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On Leonard Pairs And q-Tetrahedron Algebra $\boxtimes_q$

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Abstract. Let $\mathcal{F}$ denote an algebraically closed field of characteristic zero, fix a nonzero scalar $q \in \mathcal{F}$ that is not a root of unity. Consider the q-tetrahedron algebra $\boxtimes_q$ over $\mathcal{F}$ with standard generators $\{X_{ij} : i,j \in Z_4, j-i = 1 \text{ or } j-i = 2\}$. Let V denote finite dimensional evaluation module for $\boxtimes_q$. In this article for each $r \in Z_4$ and $X_{r+2,r} \in \boxtimes_q$ we find $A \in \boxtimes_q$ such that the pairs $A, X_{r+2,r}$, $A, X_{r+2,r+3}$, and $A, X_{r+3,r}$ act on V as Leonard pairs. Indeed we will show that A is a linear combination of $X_{r,r+1}$ and $X_{r+1,r+2}$.

1. Introduction

Leonard pairs were introduced by P. Terwilliger [6] to study the sequences of orthogonal polynomials with discrete support for which there is a dual sequence of orthogonal polynomials. Because these polynomials frequently arise in connection with the finite-dimensional representations of nice algebras and quantum groups, it is natural to find Leonard pairs associated with these algebraic objects. In [1] and [3], the author constructed a family of Leonard pairs from the equitable basis of sl_2 and the equitable generators of $U_q(sl_2)$. In this article we will use the standard generators of the q-tetrahedron algebra $\boxtimes_q$ to construct a family of Leonard pairs.

The the q-tetrahedron algebra $\boxtimes_q$ is associative, non-commutative algebra, this algebra was introduced by P. Terwilliger and T. Ito [5]. The $\boxtimes_q$ has eight generators $\{X_{ij} : i,j \in Z_4, j-i = 1 \text{ or } j-i = 2\}$.

We can view the algebra $\boxtimes_q$ as follows: the elements of Z_4 represent the vertices of the tetrahedron and for each distinct $i,j \in Z_4$, the standard generator X_{ij} of $\boxtimes_q$ represents the edge of the tetrahedron oriented from i to j. So, the generators X_{20}, X_{02} represent the same edge but opposite direction, similarly the generators X_{31}, X_{13}, the other generators $X_{01}, X_{12}, X_{23}, X_{30}$ represent the other edges but oriented in one direction.

Throughout this paper $\mathcal{F}$ denotes an algebraically closed field with characteristic zero, d is a nonnegative integer, and $q \in \mathcal{F}$ is a nonzero scalar which is not a root of unity. Also, let $\mathrm{Mat}_{d+1}(\mathcal{F})$ represents the $\mathcal{F}$-algebra of $(d+1) \times (d+1)$ matrices.

The article is organized as follows. In section 2 we recall the definitions of Leonard pairs and parameters array, and state some facts related to Leonard pairs. In section 3 we

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recall the definition of the q-tetrahedron algebra $\boxtimes_q$ and we give explanation for family of bases described by authors in [5] for finite dimensional evaluation module of $\boxtimes_q$. In sections 4, 5, and 6 we prove the result of this article, we will show that if V is finite dimensional evaluation module for $\boxtimes_q$, then for each $r \in Z_4$ and $X_{r+2,r} \in \boxtimes_q$ we can find $A \in \boxtimes_q$ such that the pairs $A, X_{r+2,r}$, $A, X_{r+2,r+3}$, and $A, X_{r+3,r}$ act on V as Leonard pairs. Also we will show that A is a linear combination of $X_{r,r+1}$ and $X_{r+1,r+2}$.

2. Leonard Pairs

In this section we recall the definitions of Leonard pairs and parameter arrays and some facts concerning them that we will use later in this article, before we state these definitions we review some concepts.

By tridiagonal matrix we mean a square matrix in which nonzero entry presents only on, immediately below or immediately above the main diagonal. A tridiagonal matrix is called irreducible if all entries present immediately below or immediately above the main diagonal are nonzeros. A square matrix is called upper bidiagonal if nonzero entry presents on or immediately above the main diagonal, and is called lower bidiagonal if nonzero entry presents on or immediately below the main diagonal.

Definition 1. *[6] Let V denote a vector space over $\mathcal{F}$ with finite positive dimension. By a* Leonard pair *on V, we mean an ordered pair A, A^*, where $A : V \to V$ and $A^* : V \to V$ are linear transformations that satisfy both (i) and (ii) below.*

(i) There exists a basis for V with respect to which the matrix representing A^* is diagonal and the matrix representing A is irreducible tridiagonal.

(ii) There exists a basis for V with respect to which the matrix representing A is diagonal and the matrix representing A^* is irreducible tridiagonal.

For more details about Leonard pairs see [2, 8–11].

In [7], Terwilliger showed that for each Leonard pair there exists corresponding sequence of scalars called parameter array, the scalars that appear in this parameter array depend on the eigenvalues of the Leonard pair.

We now recall the definition of the parameter array.

Definition 2. *[7] Let d denote a non negative integer. By a* parameter array *over $\mathcal{F}$ of diameter d, we mean a sequence of scalars $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d; \{\varphi_j\}_{j=1}^d, \{\phi_j\}_{j=1}^d)$ taken from $\mathcal{F}$ that satisfy the following conditions.*

$$\theta_i \neq \theta_j \qquad (0 \le i < j \le d), \tag{1}$$
$$\theta_i^* \neq \theta_j^* \qquad (0 \le i < j \le d), \tag{2}$$
$$\varphi_i \neq 0 \qquad (1 \le i \le d), \tag{3}$$
$$\phi_i \neq 0 \qquad (1 \le i \le d), \tag{4}$$
$$\varphi_i = \phi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{i-1} - \theta_d) \qquad (1 \le i \le d), \tag{5}$$

$$\phi_i \quad = \quad \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta^*_i - \theta^*_0)(\theta_{d-i+1} - \theta_0) \quad (1 \leq i \leq d), \tag{6}$$

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i} = \frac{\theta^*_{j-2} - \theta^*_{j+1}}{\theta^*_{j-1} - \theta^*_j} \qquad (2 \leq i, j \leq d-1). \tag{7}$$

The common value of (7) minus one is called the fundamental parameter of the Leonard pair, the fundamental parameter of the Leonard pairs appear through this article is equal $q^2 + q^{-2}$.

