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Characterizations of Nonadditive Mappings in Prime $*$-Rings Involving Bi-Skew Products

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Abstract. The paper investigates nonadditive mappings $\Omega : \Re \to \Re$ on a prime ring $\Re$ with involution $*$, characterized by satisfying one of the following conditions:

(i) $[\Omega(u), \Omega(v)]_\bullet = [u, v]_\bullet$ for all $u, v \in \Re$.

(ii) $[\Omega(u), v]_\bullet = [u, \Omega(v)]_\bullet$ for all $u, v \in \Re$.

(iii) $\Omega(u \bullet v) = \Omega(u) \bullet v$ for all $u, v \in \Re$.

Furthermore, the paper characterizes generalized bi-skew Jordan derivations within prime $*$-rings and examines the implications of these results in the context of various operator algebras.

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1. Introduction

In this paper, unless stated otherwise, $\Re$ represents a prime ring with $\mathcal{Z}(\Re)$ as its center. A ring $\Re$ is defined as prime if, for any $u, v \in \Re$, the condition $u\Re v = \{0\}$ implies that either $u = 0$ or $v = 0$.

We denote the maximal left and right rings of quotients of $\Re$ by $\mathcal{Q}_{ml}(\Re)$ and $\mathcal{Q}_{mr}(\Re)$, respectively. The maximal symmetric ring of quotients of $\Re$ is denoted by $\mathcal{Q}_{ms}(\Re)$. It is well established that $\Re \subseteq \mathcal{Q}_{ms}(\Re) \subseteq \mathcal{Q}_{ml}(\Re)$ and that $\mathcal{Q}_{ms}(\Re) = \mathcal{Q}_{ml}(\Re) \cap \mathcal{Q}_{mr}(\Re)$. Both $\mathcal{Q}_{ms}(\Re)$ and $\mathcal{Q}_{ml}(\Re)$ are also recognized as prime rings and share a common center,

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denoted by $\mathcal{C}$, which is the extended centroid of $\Re$. The set $\mathcal{C}$ is defined as $\{\lambda \in \mathcal{Q}_{ms}(\Re) \mid \lambda a = a\lambda \text{ for all } a \in \Re\}$. The ring $\Re$ is prime if and only if $\mathcal{C}$ is a field (see [1–3] for more details).

An involution '$*$' on $\Re$ is defined as an anti-automorphism of order 1 or 2. An anti-automorphism ∇ of $\Re$ is classified as being of the first kind with respect to $\mathcal{Z}(\Re)$ if it acts as the identity map on $\mathcal{Z}(\Re)$; otherwise, it is considered of the second kind on $\mathcal{Z}(\Re)$. For a semiprime ring $\Re$, if ∇ is an anti-automorphism, then a right ideal $\mathcal{I}$ of $\Re$ is dense if and only if $\nabla(\mathcal{I})$ is a dense left ideal of $\Re$, and a left ideal $\mathcal{J}$ of $\Re$ is dense if and only if $\nabla(\mathcal{J})$ is a dense right ideal of $\Re$. Consequently, with a straightforward modification of the proof in [1, Proposition 2.5.4], it follows that ∇ can be uniquely extended to an anti-automorphism of $\mathcal{Q}_{ms}(\Re)$. An anti-automorphism ∇ of $\Re$ is said to be of the first kind if it acts as the identity on $\mathcal{C}$, and of the second kind otherwise. For elements $u, v \in \Re$, the following products are defined: the Lie product $[u, v] = uv - vu$, the skew Lie product $[u, v]_* = uv - vu^*$, the skew Jordan product $u \diamond v = uv + vu^*$, the bi-skew Lie product $[u, v]_\bullet = uv^* - vu^*$, and the bi-skew Jordan product $u \bullet v = uv^* + vu^*$. These types of products have been the focus of extensive research by various authors (see [4–15]).

A map $\Omega : \Re \to \Re$ is termed a strong commutativity preserving map if it satisfies $\Omega([u, v]) = [u, v]$ for all $u, v \in \Re$. In the context of $*$-rings, Ω is referred to as a strong skew commutativity preserving map if $\Omega([u, v]_*) = [u, v]_*$ for all $u, v \in \Re$, and as a strong bi-skew commutativity preserving map if $\Omega([u, v]_\bullet) = [u, v]_\bullet$ for all $u, v \in \Re$ (see [9, 12, 14, 16]). A map $\Omega : \Re \to \Re$ is called a skew commuting map if it satisfies $[\Omega(u), v]_* = [u, \Omega(v)]_*$ for all $u, v \in \Re$ (see [17]). Similarly, Ω is termed a bi-skew commuting map if $[\Omega(u), v]_\bullet = [u, \Omega(v)]_\bullet$ for all $u, v \in \Re$.

