \in [[xyz]^*]_{\rho_F}$, there exist $c_2, d_2 \in F$ such that $[c_2d_2[xyz]^*] \leq b$ and $[c_2d_2b] \leq [xyz]^*$. As $[cd[uxy]] \leq z$, we have $[cdu] \leq [xyz]^*$. Consider:

$$[[c_2d_2[cdc_1]]d_1a] = [c_2d_2[[cdc_1]d_1a]] = [c_2d_2[cd[c_1d_1a]]] = [c_2d_2[cdu]] = [c_2d_2[xyz]^*] \leq b.$$

Then $[u]_{\rho_F} \preceq [\![x]_{\rho_F}[y]_{\rho_F}[z]_{\rho_F}]\!]^*$.

Conversely, suppose that $[u]_{\rho_F} \preceq [\![x]_{\rho_F}[y]_{\rho_F}[z]_{\rho_F}]\!]^*$; then $[u]_{\rho_F} \preceq [[xyz]^*]_{\rho_F}$. To show that $[\![u]_{\rho_F}[x]_{\rho_F}[y]_{\rho_F}]\!] \preceq [z]_{\rho_F}$, that is, $[uxy]_{\rho_F} \preceq [z]_{\rho_F}$. Let $s \in [[uxy]]_{\rho_F}$ and $t \in [z]_{\rho_F}$. Since $[xyz]^* \in [[xyz]^*]_{\rho_F}$, $u \in [u]_{\rho_F}$, and $[u]_{\rho_F} \preceq [[xyz]^*]_{\rho_F}$, there exist $c', d' \in F$ such that $[c'd'u] \leq [xyz]^*$. By $s \in [[uxy]]_{\rho_F}$, there exist $c_3, d_3 \in F$ such that $[c_3d_3[uxy]] \leq s$ and $[c_3d_3s] \leq [uxy]$. Similarly, by $t \in [z]_{\rho_F}$, there exist $c_4, d_4 \in F$ such that $[c_4d_4z] \leq t$ and $[c_4d_4t] \leq z$. Consider:

$$[[c_5d_5[c'd'c_3]]d_3s] = [c_5d_5[[c'd'c_3]d_3s]]$$

$$\begin{aligned}
&= [c_5d_5[c'd'[c_3d_3s]]]\\
&\leq [c_5d_5[c'd'[uxy]]]\\
&= [c_5d_5[[c'd'u]xy]]\\
&\leq [c_5d_5z]\\
&\leq t.
\end{aligned}$$

Hence, $\llbracket [u]_{\rho_F}[x]_{\rho_F}[y]_{\rho_F}\rrbracket \preceq [z]_{\rho_F}$.

In view of Lemma 4, the mapping $\eta : (T, [\ ], \leq, [\ ]^*) \longrightarrow (T/\rho_F, \llbracket\ \rrbracket, \preceq, \llbracket\ \rrbracket^*)$ defined by $x \mapsto [x]_{\rho_F}$ is a surjective mapping.

Definition 6. *Let φ be an implicative homomorphism from a commutative implicative n.p.o. ternary semigroup $(T_1, [\ ]_1, \leq_1, [\ ]_1^*)$ onto a commutative implicative n.p.o. ternary semigroup $(T_2, [\ ]_2, \leq_2, [\ ]_2^*)$. The kernel of φ, denoted by $\operatorname{Ker}\varphi$, is defined to be the set*

$$\operatorname{Ker}\varphi = \{x \in T_1 : \varphi(x) = 1'\},$$

where 1 and $1'$ are the identities and the greatest elements of T_1 and T_2, respectively.

Remark 1. *By Theorem 3, we get that $\operatorname{Ker}\varphi$ is a filter of T_1.*

Lemma 5. *Let $(T, [\ ], \leq, [\ ]^*)$ be a commutative implicative n.p.o. ternary semigroup and let F be a filter of T and ρ_F a congruence on T. Then the canonical homomorphism*

$$\eta : T \longrightarrow T/\rho_F$$

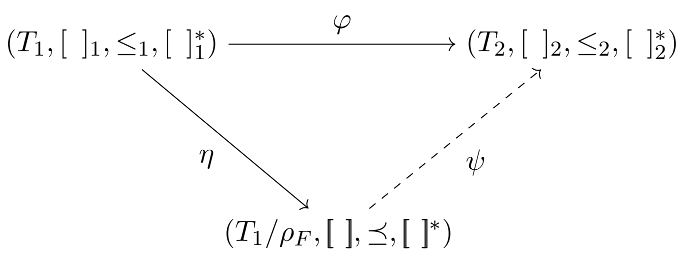
is an implicative homomorphism from $(T, [\ ], \leq, [\ ]^*)$ onto $(T/\rho_F, \llbracket\ \rrbracket, \preceq, \llbracket\ \rrbracket^*)$.

Proof. If $x, y, z \in T$, then

$$\eta([xyz]^*) = [[xyz]^*]_{\rho_F} = \llbracket [x]_{\rho_F}[y]_{\rho_F}[z]_{\rho_F}\rrbracket^* = \llbracket \eta([x]_{\rho_F})\eta([y]_{\rho_F})\eta([z]_{\rho_F})\rrbracket^*.$$

Hence, the assertion holds.