In Definition 1, the Leonard pair is described as diagonal and irreducible tridiagonal matrices. In [12], Terwilliger showed that the Leonard pair can also be described as upper bidiagonal and lower bidiagonal matrices using the parameter array associated with the Leonard pair as it appears in the following theorem.

Theorem 1. *[12] Let d denote a nonnegative integer, let B and B^* denote matrices in $\mathrm{Mat}_{d+1}(\mathcal{F})$. Assume B is lower bidiagonal and B^* is upper bidiagonal. Then the following are equivalent.*

(i) The pair B, B^* is a Leonard pair in $\mathrm{Mat}_{d+1}(\mathcal{F})$.

(ii) There exists a parameter array $(\{\theta_i\}_{i=0}^d, \{\theta^*_i\}_{i=0}^d; \{\varphi_j\}_{j=1}^d, \{\phi_j\}_{j=1}^d)$ over $\mathcal{F}$ such that

$$B(i,i) = \theta_i, \qquad B^*(i,i) = \theta^*_i \qquad (0 \leq i \leq d),$$
$$B(j, j-1)B^*(j-1, j) = \varphi_j \qquad (1 \leq j \leq d).$$

Suppose (i), (ii) *hold. Then the parameter array in* (ii) *is uniquely determined by B, B^*.*

Indeed to prove the result of this article we will describe the Leonard pairs that appear in sections $4, 5$ and 6 as upper bidiagonal and lower bidiagonal matrices and use Theorem 1 to obtain our result.

3. The q-tetrahedron algebra $\boxtimes_q$

In this section we recall the definitions of $\boxtimes_q$ algebra and its evaluation module, also we state some facts about this algebra that we will use later in this article. The material in this section can be found in [4] and [5].

Definition 3. *[5] Let $\boxtimes_q$ denote the unital associative $\mathcal{F}$-algebra that has generators*

$$\{X_{ij} : i, j \in Z_4, j - i = 1 \text{ or } j - i = 2\}.$$

and the following relations:

(i) For $i, j \in Z_4$ such that $j - i = 2$,

$$X_{ij} X_{ji} = 1.$$

(ii) For $h, i, j \in Z_4$ such that the the pair $(i-h, j-i)$ is one of $(1,1)$, $(1,2)$, $(2,1)$,

$$\frac{qX_{hi}X_{ij} - q^{-1}X_{ij}X_{hi}}{q - q^{-1}} = 1.$$

(iii) For $h, i, j, k \in Z_4$ such that $i-h = j-i = k-j = 1$,

$$X_{hi}^3 X_{jk} - [3]_q X_{hi}^2 X_{jk} X_{hi} + [3]_q X_{hi} X_{jk} X_{hi}^2 - X_{jk} X_{hi}^3 = 0.$$

*We call $\boxtimes_q$ the q-*tetrahedron *algebra.*

To prove the result of this article in sections 4, 5 and 6, we will describe the action of the generators of $\boxtimes_q$ on different bases of an evaluation module of $\boxtimes_q$. So, we now recall the definition of an evaluation module of $\boxtimes_q$.

The authors in [5] gave definition for the evaluation module of $\boxtimes_q$, and the authors in [4] described 24 bases for it. we can summarize their work as follows:

Let V be a vector space over $\mathcal{F}$ with finite positive dimension. Let $\{s_i\}_{i=0}^d$ denote a sequence of positive integers whose sum is equal the dimension of vector space V. A decomposition of V of shape $\{s_i\}_{i=0}^d$ is a sequence of subspaces $\{W_i\}_{i=0}^d$ of vector space V such that the dimension of W_i is s_i $(0 \leq i \leq d)$, and $V = \sum_{i=0}^d W_i$ (direct sum). We call d the diameter of V.

Definition 4. *[5] An evaluation module for q-tetrahedron algebra $\boxtimes_q$ is a finite dimensional, nontrivial irreducible $\boxtimes_q$-module with shape* $(1, 1, ..., 1)$.

A flag on vector space V of shape $\{s_i\}_{i=0}^d$ is a sequence of subspaces $\{W_i\}_{i=0}^d$ of V such that $W_{i-1} \subseteq W_i$, and the dimension of W_i is equal $s_0 + s_1 + ... + s_i$ for $0 \leq i \leq d$.

Let V denote finite dimensional irreducible module for $\boxtimes_q$ with diameter d. For distinct i, j in Z_4 such that $j - i = 1$ or $j - i = 2$ we define a decomposition of the vector space V called $[i, j]$. The decomposition $[i, j]$ has diameter d, and the nth component of $[i, j]$ is the eigenspace of X_{ij} with eigenvalue q^{d-2n} for $0 \leq n \leq d$.

There exists a collection of flags on vector spaces V, denoted $[i]$, $i \in Z_4$, such that for distinct $i, j \in Z_4$ the decomposition $[i, j]$ induces the flag $[i]$. By construction, the shape of the flag $[i]$ coincides with the shape of V.

Definition 5. *[4] Let V denote an evaluation module for $\boxtimes_q$ that has diameter d. Pick mutually distinct $i, j, k, l \in Z_4$. A basis $\{v_n\}_{n=0}^d$ for V is called an $[i, j, k, l]$-basis whenever:*

(i) for $0 \leq n \leq d$ the vector v_n is contained in the component n of the decomposition $[k, l]$ of V;

(ii) $\sum_{n=0}^d v_n$ is contained in component 0 of the flag $[j]$ on V.

Lemma 1. *[4] Let V denote an evaluation module for $\boxtimes_q$, and pick mutually distinct $i, j, k, l \in Z_4$. Then there exists an $[i, j, k, l]$-basis for V.*

Let V denote an evaluation module for $\boxtimes_q$. In Definition 5 and Lemma 1 we can recognize 24 bases for V. The action of standard generators of $\boxtimes_q$ on these 24 bases is described in Theorem 11.1 in [4], the authors used special matrices Z, K_q, E_q, and $G_q(t)$ to describe the action of the standard generators of $\boxtimes_q$ on these 24 bases, these matrices are described in the following definition.

