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New Integral Transforms of the Extended k- Generalized Mittag-Leffler Function with Graphical Representations

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Abstract. Different integral transforms have extensive applications in various areas of science and engineering. This paper discusses some of the new integral transforms of the extended k-generalized Mittag-Leffler function. We examine integral transforms such as the Euler-Beta, Laplace, Mohand, Aboodh, SEE, and Sadik transform. Moreover, we also tried to establish the graphical representations of these transforms.

2020 Mathematics Subject Classifications: 33B15, 33C05, 33C60, 33E12

Key Words and Phrases: The extended k-generalized Mittag-Leffler function (EkG M-L function), Laplace transform, Mohand transform, Aboodh transform, SEE transform, Sadik transform, Fractional integration, differential operator

1. Introduction:

Integral transforms are essential tools across various scientific and engineering fields, facilitating the solution of complex problems. The EkG M-L function, a versatile mathematical construct, has found widespread applications in these areas. This paper explores multiple integral transforms of EkG M-L function, including Euler-Beta transform [46, 49], Laplace transform [20], Mohand transform [32], Aboodh transform [2], SEE transform [31], and Sadik transform [44]. Through this exploration, the study aims to deepen the understanding of this function and its utility, contributing to advancements in both theoretical and applied research. The application of these integral transforms provides new insights and solutions, further enhancing the significance of the EkG M-L function in scientific and engineering domains. They play an important role in solving differential equations and analyzing systems in various fields like physics, engineering, and applied mathematics.

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The EkG M-L function, a special function discussed in [5, 21, 30, 37, 45, 54], is noteworthy for its board applicability and flexibility in addressing problems in fractional calculus and complex systems. This function generalizes the classical Mittag-Leffler function and has proven instrumental in both scientific and engineering research. Other integral transforms, such as the modified Laplace transform and formable integral transform, as discussed in [48, 55] respectively, also provide effective methods for solving integral and differential equations. By exploring these transforms, researchers uncover new characteristics and applications of the EkG M-L function, further enhancing its utility in various research domains. A comprehensive analysis of various transforms and their dualities, such as Euler, Laplace, Whittaker, and K-transforms has highlighted distinct qualities and applications of the EkG M-L function [17, 18]. This research underscores the function's versatility and theoretical significance within scientific and engineering disciplines. The study also delves into integral and series representations associated with special functions like generalized Wright hyper-geometric function and Fox's-H functions [19, 38], providing insights into the EkG M-L function's behavior and expanding its potential applications across domains. Such in-depth exploration enhances the function's utility and opens avenues for its application in advanced research. Specifically, the EkG M-L function and its extensions have been applied in modeling phenomena such as anomalous diffusion, membrane protein mobility, and visco elastic creep in glasses. In fractional calculus applications, including numerical analysis, physics, and engineering, this function demonstrate remarkable versatility and importance [23]. Furthermore, properties related to fractional calculus, such as k-Weyl fractional integral and k-extended Euler beta integral transform, have also been investigated, emphasizing the function's relevance in mathematical transformations and computations. Overall, the EkG M-L function is crucial in diverse scientific disciplines, making it a valuable tool for researchers and practitioners alike [51, 52]. In 1729, the renowned mathematician Euler introduced the integral function that later became known as the Gamma function [7, 9, 28]. This function generalizes the factorial function, extending its domain from positive integers to complex numbers. The Euler Gamma function [28] is defined as follows:

$$z! = \Gamma(z+1) = \int_0^{\infty} t^z e^{-t}\, dt, \quad z \in \mathbb{C},\, \Re(z) > 0. \tag{1}$$

This defines $z!$ as an analytic function of z, $\forall z, \quad \Re(z) \geq 0$.

The beta function was introduced by Legendre and further studied by Whittaker and Watson, with its formulation expressed as follows [12]:

$$B(z_1, z_2) = \frac{(z_1-1)!(z_2-1)!}{(z_1+z_2-1)!} \tag{2}$$

The Euler integral of the first kind , also known as the beta function, is a special function that related to the gamma function. The beta integral, typically represented with two variables, is given by [29]:

$$B(z_1, z_2) = \int_0^1 t^{z_1-1}(1-t)^{z_2-1}\,dt, \quad \text{such that } z_1, z_2 \in \mathbb{C}, \Re(z_1) > 0, \Re(z_2) > 0 \tag{3}$$

The primary characteristic of the beta function is that represents the integral of the product of two gamma functions. The beta integral is widely used in areas such as probability theory, statistics, and calculus. It is also applied to evaluate specific integrals and to compute certain special functions [40].

$$B(z_1, z_2) = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1 + z_2)}, \quad z_1, z_2 \in \mathbb{C} \setminus \mathbb{Z}_0^- \tag{4}$$

The extended gamma function, extended Beta function and the extended Gauss hyper-geometric function are used our final results [38]. The extended Gamma function $\Gamma_p^{\{K_l\}_{l\in\mathbb{N}_0}}(z)$ [10]is defined as below:

$$\Gamma_p^{\{K_l\}_{l\in\mathbb{N}_0}}(z) = \int_0^\infty t^{z-1} f\left(\{K_l\}_{l\in\mathbb{N}_0}; -t - \frac{p}{t}\right) dt \quad \text{where } \Re(z) > 0, \Re(p) \geq 0 \tag{5}$$