A map $\Omega : \Re \to \mathcal{Q}_{ms}(\Re)$ is described as $*$-linear if it holds that $\Omega(u^*) = \Omega(u)^*$ for all $u \in \Re$. Furthermore, a map $\Omega : \Re \to \Re$ is known as a derivation if it satisfies $\Omega(uv) = \Omega(u)v + u\Omega(v)$ for all $u, v \in \Re$. In $*$-rings, a map $\Omega : \Re \to \Re$ is called a skew Lie derivation if $\Omega([u, v]_*) = [\Omega(u), v]_* + [u, \Omega(v)]_*$ for all $u, v \in \Re$ (see [7, 18]). Similarly, it is called a skew Jordan derivation if $\Omega(u \diamond v) = \Omega(u) \diamond v + u \diamond \Omega(v)$ for all $u, v \in \Re$. A map $\Omega : \Re \to \Re$ is defined as a bi-skew Jordan derivation if $\Omega(u \bullet v) = \Omega(u) \bullet v + u \bullet \Omega(v)$ for all $u, v \in \Re$ (see [19–22]). Additionally, a map $\Psi : \Re \to \Re$ is termed a generalized bi-skew Jordan derivation if there exists a bi-skew Jordan derivation $\Omega : \Re \to \Re$ such that $\Psi([u, v]_\bullet) = [\Psi(u), v]_\bullet + [u, \Omega(v)]_\bullet$ for all $u, v \in \Re$ (see [22]). A derivation $\Omega : \Re \to \Re$ is called an additive $*$-derivation if Ω is both additive and $*$-linear. A map $\Omega : \Re \to \Re$ is called a left (resp. right) centralizer if $\Omega(uv) = \Omega(u)v$ (resp. $\Omega(uv) = u\Omega(v)$) for all $u, v \in \Re$ (see [23]). Similarly, Ω is termed a left (resp. right) bi-skew Jordan centralizer if $\Omega(u \bullet v) = \Omega(u) \bullet v$ (resp. $\Omega(u \bullet v) = u \bullet \Omega(v)$) for all $u, v \in \Re$.

Brešar and Miers [24, Theorem 5] proved that if $\Re$ is a semiprime ring and $\Omega : \Re \to \Re$ is an additive strong commutativity preserving map, then $\Omega(u) = \lambda u + \mu(u)$ for all $u \in \Re$, where $\lambda \in \mathcal{C}$ and $\mu : \Re \to \mathcal{C}$ is an additive map. Recently, several authors have studied strong commutativity preserving maps (see [8, 10, 11, 15]). Strong skew commutativity preserving maps have received a lot of attention from various algebraists and have been widely studied in the context of rings and algebras (see [4, 6, 9, 12, 14]). Quiet recently, Siddeeque et al. [25, Theorem 2.2] characterized surjective strong skew commutativity preserving maps in prime rings without assuming the existence of the unity and a nontrivial symmetric idempotent. They established that if $\Re$ is a prime ring with an involution '$*$' and $\Omega : \Re \to \Re$ is a surjective strong skew commutativity preserving map, then there exists $\lambda \in \{1, -1\}$ such that $\Omega(a) = \lambda a$ for all $a \in \Re$. Qi and Chen [16, Theorem 2.1] characterized surjective strong bi-skew commutativity preserving maps on prime $*$-algebras. Consequently, they proved the following result: Let $\mathfrak{A}$ be a prime $*$-algebra, over a field $\mathbb{K}$, with unity I and containing a nontrivial symmetric idempotent. Suppose that '$*$' is of the second kind on $\mathcal{Z}(\mathfrak{A})$ and $\Psi(I)\Psi(I)^* = \Psi(I)^*\Psi(I) = I$. If $\Psi : \mathfrak{A} \to \mathfrak{A}$ is a surjective strong bi-skew commutativity preserving map, then $\Psi(u) = \alpha u \Psi(I)$ for all $u \in \mathcal{A}$, where $\alpha^* = \alpha \in \mathcal{C}$ and $\alpha^2 = I$. Khong and Zhang [17] showed, under certain restrictions, that if $\Re$ is a unital $*$-ring containing a nontrivial symmetric idempotent and $\Omega : \Re \to \Re$ is a skew commutating map, then there exists $\lambda^* = \lambda \in \mathcal{Z}(\Re)$ such that $\Omega(a) = \lambda a$ for all $a \in \Re$. Recently, Siddeeque et al. [25, Theorem 2.3] characterized skew commutating maps in prime rings without assuming the existence of the unity and a nontrivial symmetric idempotent. They established that if $\Re$ is a prime ring with an involution '$*$' and $\Omega : \Re \to \Re$ is a skew commutating map, then there exists $\lambda^* = \lambda \in \mathcal{C}$ such that $\Omega(a) = \lambda a$ for all $a \in \Re$. Motivated by the above results, in Section 2 of the present paper, we will characterize surjective strong bi-skew commutativity preserving maps and bi-skew commutating maps in prime rings without assuming the existence of the unity and a nontrivial symmetric idempotent (see Theorems 2.1 and 2.2). As applications, we will characterize such maps in different operator algebras.

Skew Lie, skew Jordan and bi-skew Jordan derivations have been explored by various algebraists in the context of algebras and rings (see [7, 18, 21, 22] and their bibliographic content). Very recently, Siddeeque and Shikeh [20] characterized bi-skew Jordan derivations in prime rings and proved that every bi-skew Jordan derivation on a unital prime $*$-ring containing a nontrivial symmetric idempotent is an additive $*$-derivation. In Section 3, we will characterize generalized bi-skew Jordan derivations in prime rings (see Theorem 3.2). As applications, we will characterize generalized bi-skew Jordan derivations in different operator algebras.

2. Strong bi-skew commutativity preserving maps and bi-skew commuting maps in prime rings

We facilitate our discussion with the following lemma which plays a crucial role in the proof of our main results.

Lemma 2.1. *Let $\Re$ be a prime ring with an involution '$*$' of order 2 and let $\mathfrak{a}, \mathfrak{b} \in \mathcal{Q}_{ml}(\Re)$ such that $\mathfrak{b}u^* = u\mathfrak{a}$ for all $u \in \Re$. Then $\mathfrak{a} = \mathfrak{b} = 0$.*

Proof. If $\Re$ is noncommutative, then by [20, Lemma 2.1], the result follows. Therefore, let's assume that $\Re$ is commutative. Thus, $\alpha^* \neq \alpha$ for some $\alpha \in \Re$. Substituting αu for u in the given relation, we find that $\mathfrak{b}\alpha^* u^* = \alpha u \mathfrak{a}$ for all $u \in \Re$. Also, $\alpha \mathfrak{b} u^* = \alpha u \mathfrak{a}$ for all $u \in \Re$. Hence, $(\alpha^* - \alpha)\mathfrak{b}u^* = 0$ for all $u \in \Re$. Therefore, $\mathfrak{b} = 0$ and consequently, $\mathfrak{a} = 0$.