Definition 6. *Let Z, K_q, E_q, and $G_q(t)$ denote the matrices in $Mat_{d+1}(\mathcal{F})$ such that $Z(i,j) = \delta_{i+j,d}$ for $0 \leq i,j \leq d$, the matrix K_q is diagonal with $K_q(i,i) = q^{d-2i}$ for $0 \leq i \leq d$, the matrix E_q is upper bidiagonal with $E_q(i,i) = q^{2i-d}$ for $0 \leq i \leq d$, and $E_q(i-1,i) = q^d - q^{2i-d-2}$ for $1 \leq i \leq d$, and the matrix $G_q(t)$ is upper bidiagonal with $G_q(t)(i,i) = q^{2i-d}$ for $0 \leq i \leq d$, and $G_q(t)(i-1,i) = (q^d - q^{2i-d-2})(1 - tq^{d-2i+1})$ for $1 \leq i \leq d$.*

We remark here that we will use the notations in Definition 6 in our work in the next sections.

4. The Leonard pair A, X_{20}

The q-tetrahedron algebra has eight generators $\{X_{20}, X_{02}, X_{13}, X_{31}, X_{01}, X_{30}, X_{12}, X_{23}\}$ with relations as in definition 3. Let $S = \{X_{20}, X_{02}, X_{13}, X_{31}\}$, and let $\{A_1, A_2, A_3, A_4\} = \{X_{01}, X_{30}, X_{12}, X_{23}\}$. In this paper we show that for each $B \in S$, we can find $A = aA_1 + bA_2$ such that the pairs A, B, A, A_3, and A, A_4 are Leonard pairs.

Before we start proving our result, we recall the following lemmas which will help us in our work.

Lemma 2. *[4] Pick an integer $d \geqslant 1$ and a nonzero $t \in \mathcal{F}$ that is not among $\{q^{d-2n+1}\}_{n=1}^{d}$. Then there exists an evaluation module $V_d(t)$ for $\boxtimes_q$ such that $V_d(t)$ has a basis $\mathbf{s} = [3,2,0,1]$ for which the matrices represent X_{20}, X_{01}, X_{12}, and X_{30} are E_q, K_q, $ZE_{q^{-1}}Z$, and $G_q(t)$ respectively.*

The entries of the matrix $E_{q^{-1}}$ is given in Definition 6, the entries of $ZE_{q^{-1}}Z$ can be found using the following lemma.

Lemma 3. *[4] For $C \in \mathrm{Mat}_{d+1}(\mathcal{F})$ and $0 \leqslant i,j \leqslant d$ the following coincide*

(i) the entry (i,j) of ZCZ,

(ii) the entry $(d-i, d-j)$ of C.

Lemma 4. *[4] Let V denote an evaluation module for $\boxtimes_q$ that has diameter d. Then there exists a unique $t \in \mathcal{F}$ such that:*

(i) t is a nonzero and not among $\{q^{d-2n+1}\}_{n=1}^{d}$.

(ii) the $\boxtimes_q$-module V is isomorphic to $V_d(t)$.

For the rest of the paper we use V to represents $V_d(t)$, and t will be a nonzero and not among $\{q^{d-2n+1}\}_{n=1}^{d}$ scalar in $\mathcal{F}$.

We now start proving the our result.

Definition 7. *Let $A \in \boxtimes_q$ denote a linear combination of X_{01}, X_{12}. Write*

$$A \;=\; aX_{01} + bX_{12}.$$

For the rest of the article, by the notation $[T]_s$ we mean the matrix that represents the linear map $T : V \to V$ with respect to the basis s of V.

Lemma 5. *With reference to Lemma 2 and Definition 7, let $B_1 = X_{20}$. Then the matrices represent A and B_1 with respect to the basis $\mathbf{s}$ are lower bidiagonal and upper bidiagonal respectively with entries:*

$$[A]_{\mathbf{s}}(i,i) = aq^{d-2i} + bq^{2i-d} \qquad (0 \le i \le d),$$

$$\begin{aligned} [B_1]_{\mathbf{s}}(i,i) &= q^{2i-d} \qquad (0 \le i \le d),\\ [A]_{\mathbf{s}}(i,i-1) &= bq^{-d}(1-q^{2i}) \qquad (1 \le i \le d),\\ [B_1]_{\mathbf{s}}(i-1,i) &= q^{d}(1-q^{2i-2d-2}) \qquad (1 \le i \le d). \end{aligned}$$

Proof. The matrices that represent the action of X_{01}, X_{12} and X_{20} are given in Lemma 2, and the entries of these matrices are given in Definition 6.

Definition 8. *With reference to Lemma 5, define*

$$\begin{aligned} \alpha_i &= aq^{d-2i} + bq^{2i-d} \qquad (0 \le i \le d),\\ \alpha_i^* &= q^{2i-d} \qquad (0 \le i \le d),\\ \varphi_i &= b(q^{2i}-1)(q^{2i-2d-2}-1) \qquad (1 \le i \le d),\\ \phi_i &= a(q^{2i}-1)(q^{2i-2d-2}-1) \qquad (1 \le i \le d) \end{aligned}$$

Note that $\alpha_i = [A]_{\mathbf{s}}(i,i)$, $\alpha_i^* = [B_1]_{\mathbf{s}}(i,i)$ for $(0 \le i \le d)$, and $\varphi_i = [A]_{\mathbf{s}}(i,i-1)[B_1]_{\mathbf{s}}(i-1,i)$ for $(1 \le i \le d)$.

Now, by Theorem 1, if we find the conditions on the sequence of scalars $(\{\alpha_i\}_{i=0}^{d}, \{\alpha_i^*\}_{i=0}^{d}; \{\varphi_j\}_{j=1}^{d}, \{\phi_j\}_{j=1}^{d})$ in which the sequence is a parameter array, then these conditions imply that the pair A, B_1 is a Leonard pair.