Theorem 4. *Let $(T_1, [\ ]_1, \leq_1, [\ ]_1^*)$ and $(T_2, [\ ]_2, \leq_2, [\ ]_2^*)$ be any two commutative implicative n.p.o. ternary semigroups, with 1 and $1'$ are the identities and the greatest elements of T_1 and T_2, respectively. Let $\varphi : T_1 \longrightarrow T_2$ be an implicative homomorphism from T_1 onto T_2, with $F = \operatorname{Ker}\varphi$, and let $\eta : T_1 \longrightarrow T_1/\rho_F$ be a canonical homomorphism from T_1 onto T_1/ρ_F. Then there exists an implicative homomorphism $\psi : T_1/\rho_F \longrightarrow T_2$ from T_1/ρ_F onto T_2 such that the following diagram is commutative:*



Moreover, if $\operatorname{Ker}\eta = \varphi^{-1}(1')$ *, then* ψ *is an implicative isomorphism, that is*

$$(T_1/\rho_F, [\ ], \preceq, [\ ]^*) \cong (T_2, [\ \ ]_2, \leq_2, [\ \ ]_2^*).$$

Proof. Define $\psi : T_1/\rho_F \longrightarrow T_2$ by

$$\psi([x]_{\rho_F}) = \varphi(x)$$

for any $x \in T_1$. We have ψ is well-defined. Indeed, let $[x]_{\rho_F}, [y]_{\rho_F} \in T_1/\rho_F$ be such that $[x]_{\rho_F} = [y]_{\rho_F}$. Since $x\rho_F y$, there exist $c, d \in F$ such that $[cdx]_1 \leq_1 y$ and $[cdy]_1 \leq_1 x$. By Theorem 3, $\varphi([cdx]_1) \leq_2 \varphi(y)$ and $\varphi([cdy]_1) \leq_2 \varphi(x)$. Since $c, d \in F$, we have $\varphi(c) = 1'$ and $\varphi(d) = 1'$. From

$$\varphi([cdx]_1) = [\varphi(c)\varphi(d)\varphi(x)]_2 = [1'1'\varphi(x)]_2 = \varphi(x)$$

and

$$\varphi([cdy]_1) = [\varphi(c)\varphi(d)\varphi(y)]_2 = [1'1'\varphi(y)]_2 = \varphi(y)$$

it follows that $\varphi(x) \leq_2 \varphi(y)$ and $\varphi(y) \leq_2 \varphi(x)$. Hence, $\varphi(x) = \varphi(y)$.

Next, we have ψ is an implicative homomorphism. In fact, for $[x]_{\rho_F}, [y]_{\rho_F}, [z]_{\rho_F} \in T_1/\rho_F$, we have

$$\begin{aligned}\psi([[x]_{\rho_F}[y]_{\rho_F}[z]_{\rho_F}]^*) &= \psi([[xyz]_1^*]_{\rho_F}) \\ &= \varphi([xyz]_1^*) \\ &= [\varphi(x)\varphi(y)\varphi(z)]_2^* \\ &= [\psi([x]_{\rho_F})\psi([y]_{\rho_F})\psi([z]_{\rho_F})]_2^*.\end{aligned}$$

If $t \in T_2$, since φ is onto, then $\varphi(s) = t$ for some $s \in T_1$. Consequently,

$$\psi(\eta(s)) = \psi([s]_{\rho_F}) = \varphi(s) = t.$$

For any $s \in T_1$, we have

$$\psi \circ \eta(s) = \psi(\eta(s)) = \psi([s]_{\rho_F}) = \varphi(s).$$

This shows that the diagram is commutative.

Finally, we assume that $\operatorname{Ker}\eta = \varphi^{-1}(1')$. To show that ψ is an implicative isomorphism, we can only show ψ is one-to-one. Let $[x]_{\rho_F}, [y]_{\rho_F} \in T_1/\rho_F$ be such that $\psi([x]_{\rho_F}) = \psi([y]_{\rho_F})$; then $\varphi(x) = \varphi(y)$. By Theorem 2 (6), we have

$$\varphi([1xy]_1^*) = [\varphi(1)\varphi(x)\varphi(y)]_2^* = [1'\varphi(x)\varphi(x)]_2^* = 1'.$$

Hence, $[1xy]_1^* \in \varphi^{-1}(1') = \operatorname{Ker}\ \eta$. Similarly, $[1yx]_1^* \in \varphi^{-1}(1') = \operatorname{Ker}\ \eta$. Since $\operatorname{Ker}\eta \subseteq \operatorname{Ker}\varphi$, $[1xy]_1^*$, $[1yx]_1^* \in \operatorname{Ker}\varphi = F$. Let us denote $[1xy]_1^*$ by c and $[1yx]_1^*$ by d. Since $c \leq [1xy]_1^*$, $[c1x]_1 \leq_1 y$. Similarly, by $d \leq [1yx]_1^*$, we have $[d1y]_1 \leq_1 x$. Hence,

$$[[c1d]_1 1x]_1 = [c1[d1x]_1]_1 = [c1[xd1]_1]_1 = [[c1x]_1 d1]_1 \leq_1 [yd1]_1 \leq_1 y$$

and

$$[[c1d]_1 1y]_1 = [c1[d1y]_1]_1 \leq_1 [c1x]_1 \leq_1 x.$$

So $x\rho_F y$, and $[x]_{\rho_F} = [y]_{\rho_F}$. Therefore, ψ is one-to-one.

5. Conclusions

In this paper, we introduce the definition of n.p.o. ternary semigroups and implicative n.p.o. ternary semigroups in Definition 1 and Definition 2, respectively. We observe that an n.p.o. ternary semigroup with identity need not to be implicative and the greatest element of an implicative n.p.o. ternary semigroup need not to be identity. Throughout this paper, we assume that implicative n.p.o. ternary semigroups consists an element 1 which is both the greatest element and the multiplicative identity. Then we define an implicative homomorphism between two implicative n.p.o. ternary semigroups in Definition 3. The algebraic properties of such homomorphism are presented in Theorem 3. In the last section, we consider commutative implicative n.p.o. ternary semigroups. Under certain conditions, the diagram of homomorphism is presented in Theorem 4.

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