Lemma 2.2. *Let $\Re$ be a prime PI-ring with an anti-automorphism ∇. Then ∇ is of first kind if and only if ∇ is of the first kind on $\mathcal{Z}(\Re)$.*

Proof. By [26, Corollary 1], $\mathcal{Q}_{ml}(\Re) = \Re\mathcal{C} = \{\frac{u}{\alpha} | \ u \in \Re \ \text{ and } \ 0 \neq \alpha \in \mathcal{Z}(\Re)\}$. Therefore if ∇ is of the first kind on $\mathcal{Z}(\Re)$, then ∇ can be uniquely extended to an anti-automorphism of $\Re\mathcal{C}$, denoted by ∇ also, by defining $\nabla(\frac{a}{\alpha}) = \frac{\nabla(u)}{\alpha}$. Hence ∇ is of the first kind. The converse holds trivially.

Lemma 2.3. *[20, Lemma 2.2] Let $\Re$ be a ring with an involution '$*$' and let $\Omega : \Re \times \Re \to \mathcal{G}$ be a map, where $\mathcal{G}$ is an additive group. Suppose $f, h : \Re \to \mathcal{G}$ are maps such that $\Omega(u, v) = f(uv^*) + h(vu^*)$ for all $u, v \in \Re$. Then $\Omega(uw, v) = \Omega(u, vw^*)$ for all $u, w, v \in \Re$.*

The following result provides a characterization of strong bi-skew commutativity preserving maps in prime rings without assuming the existence of the unity and a nontrivial symmetric idempotent, thereby generalizing, improving and extending [16, Theorem 2.1] to prime rings.

Theorem 2.1. *Let $\Re$ be a prime ring with an involution '$*$' of order 2, and let $\Omega : \Re \to \Re$ be a surjective strong bi-skew commutativity preserving map. Then there exists $q \in \mathcal{Q}_{ms}(\Re)$ such that $qq^* = 1$ and $\Omega(u) = uq$ for all $u \in \Re$ provided that either $dim_{\mathcal{C}}\Re\mathcal{C} > 4$ or both $char(\Re) \neq 2$ and '$*$' is of the second kind.*

Proof. By the given hypothesis , we have

$$[\Omega(u), \Omega(v)]_\bullet = [u, v]_\bullet \tag{2.1}$$

for all $u, v \in \Re$. Firstly we establish some facts about Ω.

Fact I. Ω is additive. For every $u, v, w \in \Re$, we have

$$\begin{aligned}
[\Omega(u), \Omega(v+w) - \Omega(v) - \Omega(w)]_\bullet &= [\Omega(u), \Omega(v+w)]_\bullet - [\Omega(u), \Omega(v)]_\bullet - [\Omega(u), \Omega(w)]_\bullet \\
&= [u, v+w]_\bullet - [u, v]_\bullet - [u, w]_\bullet \\
&= 0.
\end{aligned}$$

Therefore by the surjectiveness of Ω, we find that

$$[u, \Omega(v+w) - \Omega(v) - \Omega(w)]_\bullet = 0$$

for all $u, v, w \in \mathcal{R}$. Now in view of Lemma 2.1, it follows that $\Omega(u+v) = \Omega(u) + \Omega(v)$ for all $u, v \in \Re$, that is, Ω is additive.

Fact II. Ω is injective. Let $a \in \Re$ be such that $\Omega(a) = 0$. then

$$[a, v]_\bullet = [\Omega(a), \Omega(v)]_\bullet = 0$$

for all $v \in \Re$. Hence by Lemma 2.1, $a = 0$. Thus Ω is injective.

Therefore from (2.1), we have

$$\Omega(u)v^* + \Omega^{-1}(v)u^* - u\Omega^{-1}(v)^* - v\Omega(u)^* = 0, \tag{2.2}$$

for all $u, v \in \Re$. Next, we advance by examining the following two scenarios:
Case I. $\dim_{\mathcal{C}}\Re\mathcal{C} > 4$.

In view of [27, Theorem 3.5], it follows that there exists $q \in \mathcal{Q}_{ml}(\Re)$ such that $\Omega(u) = uq$ for all $u \in \Re$ and $\Omega^{-1}(v)^* = qv^*$ for all $v \in \Re$. Note that $qv^* = \Omega^{-1}(v)^* \in \Re$ for all $v \in \Re$. Hence $q\Re \subseteq \Re$. Consequently, $q \in \mathcal{Q}_{ms}(\Re)$. Therefore $\Omega^{-1}(u) = uq^*$ for all $u \in \Re$ and hence $\Omega(u)q^* = u$ for all $u \in \Re$. Using $\Omega(u) = uq$ in the last relation, we get $uqq^* = u$ for all $u \in \Re$. This entails that $qq^* = 1$.
Case I. $\dim_{\mathcal{C}}\Re\mathcal{C} \leq 4$.