So, in the next work we will find when the sequence $(\{\alpha_i\}_{i=0}^{d}, \{\alpha_i^*\}_{i=0}^{d}; \{\varphi_j\}_{j=1}^{d}, \{\phi_j\}_{j=1}^{d})$ satisfies the conditions $1-7$ in Definition 2.

Lemma 6. *With reference to Definition 8, $\alpha_k \ne \alpha_i$ for $k \ne i$ $(0 \le i,k \le d)$ if and only if $a - bq^{2(h-d)} \ne 0$ for $0 < h < 2d$.*

Proof.
$$\begin{aligned} \alpha_k - \alpha_i &= a(q^{d-2k} - q^{d-2i}) + b(q^{2k-d} - q^{2i-d})\\ &= aq^{d-2k}(1-q^{2(k-i)}) + bq^{2i-d}(q^{2(k-i)}-1)\\ &= (1-q^{2(k-i)})(aq^{d-2k} - bq^{2i-d})\\ &= q^{d-2k}(1-q^{2(k-i)})(a - bq^{2(i+k-d)}) \end{aligned}$$

$$= q^{d-2k}(1-q^{2(k-i)})(a-bq^{2(h-d)}),$$

where $h = k+i$.

Note that $0 < h < 2d$, its clear that $\alpha_k = \alpha_i$ if and only if $q^{d-2k}(1-q^{2(k-i)})(a-bq^{2(h-d)}) = 0$,

but $1-q^{2(k-i)} \neq 0$ because q is not a root of unity, hence, $\alpha_k \neq \alpha_i$ if and only if $a - bq^{2(h-d)} \neq 0$ for $0 < h < 2d$.

Lemma 7. *With reference to Definition 8, $\alpha^*_k \neq \alpha^*_i$ for $k \neq i$ $(0 \leq i, k \leq d)$.*

Proof. $\alpha^*_k - \alpha^*_i = 0$ if and only if $q^{2k-d} - q^{2i-d} = 0$ if and only if $q^{2k-d}(1-q^{2(i-k)}) = 0$, but q is not a root of unity. Hence the result hold.

Lemma 8. *With reference to Definition 8, $\varphi_i \neq 0$ if and only if $b \neq 0$, and $\phi_i \neq 0$ if and only if $a \neq 0$ for $1 \leq i \leq d$.*

Proof. Clear, since q is not a root of unity.

Lemma 9. *With reference to Definition 8,*

$$\varphi_i \quad = \quad \phi_1 \sum_{k=0}^{i-1} \frac{\alpha_k - \alpha_{d-k}}{\alpha_0 - \alpha_d} + (\alpha^*_i - \alpha^*_0)(\alpha_{i-1} - \alpha_d) \qquad (1 \leq i \leq d).$$

Proof. Note that

$$\alpha_k - \alpha_{d-k} = (aq^{d-2k} + bq^{2k-d}) - (aq^{2k-d} + bq^{d-2k}) = (a-b)(q^{d-2k} - q^{2k-d}),$$

and

$$\alpha_0 - \alpha_d = (aq^d + bq^{-d}) - (aq^{-d} + bq^d) = (a-b)(q^d - q^{-d}).$$

So,

$$\sum_{k=0}^{i-1} \frac{\alpha_k - \alpha_{d-k}}{\alpha_0 - \alpha_d} = \sum_{k=0}^{i-1} \frac{q^{d-2k} - q^{2k-d}}{q^d - q^{-d}} = \frac{(q^{2(d-i+1)} - 1)(q^{2i} - 1)}{(q^{2d} - 1)(q^2 - 1)}.$$

And,

$$\alpha^*_i - \alpha^*_0 = q^{2i-d} - q^{-d} = q^{-d}(q^{2i} - 1),$$

$$\alpha_{i-1} - \alpha_d = (aq^{d-2i+2} + bq^{2i-d-2}) - (aq^{-d} + bq^d) = aq^{-d}(q^{2(d-i+1)} - 1) + bq^d(q^{2(i-d-1)} - 1),$$

$$\phi_1 = aq^{-2d}(q^2-1)(1-q^{2d}).$$

Now, simplify to get the result.

Lemma 10. *With reference to Definition 8,*

$$\phi_i \quad = \quad \varphi_1 \sum_{k=0}^{i-1} \frac{\alpha_k - \alpha_{d-k}}{\alpha_0 - \alpha_d} + (\alpha^*_i - \alpha^*_0)(\alpha_{d-i+1} - \alpha_0) \qquad (1 \leq i \leq d).$$

Proof. Similar to proof of Lemma 9.

Lemma 11. *With reference to Definition 8,*

$$\frac{\alpha_{h-2}-\alpha_{h+1}}{\alpha_{h-1}-\alpha_h}=\frac{\alpha^*_{k-2}-\alpha^*_{k+1}}{\alpha^*_{k-1}-\alpha^*_k}=q^2+q^{-2}+1 \qquad (2\leq h,k\leq d-1).$$

Proof.

$$\alpha^*_{k-2}-\alpha^*_{k+1}=q^{2(k-2)-d}-q^{2(k+1)-d}=q^{2k-d-4}(1-q^6),$$

and

$$\alpha^*_{k-1}-\alpha^*_k=q^{2(k-1)-d}-q^{2(k)-d}=q^{2k-d-2}(1-q^2).$$

Hence,

$$\frac{\alpha^*_{k-2}-\alpha^*_{k+1}}{\alpha^*_{k-1}-\alpha^*_k}=q^{-2}\frac{1-q^6}{1-q^2}=q^2+q^{-2}+1 \qquad (2\leq k\leq d-1).$$

Similar proof for α.

Lemma 12. *With reference to Definition 8, let a and b be scalars in $\mathcal{F}$. Then the sequence of scalars $(\{\alpha_i\}_{i=0}^d, \{\alpha^*_i\}_{i=0}^d; \{\varphi_j\}_{j=1}^d, \{\phi_j\}_{j=1}^d)$ is a parameter array if and only if $a\neq 0$, $b\neq 0$ and $a-bq^{2(i-d)}\neq 0$ for $1\leq i\leq 2d-1$.*

Proof. Note that the conditions $1-7$ of the parameter array in Definition 2 hold for the sequence $(\{\alpha_i\}_{i=0}^d, \{\alpha^*_i\}_{i=0}^d; \{\varphi_j\}_{j=1}^d, \{\phi_j\}_{j=1}^d)$ from Lemmas 6, 7, 8, 9, 10, 11 respectively if and only if $a\neq 0$, $b\neq 0$ and $a-bq^{2(i-d)}\neq 0$ for $1\leq i\leq 2d-1$.

Theorem 2. *Assume $d\geq 2$, let V denote an evaluation module for $\boxtimes_q$ with dimension $d+1$. Let $A\in\boxtimes_q$ denote an arbitrary linear combination of X_{01} and X_{12}, let $B_1\in\boxtimes_q$ such that $B_1=X_{20}$, let a and b be scalars in $\mathcal{F}$. Write $A=aX_{01}+bX_{12}$. Then the pair A, B_1 acts on V as a Leonard pair if and only if $a\neq 0$, $b\neq 0$ and $a-bq^{2(i-d)}\neq 0$ for $1\leq i\leq 2d-1$.*

Proof. The action of the pair A, B_1 on the basis $\mathbf{s}$ is described in Lemma 5, the matrices represent A and B_1 with respect to the basis $\mathbf{s}$ are lower bidiagonal and upper bidiagonal respectively in which $\alpha_i=[A]_{\mathbf{s}}(i,i)$, $\alpha^*_i=[B_1]_{\mathbf{s}}(i,i)$, and $\varphi_i=[A]_{\mathbf{s}}(i,i-1)[B_1]_{\mathbf{s}}(i-1,i)$. In Lemma 12 we show that the sequence of scalars $(\{\alpha_i\}_{i=0}^d, \{\alpha^*_i\}_{i=0}^d; \{\varphi_j\}_{j=1}^d, \{\phi_j\}_{j=1}^d)$ is a parameter array if and only if $a\neq 0$, $b\neq 0$ and $a-bq^{2(i-d)}\neq 0$ for $1\leq i\leq 2d-1$. Hence, the result hold by Theorem 1.

5. The Leonard Pair A, X_{30}

Lemma 13. *With reference to Lemma 2 and Definition 7, let $B_2 = X_{30}$. Then the matrices represent A and B_2 with respect to the basis $\mathbf{s}$ are lower bidiagonal and upper bidiagonal respectively with entries:*

$$[A]_{\mathbf{s}}(i,i) = aq^{d-2i} + bq^{2i-d} \qquad (0 \le i \le d),$$

$$[B_2]_{\mathbf{s}}(i,i) = q^{2i-d} \qquad (0 \le i \le d),$$
$$[A]_{\mathbf{s}}(i,i-1) = bq^{-d}(1-q^{2i}) \qquad (1 \le i \le d),$$
$$[B_2]_{\mathbf{s}}(i-1,i) = q^d(1-q^{2i-2d-2})(1-tq^{d-2i+1}) \qquad (1 \le i \le d).$$

Proof. The matrices that represent the action of X_{01}, X_{12} and X_{30} are given in Lemma 2, and the entries of these matrices are given in Definition 6.

Definition 9. *With reference to Lemma 13, define*

$$\begin{aligned} \Psi_i &= b(1-q^{2i-d})(1-q^{2i-2d-2})(1-tq^{d-2i+1}) \qquad (1 \le i \le d), \\ \Lambda_i &= q^{-d-1}(q^{2d-2i+2}-1)(q^{2i}-1)(bt-aq^{2i-d-1}) \qquad (1 \le i \le d) \end{aligned}$$

Note that $\alpha_i = [A]_{\mathbf{s}}(i,i)$, $\alpha_i^* = [B_2]_{\mathbf{s}}(i,i)$ for $(0 \le i \le d)$, where α_i and α_i^* appear in Definition 8. And $\Psi_i = [A]_{\mathbf{s}}(i,i-1)[B_2]_{\mathbf{s}}(i-1,i)$ for $(1 \le i \le d)$.

Now, by Theorem 1, if we find the conditions on the sequence of scalars $(\{\alpha_i\}_{i=0}^d, \{\alpha_i^*\}_{i=0}^d; \{\Psi_j\}_{j=1}^d, \{\Lambda_j\}_{j=1}^d)$ in which the sequence is a parameter array, then these conditions imply that the pair A, B_2 is a Leonard pair.

So, we now need to find when the sequence $(\{\alpha_i\}_{i=0}^d, \{\alpha_i^*\}_{i=0}^d; \{\Psi_j\}_{j=1}^d, \{\Lambda_j\}_{j=1}^d)$ satisfies the seven conditions of the parameter array in Definition 2. From Lemmas 6, 7, and 11 we know when the conditions 1, 2, and 7 hold. in the next work we will find when the conditions $3-6$ of Definition 2 hold.

Lemma 14. *With reference to Definitions 9, $\Psi_i \neq 0$ if and only if $b \neq 0$ and $t \neq q^{2i-d-1}$ for $1 \le i \le d$. And $\Lambda_i \neq 0$ if and only if $bt \neq aq^{2i-d-1}$ for $1 \le i \le d$.*

Proof. Since q is not a root of unity, this implies that $\Psi_i = 0$ if and only if $b = 0$ or $1 - tq^{d-2i+1} = 0$, solve for t to get the result for Ψ_i. Similar work for Λ_i.

Lemma 15. *With reference to Definitions 8 and 9*

$$\Psi_i = \Lambda_1 \sum_{k=0}^{i-1} \frac{\alpha_k - \alpha_{d-k}}{\alpha_0 - \alpha_d} + (\alpha_i^* - \alpha_0^*)(\alpha_{i-1} - \alpha_d) \qquad (1 \le i \le d).$$

Proof. Similar to proof of Lemma 9.

Lemma 16. *With reference to Definitions 8 and 9,*

$$\Lambda_i = \Psi_1 \sum_{h=0}^{i-1} \frac{\alpha_h - \alpha_{d-h}}{\alpha_0 - \alpha_d} + (\alpha_i^* - \alpha_0^*)(\alpha_{d-i+1} - \alpha_0) \quad (1 \le i \le d).$$

Proof. Similar to proof of Lemma 9.