In this case $\Re$ is a PI-ring and $\mathcal{Q}_{ml}(\Re) = \mathcal{Q}_{ms}(\Re) = \Re\mathcal{C}$. According to the given hypothesis char$(\Re) \neq 2$ and '$*$' is of the second kind. Hence by Lemma 2.2, there is $\zeta \in \mathcal{Z}(\Re)$ such that $\zeta^* \neq \zeta$. Let $\alpha = \zeta^* - \zeta$ and $\beta = \alpha^2$. Then $\alpha^* = -\alpha$ and $\beta^* = \beta$. Setting $v = \alpha$ in (2.2), we get

$$\alpha(\Omega(u) + \Omega(u)^*) = \Omega^{-1}(\alpha)u^* - u\Omega^{-1}(\alpha)^*, \tag{2.3}$$

for all $u \in \Re$. Also taking $v = \beta$ in (2.2), we get

$$\beta(\Omega(u) - \Omega(u)^*) = u\Omega^{-1}(\beta)^* - \Omega^{-1}(\beta)u^* \tag{2.4}$$

for all $u \in \Re$. Multiplying both sides of (2.3) by β and (2.4) by α, we get

$$\alpha\beta(\Omega(u) + \Omega(u)^*) = \beta\Omega^{-1}(\alpha)u^* - \beta u\Omega^{-1}(\alpha)^*, \tag{2.5}$$

and

$$\alpha\beta(\Omega(u) - \Omega(u)^*) = \alpha u\Omega^{-1}(\beta)^* - \alpha\Omega^{-1}(\beta)u^* \tag{2.6}$$

for all $u \in \Re$, respectively. Adding (2.5) and (2.6), we find that

$$2\alpha\beta\Omega(u) = \alpha u\Omega^{-1}(\beta)^* - \alpha\Omega^{-1}(\beta)u^* + \beta\Omega^{-1}(\alpha)u^* - \beta u\Omega^{-1}(\alpha)^*, \tag{2.7}$$

for all $u \in \Re$. Now $2\alpha\beta$ is invertible in $\mathcal{C}$. Hence, we have

$$\Omega(u) = uq + q_1u^*, \tag{2.8}$$

for all $u \in \Re$, where $q = (2\alpha\beta)^{-1}(\alpha\Omega^{-1}(\beta)^* - \beta\Omega^{-1}(\alpha)^*) \in \Re\mathcal{C}$ and $q_1 = (2\alpha\beta)^{-1}(\beta\Omega^{-1}(\alpha) - \alpha\Omega^{-1}(\beta)) \in \Re\mathcal{C}$. Using this in (2.1), we get

$$(uq + q_1u^*)(q^*v^* + vq_1^*) - (vq + q_1v^*)(q^*u^* + uq_1^*) = uv^* - vu^*, \tag{2.9}$$

for all $u, v \in \Re$. Replacing v by αv in (2.9), we get

$$(uq + q_1u^*)(-q^*v^* + vq_1^*) - (vq - q_1v^*)(q^*u^* + uq_1^*) = -uv^* - vu^*, \tag{2.10}$$

for all $u, v \in \Re$. Subtracting (2.10) and (2.9), we find that

$$(uq + q_1u^*)q^*v^* - q_1v^*(q^*u^* + uq_1^*) = uv^*, \tag{2.11}$$

for all $u, v \in \Re$.Alter u by αu in (2.11), we have

$$(uq - q_1u^*)q^*v^* - q_1v^*(-q^*u^* + uq_1^*) = uv^*, \tag{2.12}$$

for all $u, v \in \Re$. Subtracting (2.12) from (2.11), we obtain

$$q_1u^*q^*v^* = q_1v^*q^*u^*, \tag{2.13}$$

for all $u, v \in \Re$. Setting $u = \alpha$ in (2.13), we get $q_1[q^*, v] = 0$ for all $v \in \Re$. Therefore, $q_1 = 0$ or $q \in \mathcal{C}$. If $q_1 = 0$, then from (2.11), we find that $u(qq^* - 1)v = 0$ for all $u, v \in \Re$. Consequently, $qq^* = 1$. Hence we assume that $q \in \mathcal{C}$. Adding (2.11) and (2.12), we obtain

$$qq^*uv - q_1vuq_1^* = uv, \tag{2.14}$$

for all $u, v \in \Re$. Putting $u = v = \alpha$ in (2.14), we find that

$$qq^* - q_1q_1^* = 1. \tag{2.15}$$

Hence from (2.14), we have $q_1q_1^*uv = q_1vuq_1^*$ for all $u, v \in \Re$. Taking $v = \alpha$ in the last relation, we have $q_1[q_1^*, u] = 0$ for all $u \in \Re$. Therefore $q_1 \in \mathcal{C}$ and hence from (2.14), we find that $qq^*uv - q_1q_1^*vu = uv$ for all $u, v \in \Re$. Using (2.15), we get $q_1q_1^*(uv + vu) = 0$ for all $u, v \in \Re$. Putting $u = v = \alpha$ in the previous relation, we get $q_1q_1^* = 0$. Since $\mathcal{C}$ is a field, we conclude that $q_1 = 0$. From (2.8), we infer that $\Omega(u) = uq$ for all $u \in \Re$.

The following example shows that Theorem 2.1 does not hold if $\dim_{\mathcal{C}}\Re\mathcal{C} = 4$ and '$*$' is of the first kind.

Example 2.1. *Consider the ring $\mathcal{M}_2(\mathbb{K})$ of all 2×2 matrices over any field $\mathbb{K}$ with involution $*$ as the usual transpose map. Let $\lambda \in \mathbb{K}$ be fixed such that $\lambda \notin \{-1, 0, 1\}$. Define the map $\Omega : \mathcal{M}_2(\mathbb{K}) \to \mathcal{M}_2(\mathbb{K})$ by $\Omega\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} \lambda a & \frac{b}{\lambda} \\ \frac{c}{\lambda} & \lambda d \end{bmatrix}$. Then it can be easily seen that $\Omega([u, v]_\bullet) = [u, v]_\bullet$ holds for all $u, v \in \mathcal{M}_2(\mathbb{K})$. However, Ω is not of the form as described in Theorem 2.1.*

The following result gives a characterization of bi-skew commuting maps in prime rings without assuming the existence of the unity and a nontrivial symmetric idempotent.

Theorem 2.2. *Let $\Re$ be a prime ring with an involution '$*$' of order 2 and let $\Omega : \Re \to \Re$ be a bi-skew commuting map. Then there exists $q^* = q \in \mathcal{Q}_{ms}(\Re)$ such that $\Omega(u) = uq$ for all $u \in \Re$ provided that either $\dim_{\mathcal{C}}\Re\mathcal{C} > 4$ or both $char(\Re) \neq 2$ and '$*$' is of the second kind.*

Proof. According to the given hypothesis

$$[\Omega(u), v]_\bullet = [u, \Omega(v)]_\bullet \tag{2.16}$$

for all $u, v \in \Re$. First we show Ω is additive. For every $u, v, w \in \Re$, we have

$$[\Omega(u + w) - \Omega(u) - \Omega(w), v]_\bullet \quad = [\Omega(u + w), v]_\bullet - [\Omega(u), v]_\bullet - [\Omega(w), v]_\bullet$$

$$= [u+w,\Omega(v)]_\bullet - [u,\Omega(v)]_\bullet - [w,\Omega(v)]_\bullet$$
$$= 0$$

Applying Lemma 2.1, we infer that $\Omega(u+v) = \Omega(u) + \Omega(v)$ for all $a, b \in \Re$, that is, Ω is additive. Now (2.16), can be rewritten as

$$u\Omega(v)^* + v\Omega(u)^* - \Omega(u)v^* - \Omega(v)u^* = 0 \tag{2.17}$$

for all $u, v \in \Re$. Now we proceed by considering the following two cases:-
Case 1: $\dim(\Re\mathcal{C}) > 4$.

According to [27, Theorem 3.5] there exists $q \in Q_{ml}(\Re)$ such that

$$\Omega(u) = uq \quad \text{for all} \quad u \in \Re \tag{2.18}$$

and

$$\Omega(v)^* = qv^* \quad \text{for all} \quad v \in \Re. \tag{2.19}$$

Since $qv^* = \Omega(v)^* \in \Re$ for all $v \in \Re$. Therefore $q\Re \subseteq \Re$ and hence $q \in Q_{mr}(\Re)$. Consequently, $q \in Q_{ms}(\Re)$. Thus from (2.19), we find that

$$\Omega(u) = uq^* \quad \text{for all} \quad u \in \Re. \tag{2.20}$$

From (2.18) and (2.20) , we have $uq = uq^*$ for all $u \in \Re$. Hence $q^* = q$.
Case II: $\dim(\Re) \leq 4$.

In this case by [26, Theorem 2], $\mathcal{Z}(\mathcal{A}) \neq \{0\}$. Also, by the given hypothesis $\mathrm{char}(\Re) \neq 2$ and $\zeta^* \neq \zeta$ for some $\zeta \in \mathcal{Z}(\mathcal{A})$. Let $\alpha = \zeta^* - \zeta$ and $\beta = \alpha^2$. Then $\alpha^* = -\alpha$ and $\beta^* = \beta$. Setting $v = \alpha$ in (2.17), we get

$$\alpha(\Omega(u) + \Omega(u)^*) = \Omega(\alpha)u^* - u\Omega(\alpha)^*, \tag{2.21}$$

for all $u \in \Re$. Also taking $v = \beta$ in (2.17), we get

$$\beta(\Omega(u) - \Omega(u)^*) = u\Omega(\beta)^* - \Omega(\beta)u^* \tag{2.22}$$

for all $u \in \Re$. Multiplying both sides of (2.21) by β and (2.22) by α, we get

$$\alpha\beta(\Omega(u) + \Omega(u)^*) = \beta\Omega(\alpha)u^* - \beta u\Omega(\alpha)^*, \tag{2.23}$$

and

$$\alpha\beta(\Omega(u) - \Omega(u)^*) = \alpha u\Omega(\beta)^* - \alpha\Omega(\beta)u^* \tag{2.24}$$

for all $u \in \Re$, respectively. Adding (2.23) and (2.24), we find that

$$\Omega(u) = uq + q_1u^*, \tag{2.25}$$

for all $u \in \Re$, where $q = (2\alpha\beta)^{-1}(\alpha\Omega(\beta)^* - \beta\Omega(\alpha)^*) \in \Re\mathcal{C}$ and $q_1 = (2\alpha\beta)^{-1}(\beta\Omega(\alpha) - \alpha\Omega(\beta)) \in \Re\mathcal{C}$. Using this in (2.17), we have

$$u(vq_1^* + q^*v^*) + v(uq_1^* + q^*u^*) - (uq + q_1u^*)v^* - (vq + q_1v^*)u^* = 0 \tag{2.26}$$

for all $u, v \in \Re$.Alter v by αv in (2.26), we get

$$u(vq_1^* - q^*v^*) + v(uq_1^* + q^*u^*) + (uq + q_1u^*)v^* - (vq - q_1v^*)u^* = 0 \tag{2.27}$$

for all $u, v \in \Re$. Adding (2.26) and (2.27), we find that

$$uvq_1^* + v(uq_1^* + q^*u^*) - vqu^* = 0 \tag{2.28}$$

for all $u, v \in \Re$.Alter u by αu in (2.28), we get

$$uvq_1^* + v(uq_1^* - q^*u^*) + vqu^* = 0 \tag{2.29}$$

for all $u, v \in \Re$. Adding (2.28) and (2.29), we get $(uv + vu)q_1^* = 0$ for all $u, v \in \Re$. Consequently, $q_1 = 0$ and hence from (2.28), we see that $vq^*u = vqu$ for all $u, v \in \Re$. Thus $q^* = q$. Hence $\Omega(u) = qu$ for all $u \in \Re$, where $q^* = q \in \Re\mathcal{C}$.