Lemma 17. *With reference to Definitions 8 and 9, let a, b and t be scalars in $\mathcal{F}$. Then the sequence of scalars $(\{\alpha_i\}_{i=0}^d, \{\alpha_i^*\}_{i=0}^d; \{\Psi_j\}_{j=1}^d, \{\Lambda_j\}_{j=1}^d)$ is a parameter array if and only if $b \neq 0$, $t \neq q^{2i-d-1}$, $bt \neq aq^{2i-d-1}$ for $1 \leq i \leq d$, and $a - bq^{2(i-d)} \neq 0$ for $1 \leq i \leq 2d-1$.*

Proof. Note that the conditions $1-7$ of the parameter array in Definition 2 hold for the sequence $(\{\alpha_i\}_{i=0}^d, \{\alpha_i^*\}_{i=0}^d; \{\Psi_j\}_{j=1}^d, \{\Lambda_j\}_{j=1}^d)$ from Lemmas 6, 7, 14, 15, 16, 11 respectively if and only if $b \neq 0$, $t \neq q^{2i-d-1}$, $bt \neq aq^{2i-d-1}$ for $1 \leq i \leq d$, and $a - bq^{2(i-d)} \neq 0$ for $1 \leq i \leq 2d-1$.

Theorem 3. *Assume $d \geq 2$, let V denote an evaluation module for $\boxtimes_q$ with dimension $d+1$. Let $A \in \boxtimes_q$ denote an arbitrary linear combination of X_{01} and X_{12}, let $B_2 \in \boxtimes_q$ such that $B_2 = X_{30}$, let a, b and $t \neq 0$ be scalars in $\mathcal{F}$. Write $A = aX_{01} + bX_{12}$. Then the pair A, B_2 acts on V as a Leonard pair if and only if $b \neq 0$, $t \neq q^{2i-d-1}$, $bt \neq aq^{2i-d-1}$ for $1 \leq i \leq d$, and $a - bq^{2(i-d)} \neq 0$ for $1 \leq i \leq 2d-1$.*

Proof. The action of the pair A, B_2 on the basis $\mathbf{s}$ is described in Lemma 13, the matrices represent A and B_2 with respect to the basis $\mathbf{s}$ are lower bidiagonal and upper bidiagonal respectively in which $\alpha_i = [A]_{\mathbf{s}}(i,i)$, $\alpha_i^* = [B_2]_{\mathbf{s}}(i,i)$, and $\Psi_i = [A]_{\mathbf{s}}(i,i-1)[B_2]_{\mathbf{s}}(i-1,i)$. In Lemma 17 we show that the sequence of scalars $(\{\alpha_i\}_{i=0}^d, \{\alpha_i^*\}_{i=0}^d; \{\Psi_j\}_{j=1}^d, \{\Lambda_j\}_{j=1}^d)$ is a parameter array if and only if $b \neq 0$, $t \neq q^{2i-d-1}$, $bt \neq aq^{2i-d-1}$ for $1 \leq i \leq d$, and $a - bq^{2(i-d)} \neq 0$ for $1 \leq i \leq 2d-1$. Hence, the result hold by Theorem 1.

6. The Leonard Pair A, X_{23}

Lemma 18. *[4] With reference to Definition 3 and Lemma 1, Let V denote an evaluation module for $\boxtimes_q$. Then for the basis $\mathbf{v} = [3,0,2,1]$ of V the matrices represent X_{23}, X_{12}, and X_{01} are $G_{q^{-1}}(t)$, $K_{q^{-1}}$, and ZE_qZ respectively.*

Lemma 19. *With reference to Lemma 18 and Definition 7, let $B_3 = X_{23}$. Then the matrices represent A and B_3 with respect to the basis $\mathbf{v}$ are lower bidiagonal and upper bidiagonal respectively with entries:*

$$[A]_{\mathbf{v}}(i,i) = aq^{d-2i} + bq^{2i-d} \qquad (0 \leq i \leq d),$$

$$[B_3]_{\mathbf{v}}(i,i) = q^{d-2i} \qquad (0 \leq i \leq d),$$
$$[A]_{\mathbf{v}}(i,i-1) = aq^d(1-q^{-2i}) \qquad (1 \leq i \leq d),$$
$$[B_3]_{\mathbf{v}}(i-1,i) = q^{-d}(1-q^{2d-2i+2})(1-tq^{2i-d-1}) \qquad (1 \leq i \leq d).$$

Proof. The matrices that represent the action of X_{01}, X_{12} and X_{23} are given in Lemma 18, and the entries of these matrices are given in Definition 6.

Definition 10. *With reference to Lemma 19, define*

$$\begin{aligned}\Upsilon_i &= a(1-q^{2d-2i+2})(1-q^{-2i})(1-tq^{2i-d-1}) \qquad (1 \le i \le d),\\ \Omega_i &= q^{-d-1}(q^{2d-2i+2}-1)(q^{2i}-1)(at-bq^{d-2i+1}) \qquad (1 \le i \le d).\end{aligned}$$

Note that $\alpha_i = [A]_{\mathbf{v}}(i,i)$, $\alpha_i^* = [B_3]_{\mathbf{v}}(i,i)$ for $(0 \le i \le d)$, where α_i and α_i^* appear in Definition 8. And $\Upsilon_i = [A]_{\mathbf{v}}(i,i-1)[B_3]_{\mathbf{v}}(i-1,i)$ for $(1 \le i \le d)$.

Now, by Theorem 1, if we find the conditions on the sequence of scalars $(\{\alpha_i\}_{i=0}^d, \{\alpha_i^*\}_{i=0}^d; \{\Upsilon_j\}_{j=1}^d, \{\Omega_j\}_{j=1}^d)$ in which the sequence is a parameter array, then these conditions imply that the pair A, B_3 is a Leonard pair.

So, we now need to find when the sequence $(\{\alpha_i\}_{i=0}^d, \{\alpha_i^*\}_{i=0}^d; \{\Upsilon_j\}_{j=1}^d, \{\Omega_j\}_{j=1}^d)$ satisfies the seven conditions of the parameter array in Definition 2. From Lemmas 6, 7, and 11 we know when the conditions 1, 2, and 7 hold. in the next work we will find when the conditions $3-6$ of Definition 2 hold.