The following example shows that Theorem 2.2 does not hold if $\dim_{\mathcal{C}}\Re\mathcal{C} = 4$ and '$*$' is of the first kind.

Example 2.2. *Let $\mathcal{M}_2(\mathbb{K})$ be the ring of all square matrices of order 2 over any field $\mathbb{K}$ with involution '$*$' as the usual transpose. Define the map $\Omega : \mathcal{M}_2(\mathbb{K}) \to \mathcal{M}_2(\mathbb{K})$ by $\Omega\left(\begin{bmatrix}\zeta_1 & \zeta_2\\ \zeta_3 & \zeta_4\end{bmatrix}\right) = \begin{bmatrix}\zeta_1 - \zeta_4 & \zeta_2 - \zeta_3\\ \zeta_2 - \zeta_3 & \zeta_1 - \zeta_4\end{bmatrix}$. Then it can be easily seen that $[\Omega(a), b]_\bullet = [a, \Omega(b)]_\bullet$ for all $a, b \in \Re$. However, Ω is not of the form as described in Theorem 2.2.*

3. Generalized bi-skew Jordan derivations in prime rings

The following result gives a characterization of additive bi-skew Jordan derivations in prime rings without assuming the existence of a nontrivial symmetric idempotent.

Theorem 3.1. *Let $\Re$ be a noncommutative prime ring with an involution '$*$' and let $\Omega : \Re \to \mathcal{Q}_{ms}(\Re)$ be an additive bi-skew Jordan derivation. Suppose that either $\dim_{\mathcal{C}}\Re\mathcal{C} > 4$ or $\Re$ is unital. Then Ω is a $*$-derivation unless $\dim_{\mathcal{C}}\Re\mathcal{C} = 4$ and $char(\Re) = 2$.*

Proof. Since $\Omega : \Re \to \mathcal{Q}_{ms}(\Re)$ is an additive bi-skew Jordan derivation, therefore $\Omega(u \bullet v) = \Omega(u) \bullet v + u \bullet \Omega(v)$ for all $u, v \in \Re$. Now $a \bullet b = b \bullet a$ for any $a, b \in \Re$. Hence if $\Re$ is 2-torsion free, then replacing v by u in the previous expression, we have $\Omega(uu^*) = \Omega(u) \bullet u$ for all $u \in \Re$. Also if $\Omega : \Re \to \mathcal{Q}_{ms}(\Re)$ is an additive map satisfying $\Omega(uu^*) = \Omega(u) \bullet u$ for all $u \in \Re$, then replacing u by $u + v$ in the last expression, we find that $\Omega(u \bullet v) = \Omega(u) \bullet v + u \bullet \Omega(v)$ for all $u, v \in \Re$. Thus if $\Re$ is a 2-torsion free prime ring with an involution '$*$', then an additive map $\Omega : \Re \to \mathcal{Q}_{ms}(\Re)$ is a bi-skew Jordan derivation if and only if $\Omega(uu^*) = \Omega(u) \bullet u$ for all $u \in \Re$. Therefore in view of Theorem 2.1 of [20] and the arguments given in Case I of its proof, we conclude that Ω is a $*$-derivation unless $\dim_{\mathcal{C}}\Re\mathcal{C} = 4$ and $\text{char}(\Re) = 2$.

Proposition 3.1. *Let $\Re$ be a noncommutative prime ring with an involution '$*$' and let $\Omega : \Re \to \mathcal{Q}_{ms}(\Re)$ be a left (right) bi-skew Jordan centralizer. Then there exists $\lambda^* = \lambda \in \mathcal{C}$ such that $\Omega(u) = \lambda u$ for all $u \in \Re$.*

Proof. We give the details of the proof only when Ω is a left bi-skew Jordan centralizer. The case when Ω is a right bi-skew Jordan centralizer can be proved by using similar arguments.
Suppose

$$\Omega(u \bullet v) = \Omega(u) \bullet v \tag{3.1}$$

for all $u, v \in \Re$. Alter v by $v + w$ in (3.1), we have

$$\Omega(u \bullet v + u \bullet w) = \Omega(u) \bullet v + \Omega(u) \bullet w \tag{3.2}$$

for all $u, v, w \in \Re$. That is,

$$\Omega(v \bullet u + v \bullet w) = \Omega(v) \bullet u + \Omega(v) \bullet w \tag{3.3}$$

for all $u, v, w \in \Re$. Also alter u by $u + w$ in (3.1), we have

$$\Omega(u \bullet v + w \bullet v) = \Omega(u + w) \bullet v \tag{3.4}$$

for all $u, v, w \in \Re$. Comparing (3.3) and (3.4), we find that

$$\Omega(u + w) \bullet v = \Omega(v) \bullet u + \Omega(v) \bullet w \tag{3.5}$$

for all $u, v, w \in \Re$. Putting $w = 0$ in (3.5), we get

$$\Omega(u) \bullet v = \Omega(v) \bullet u \tag{3.6}$$

for all $u, v \in \Re$. Therefore from (3.5), we find that

$$(\Omega(u + w) - \Omega(u) - \Omega(w)) \bullet v = 0 \tag{3.7}$$

for all $u, v, w \in \Re$. In view of Lemma 2.1, it follows that $\Omega(u + w) = \Omega(u) + \Omega(w)$ for all $u, v \in \Re$. Thus Ω is additive.

Now (3.1) can be rewritten as

$$\Omega(uv^* + vu^*) = \Omega(u)v^* + v\Omega(u)^* \tag{3.8}$$

for all $u, v \in \Re$. Consider the map $\Phi : \Re^2 \to \mathcal{Q}_{ms}(\Re)$ given by

$$\Phi(u, v) = \Omega(uv^*) + \Omega(vu^*).$$

In view of Lemma 2.3, it follows that Φ satisfies the following relation

$$\Phi(uw, v) = \Phi(u, vw^*)$$

for all $u, v, w \in \Re$. Using (3.8), we obtain

$$(\Omega(uw) - \Omega(u)w)v^* + v(\Omega(uw)^* - w^*\Omega(u)^*) = 0 \tag{3.9}$$

for all $u, v, w \in \Re$. Applying Lemma 2.1, we deduce that $\Omega(uw) = \Omega(u)w$ for all $u, w \in \Re$. By [23, Lemma 2.1] there exists $q \in \mathcal{Q}_{mr}(\Re)$ such that $\Omega(u) = qu$ for all $u \in \Re$. Now in view of [1, Proposition 2.1.7], it follows that there exists a nonzero dense right ideal $\mathcal{K}$ of $\Re$ such that $\mathcal{K}q \subseteq \Re$. Hence from (3.8), we have

$$qvu^* = v(qu)^* \text{ for all } u \in \mathcal{K} \text{ and } v \in \Re. \tag{3.10}$$

For each fixed u this is a GPI. Hence by [1, Theorem 6.4.4] $qvu^* = v(qu)^*$ for all $v \in \mathcal{Q}_{mr}(\Re)$. Therefore putting $v = 1$, we find that $qu^* = (qu)^*$ for all $u \in \mathcal{K}$. Thus from (3.10), we have $qvu^* = vqu^*$ for all $v \in \Re$ and $u \in \mathcal{K}$. Consequently, $q \in \mathcal{C}$. Now from (3.10) it can be easily seen that $q^* = q$. This completes the proof.

Now we are ready to provide a characterization of generalized bi-skew Jordan derivations in prime rings.

Theorem 3.2. *Let $\Re$ be a unital prime $*$-ring containing a nontrivial symmetric idempotent. Suppose that $\Phi : \Re \to \mathcal{Q}_{ms}(\Re)$ is a generalized bi-skew Jordan derivation with $\Omega : \Re \to \mathcal{Q}_{ms}(\Re)$ as associated bi-skew Jordan derivation. Then Ω is an additive $*$-derivation and there exists $\lambda^* = \lambda \in \mathcal{C}$ such that $\Phi(u) = \lambda u + \Omega(u)$ unless $dim_{\mathcal{C}}\Re\mathcal{C} = 4$ and $char(\Re) = 2$.*

Proof. By the given hypothesis

$$\Phi(u \bullet v) = \Phi(u) \bullet v + u \bullet \Omega(v)$$

and

$$\Omega(u \bullet v) = \Omega(u) \bullet v + u \bullet \Omega(v)$$

for all $u, v \in \Re$. Therefore,

$$\Psi(u \bullet v) = \Psi(u) \bullet v$$

for all $u, v \in \Re$, where $\Psi : \Re \to \mathcal{Q}_{ms}(\Re)$ is a map given by $\Psi(u) = (\Phi - \Omega)(u)$. In view of Proposition 3.1, it follows that there exists $\lambda^* = \lambda \in \mathcal{C}$ such that $\Psi(u) = \lambda u$ for all $u \in \Re$. Consequently, $\Phi(u) = \lambda u + \Omega(u)$ for all $u \in \Re$. Finally, by [20], Ω is an additive $*$-derivation. This completes the proof.

The following result gives a characterization of generalized additive bi-skew Jordan derivations in prime rings without assuming the existence of a nontrivial symmetric idempotent.

Theorem 3.3. *Let $\Re$ be a noncommutative prime ring with an involution '$*$' and let $\Phi : \Re \to \mathcal{Q}_{ms}(\Re)$ be a generalized bi-skew Jordan derivation with $\Omega : \Re \to \mathcal{Q}_{ms}(\Re)$ as associated bi-skew Jordan derivation such that both Φ and Ω are additive. Suppose that either $dim_{\mathcal{C}}\Re\mathcal{C} > 4$ or $\Re$ is unital. Then Ω is a $*$-derivation and there exists $\lambda^* = \lambda \in \mathcal{C}$ such that $\Phi(u) = \lambda u + \Omega(u)$ unless $dim_{\mathcal{C}}\Re\mathcal{C} = 4$ and $char(\Re) = 2$.*

Proof. By the given hypothesis

$$\Phi(u \bullet v) = \Phi(u) \bullet v + u \bullet \Omega(v)$$

and

$$\Omega(u \bullet v) = \Omega(u) \bullet v + u \bullet \Omega(v)$$

for all $u, v \in \Re$. Therefore,

$$\Psi(u \bullet v) = \Psi(u) \bullet v$$

for all $u, v \in \Re$, where $\Psi : \Re \to \mathcal{Q}_{ms}(\Re)$ is a map given by $\Psi(u) = (\Phi - \Omega)(u)$. In view of Proposition 3.1, it follows that there exists $\lambda^* = \lambda \in \mathcal{C}$ such that $\Psi(u) = \lambda u$ for all $u \in \Re$. Consequently, $\Phi(u) = \lambda u + \Omega(u)$ for all $u \in \Re$. Finally, by Theorem 3.1, Ω is an $*$-derivation. This completes the proof.

4. Applications to Some Operator Algebras

As applications of the outcomes presented in the preceding sections, we aim to delineate strong biskew commutativity-preserving maps, bi-skew commuting maps, and generalized bi-skew Jordan derivations within standard operator algebras and factor von Neumann algebras. Throughout this section, all vector spaces and algebras are defined over the field $\mathbb{C}$ of complex numbers.