Lemma 20. *With reference to Definition 10, $\Upsilon_i \neq 0$ if and only if $a \neq 0$ and $t \neq q^{d-2i+1}$ for $1 \le i \le d$. And $\Omega_i \neq 0$ if and only if $at \neq bq^{d-2i+1}$ for $1 \le i \le d$.*

Proof. Since q is not a root of unity, this implies that $\Upsilon_i = 0$ if and only if $a = 0$ or $1 - tq^{2i-d-1} = 0$, solve for t to get the result for Υ_i. Similar work for Ω_i.

Lemma 21. *With reference to Definitions 8 and 10,*

$$\Upsilon_i = \Omega_1 \sum_{k=0}^{i-1} \frac{\alpha_k - \alpha_{d-k}}{\alpha_0 - \alpha_d} + (\alpha_i^* - \alpha_0^*)(\alpha_{i-1} - \alpha_d) \qquad (1 \le i \le d).$$

Proof. Similar to proof of Lemma 9.

Lemma 22. *With reference to Definitions 8 and 10,*

$$\Omega_i = \Upsilon_1 \sum_{k=0}^{i-1} \frac{\alpha_k - \alpha_{d-k}}{\alpha_0 - \alpha_d} + (\alpha_i^* - \alpha_0^*)(\alpha_{d-i+1} - \alpha_0) \qquad (1 \le i \le d).$$

Proof. Similar to proof of Lemma 9.

Lemma 23. *With reference to Definition 8 and 10, let a, b and t be scalars in $\mathcal{F}$. Then the sequence of scalars $(\{\alpha_i\}_{i=0}^d, \{\alpha_i^*\}_{i=0}^d; \{\Upsilon_j\}_{j=1}^d, \{\Omega_j\}_{j=1}^d)$ is a parameter array if and only if $a \neq 0$ and $t \neq q^{d-2i+1}$, $at \neq bq^{d-2i+1}$ for $1 \le i \le d$, and $a - bq^{2(i-d)} \neq 0$ for $1 \le i \le 2d-1$.*

Proof. Note that the conditions $1-7$ of the parameter array in Definition 2 hold for the sequence $(\{\alpha_i\}_{i=0}^d, \{\alpha_i^*\}_{i=0}^d; \{\Upsilon_j\}_{j=1}^d, \{\Omega_j\}_{j=1}^d)$ from Lemmas 6, 7, 20, 21, 22, 11 respectively if and only if $a \neq 0$ and $t \neq q^{d-2i+1}$, $at \neq bq^{d-2i+1}$ for $1 \le i \le d$, and $a - bq^{2(i-d)} \neq 0$ for $1 \le i \le 2d-1$.

Theorem 4. *Assume $d \geq 2$, let V denote an evaluation module for $\boxtimes_q$ with dimension $d+1$. Let $A \in \boxtimes_q$ denote an arbitrary linear combination of X_{01} and X_{12}, let $B_3 \in \boxtimes_q$ such that $B_3 = X_{23}$, let a, b and $t \neq 0$ be scalars in $\mathcal{F}$. Write $A = aX_{01} + bX_{12}$. Then the pair A, B_3 acts on V as a Leonard pair if and only if $a \neq 0$ and $t \neq q^{d-2i+1}$, $at \neq bq^{d-2i+1}$ for $1 \leq i \leq d$, and $a - bq^{2(i-d)} \neq 0$ for $1 \leq i \leq 2d-1$.*

Proof. The action of the pair A, B_3 on the basis $\mathbf{v}$ is described in Lemma 19, the matrices represent A and B_3 with respect to the basis $\mathbf{v}$ are lower bidiagonal and upper bidiagonal respectively in which $\alpha_i = [A]_{\mathbf{v}}(i,i)$, $\alpha_i^* = [B_3]_{\mathbf{v}}(i,i)$, and $\Upsilon_i = [A]_{\mathbf{v}}(i,i-1)[B_3]_{\mathbf{v}}(i-1,i)$. In Lemma 23 we show that the sequence of scalars $(\{\alpha_i\}_{i=0}^d, \{\alpha_i^*\}_{i=0}^d; \{\Upsilon_j\}_{j=1}^d, \{\Omega_j\}_{j=1}^d)$ is a parameter array if and only if $a \neq 0$ and $t \neq q^{d-2i+1}$, $at \neq bq^{d-2i+1}$ for $1 \leq i \leq d$, and $a - bq^{2(i-d)} \neq 0$ for $1 \leq i \leq 2d-1$. Hence, the result hold by Theorem 1.

Theorem 5. *Assume $d \geq 2$, let V denote an evaluation module for $\boxtimes_q$ with dimension $d+1$. Let $A \in \boxtimes_q$ denote an arbitrary linear combination of X_{01} and X_{12}, let $B_1, B_2, B_3 \in \boxtimes_q$ such that $B_1 = X_{20}$, $B_2 = X_{30}$, and $B_3 = X_{23}$, let a, b and $t \neq 0$ be scalars in $\mathcal{F}$. Write $A = aX_{01} + bX_{12}$. Then the pairs A, B_1, A, B_2, and A, B_3 act on V as Leonard pairs if and only if $a \neq 0$, $b \neq 0$, $t \neq 0$, $t \neq q^{d-2i+1}$, $a^{-1}bt \neq q^{2i-d-1}$, $b^{-1}at \neq q^{d-2i+1}$ for $1 \leq i \leq d$, and $a - bq^{2(i-d)} \neq 0$ for $1 \leq i \leq 2d-1$.*

Proof. Clear from Theorems 2, 3, and 4.

Lemma 24. *[4] Consider the $\boxtimes_q$-module $V_d(t)$. Pick mutually distinct $i, j, k, l \in Z_4$. Then for each standard generator X_{rs} the following are the same:*

(i) the matrix that represents X_{rs} with respect to an $[i,j,k,l]$-basis for $V_d(t)$;

(ii) the matrix that represents $X_{r+1,s+1}$ with respect to an $[i+1, j+1, k+1, l+1]$-basis for $V_d(t^{-1})$.