Suppose $\mathbb{H}$ represents a Hilbert space, with $\mathcal{B}(\mathbb{H})$ denoting the algebra comprising all bounded linear operators on $\mathbb{H}$, and $\mathcal{F}(\mathbb{H})$ representing the ideal consisting of all finite rank operators within $\mathcal{B}(\mathbb{H})$. The map $T \mapsto T^*$, which takes an operator to its Hilbert adjoint operator, is an involution on $\mathcal{B}(\mathbb{H})$. Here $\mathbb{C}$, the field of complex numbers, is equipped with the conjugate involution and $\mathcal{B}(\mathbb{H})$ forms a $*$-algebra. Therefore $\mathcal{B}(\mathbb{H})$ is an algebra of characteristic zero and the map $T \mapsto T^*$, where T^* denotes the Hilbert adjoint operator of T, is an involution of the second kind on $\mathcal{Z}(\mathcal{B}(\mathbb{H}))$. A subset $\mathcal{M}$ of $\mathcal{B}(\mathbb{H})$ is said to be closed under adjoint operation if $u \in \mathcal{M}$ implies that $u^* \in \mathcal{M}$, that is, $\mathcal{M}^* \subseteq \mathcal{M}$.

Standard operator algebras:A subalgebra $\mathcal{S}$ of $\mathcal{B}(\mathbb{H})$ earns the label of a standard operator algebra if it includes the identity operator and encompasses $\mathcal{F}(\mathbb{H})$. It's evident that $\mathcal{B}(\mathbb{H})$ itself is a standard operator algebra. Furthermore, every standard operator algebra qualifies as a prime algebra. Moreover, for any standard operator algebra $\mathcal{S}$, its center $\mathcal{Z}(\mathcal{S})$ is $\mathbb{C}I$. A self-adjoint standard operator algebra $\mathcal{S}$ represents an algebra of characteristic zero, and a mapping $T \mapsto T^*$, where T^* denotes the Hilbert adjoint operator of T, serves as an involution of the second kind on $\mathcal{Z}(\mathcal{S})$. Leveraging the results derived in the preceding section, the following corollaries emerge.

Corollary 4.1. *Let $\mathcal{S}$ be a self-adjoint standard operator algebra on a Hilbert space $\mathbb{H}$. Suppose that $\chi : \mathcal{S} \to \mathcal{S}$ is a surjective map. Then χ is strong bi-skew commutativity preserving map if and only if there exists $\lambda \in \mathcal{Q}_{ms}(\mathcal{S})$ with $\lambda\lambda^* = 1$ such that $\chi(a) = \lambda a$ for all $a \in \mathcal{S}$.*

Corollary 4.2. *Let $\mathcal{S}$ be a self-adjoint standard operator algebra on a Hilbert space $\mathbb{H}$. Then $\chi : \mathcal{S} \to \mathcal{S}$ is a bi-skew commuting map if and only if there exists $\lambda \in \mathbb{R}$ such that $\chi(a) = \lambda a$ for all $a \in \mathcal{S}$.*

Corollary 4.3. *Let $\mathcal{S}$ be a self-adjoint standard operator algebra on a Hilbert space $\mathbb{H}$. Suppose that $\Phi : \mathcal{S} \to \mathcal{S}$ is a generalized bi-skew Jordan derivation with $\Omega : \Re \to \mathcal{S}$ as associated bi-skew Jordan derivation. Then $\Omega : \mathcal{S} \to \mathcal{S}$ is an additive $*$-derivation and there exists $\lambda \in \mathbb{R}$ such that $\Phi(u) = \lambda u + \Omega(u)$.*

Factor von Neumann algebras A von Neumann algebra $\mathcal{N}$ is a subalgebra of $\mathcal{B}(\mathbb{H})$ which satisfies the double commutant property, that is, $\mathcal{N}'' = \mathcal{N}$ where

$$\mathcal{N}' = \{T \in \mathcal{B}(\mathbb{H}) \mid TF = FT \text{ for all } F \in \mathcal{N}\} \text{ and } \mathcal{M}'' = (\mathcal{M}')'.$$

It is clear that a von Neumann algebra is unital. A von Neumann algebra $\mathcal{N}$ is an algebra of characteristic zero and a map $T \mapsto T^*$, where T^* denotes the Hilbert adjoint operator of T, is an involution of the second kind on $\mathcal{Z}(\mathcal{N})$. A von Neumann algebra $\mathcal{N}$ is called a factor von Neumann algebra if its center is trivial, that is, $\mathcal{Z}(\mathcal{N}) = \mathbb{C}I$. Every factor von Neumann algebra is a prime algebra.

Corollary 4.4. *Let $\mathcal{N}$ be a factor von Neumann algebra. Suppose that $\chi : \mathcal{N} \to \mathcal{N}$ is a surjective map. Then χ is strong bi-skew commutativity preserving map if and only if there exists $\lambda \in \mathcal{Q}_{ms}(\mathcal{N})$ with $\lambda\lambda^* = 1$ such that $\chi(a) = \lambda a$ for all $a \in \mathcal{N}$.*

Corollary 4.5. *Let $\mathcal{N}$ be a factor von Neumann algebra. Then $\chi : \mathcal{N} \to \mathcal{N}$ is a bi-skew commuting map if and only if there exists $\lambda \in \mathbb{R}$ such that $\chi(u) = \lambda u$ for all $u \in \mathcal{N}$.*

Corollary 4.6. *Let $\mathcal{N}$ be a factor von Neumann algebra. Suppose that $\Phi : \mathcal{N} \to \mathcal{N}$ is a generalized bi-skew Jordan derivation with $\Omega : \mathcal{N} \to \mathcal{N}$ as associated bi-skew Jordan derivation. Then $\Omega : \mathcal{N} \to \mathcal{N}$ is an additive $*$-derivation and there exists $\lambda \in \mathbb{R}$ such that $\Phi(u) = \lambda u + \Omega(u)$ for all $u \in \mathcal{N}$.*

Conflict of Interest: The authors declare that they have no Conflict of interest

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