Theorem 6. *Assume $d \geq 2$, let V denote an evaluation module for $\boxtimes_q$ with dimension $d+1$. Let $A \in \boxtimes_q$ denote an arbitrary linear combination of X_{12} and X_{23}, let $B_1, B_2, B_3 \in \boxtimes_q$ such that $B_1 = X_{31}$, $B_2 = X_{01}$, and $B_3 = X_{30}$, let a, b and $t \neq 0$ be scalars in $\mathcal{F}$. Write $A = aX_{12} + bX_{23}$. Then the pairs A, B_1, A, B_2, and A, B_3 act on V as Leonard pairs if and only if $a \neq 0$, $b \neq 0$, $t \neq 0$, $t^{-1} \neq q^{d-2i+1}$, $a^{-1}bt^{-1} \neq q^{2i-d-1}$, $b^{-1}at^{-1} \neq q^{d-2i+1}$ for $1 \leq i \leq d$, and $a - bq^{2(i-d)} \neq 0$ for $1 \leq i \leq 2d-1$.*

Proof. Similar to proof of Theorem 5 but replace t by t^{-1} and replace the bases $[i,j,k,l]$ that appear in the proof of Theorem 5 by the bases $[i+1, j+1, k+1, l+1]$, and use Lemma 24 to find the action of the standard generators of $\boxtimes_q$ on the new bases.

Theorem 7. *Assume $d \geq 2$, let V denote an evaluation module for $\boxtimes_q$ with dimension $d+1$. Let $A \in \boxtimes_q$ denote an arbitrary linear combination of X_{23} and X_{30}, let $B_1, B_2, B_3 \in \boxtimes_q$ such that $B_1 = X_{02}$, $B_2 = X_{12}$, and $B_3 = X_{01}$, let a, b and $t \neq 0$ be scalars in $\mathcal{F}$.*

Write $A = aX_{23} + bX_{30}$. Then the pairs A, B_1, A, B_2, and A, B_3 act on V as Leonard pairs if and only if $a \neq 0$, $b \neq 0$, $t \neq 0$, $t \neq q^{d-2i+1}$, $a^{-1}bt \neq q^{2i-d-1}$, $b^{-1}at \neq q^{d-2i+1}$ for $1 \leq i \leq d$, and $a - bq^{2(i-d)} \neq 0$ for $1 \leq i \leq 2d-1$.

Proof. Similar to proof of Theorem 5 but replace the bases $[i, j, k, l]$ that appear in the proof of Theorem 5 by the bases $[i+2, j+2, k+2, l+2]$, and use Lemma 24 to find the action of the standard generators of $\boxtimes_q$ on the new bases.

Theorem 8. *Assume $d \geq 2$, let V denote an evaluation module for $\boxtimes_q$ with dimension $d+1$. Let $A \in \boxtimes_q$ denote an arbitrary linear combination of X_{12} and X_{23}, let $B_1, B_2, B_3 \in \boxtimes_q$ such that $B_1 = X_{31}$, $B_2 = X_{01}$, and $B_3 = X_{30}$, let a, b and $t \neq 0$ be scalars in $\mathcal{F}$. Write $A = aX_{12} + bX_{23}$. Then the pairs A, B_1, A, B_2, and A, B_3 act on V as Leonard pairs if and only if $a \neq 0$, $b \neq 0$, $t \neq 0$, $t^{-1} \neq q^{d-2i+1}$, $a^{-1}bt^{-1} \neq q^{2i-d-1}$, $b^{-1}at^{-1} \neq q^{d-2i+1}$ for $1 \leq i \leq d$, and $a - bq^{2(i-d)} \neq 0$ for $1 \leq i \leq 2d-1$.*

Proof. Similar to proof of Theorem 5 but replace t by t^{-1} and replace the bases $[i, j, k, l]$ that appear in the proof of Theorem 5 by the bases $[i+3, j+3, k+3, l+3]$, and use Lemma 24 to find the action of the standard generators of $\boxtimes_q$ on the new bases.

References

[1] H. Alnajjar. Leonard pairs associated with equitable generators of the quantum algebra $U_q(sl_2)$. *Linear and Multilinear Algebra*, 59:1127–1142, 2011.

[2] H. Alnajjar. A linear map that acts as a leonard pair with each of the generators of $U(sl_2)$. *International Journal of Mathematics and Mathematical Science*, 2020, 2020.

[3] H. Alnajjar and B. Curtin. Leonard pairs from the equitable basis of sl_2. *ELA*, 20:490–505, 2010.

[4] T. Ito, H. Rosengren, and P. Terwilliger. Evaluation modules for the q-tetrahedron algebra. *Linear Algebra Appl.*, 451:107–168, 2014.

[5] T. Ito and P. Terwilliger. The q-tetrahedron algebra and its finite dimensional irreducible modules. *Comm. Algebra*, 35:3415–3439, 2007.

[6] P. Terwilliger. The subconstituent algebra of an association scheme. iii. *J. Algebraic Combin.*, 2(2):177–210, 1993.

[7] P. Terwilliger. Two linear transformations each tridiagonal with respect to an eigenbasis of the other. *Linear Algebra Appl.*, 330:149–203, 2001.

[8] P. Terwilliger. Leonard pairs from 24 points of view. *Rocky Mountain J. Math.*, 32(2):827–888, 2002.

[9] P. Terwilliger. Introduction to leonard pairs. opsfa rome 2001. *J. Comput. Appl. Math.*, 153(2):463–475, 2003.

[10] P. Terwilliger. Leonard pairs and the q-racah polynomials. *Linear Algebra Appl.*, 387:235–276, 2004.

[11] P. Terwilliger. Two linear transformations each tridiagonal with respect to an eigenbasis of the other. comments on the parameter array. *Des. Codes Cryptogr.*, 34:307–332, 2005.

[12] P. Terwilliger. Two linear transformations each tridiagonal with respect to an eigenbasis of the other: the TD-D and the LB-UB canonical form. *J. Algebra*, 291(1):1–45, 2005.