The extended Beta function $B_k(x, y; p)$ [16] defined as below:

$$B_k^{\{K_l\}_{l\in\mathbb{N}_0}}(x, y; p) = \int_0^1 t^{x-1}(1-t)^{y-1} f\left(\{K_l\}_{l\in\mathbb{N}_0}; \frac{-p}{t(1-t)}\right) dt \tag{6}$$

where $\min\{\Re(x), \Re(y)\} > 0$ and $\Re(p) \geq 0$

and the extended Gauss hyper-geometric function [13] is defined as below:

$$F_p^{(\{K_l\}_{l\in\mathbb{N}_0})}(\alpha, \beta, \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n\, B_k(\beta + n, \gamma - \beta; p)}{B(\beta, \gamma - \beta)} \frac{z^n}{n!} \tag{7}$$

where $|z| < 1$, $\Re(\gamma) > \Re(\beta) > 0$, $\Re(p) \geq 0$

Moreover, the sequence $\{K_l\}_{l\in\mathbb{N}_0}$ mentioned above can be reduce to novel extensions of Gamma, Beta and hyper-geometric functions [13].

Specially, when

$$K_l = \frac{(x)_l}{(y)_l} \quad \text{if } l \in \mathbb{N}_0 \tag{8}$$

In 2011, Özergin et al. [36] introduced the definitions of the extended Gamma function $\Gamma_p^{(\alpha,\beta)}(z)$, the extended Beta function $B^{(\alpha,\beta)}(x, y; p)$ and the extended hyper-geometric function $F_p^{(\alpha,\beta)}(a, b, c; z)$ as below:

$$\Gamma_p^{(\alpha,\beta)}(z) = \int_0^\infty t^{z-1}\, {}_1F_1(\alpha, \beta; -t - \frac{p}{t})\, dt \tag{9}$$

where $\min\{\Re(z), \Re(p), \Re(\alpha)\} > 0$ and $\Re(p) \geq 0$

$$B^{(\alpha,\beta)}(x,y;p)=\int_0^1 t^{x-1}(1-t)^{y-1}\,{}_1F_1(\alpha;\beta;-\frac{p}{t(1-t)})\,dt, \tag{10}$$

where $\min\{\Re(x),\Re(y),\Re(\alpha),\Re(\beta)\}>0$ and $\Re(p)\geq 0$

$$F_p^{(\alpha,\beta)}(a,b,c;z)=\frac{1}{B(b,c-b;z)}\sum_{n=0}^{\infty}\frac{(a)_nB^{(\alpha,\beta)}(b+c,c-b;p)z^n}{n!}, \tag{11}$$

where $(|z|<1;\min\{\Re(\alpha),\Re(\beta)\}>0;\Re(c)>\Re(b)>0;\Re(p)\geq 0)$

Additionally, when $p=0$ then $K_l=0;\quad l\in\mathbb{N}$. Consequently, the equations 5 - 7 reduce to classical gamma, beta, and Gauss hyper-geometric functions [35, 50].

In this paper, several new transforms, such as the Euler-Beta transform [46, 49], Laplace transform [20], Mohand Transform [32], Aboodh Transform [2], SEE (Sadiq, Emad, and Eman) Transform [31], and Sadik Transform [44] of EkG M-L function are being introduced. Additionally, the graphs of these transforms are analyzed. The following well-known facts and results are being used throughout this paper.

The EkG M-L function is as follows [21];

$$E_{(k,l,m)}^{(\rho,\sigma,c)}(x;p)=\sum_{n=0}^{\infty}\frac{B_k(\rho+n\sigma k,c-\rho;p)}{B_k(\rho,c-\rho)}\frac{(c)_{(n\sigma,k)}}{\Gamma_k(nl+m)}\frac{x^n}{n!} \tag{12}$$

where $k>0$; $x,l,m,\rho\in\mathbb{C}$, $\Re(l)>0$, $\Re(m)>0$, $\Re(\rho)>0$, $\sigma\in(0,1)\cup\mathbb{N}$; $p\geq 0$

- Euler-Beta Transform: This transform is also known as Erdelyi − Kober fractional representation [46, 49] and is defined as

$$B\{f(z);m,b\}=\int_0^1 z^{m-1}(1-z)^{b-1}f(z)\,dz,\quad \Re(m),\Re(b)>0 \tag{13}$$

- Laplace Transform: The Laplace transform [20] of the function $f(z)$ is defined as

$$L\{f(z)\}=\int_0^{\infty}e^{-sz}f(z)\,dz,\quad \Re(s)>0 \tag{14}$$

- Mohand Transform: This transform is denoted by the operator M [32] and is defined as

$$M\{f(z);p\}=p^2\int_0^{\infty}e^{-pz}f(z)\,dz \tag{15}$$

- Aboodh Transform: Aboodh transform [2] is denoted by the operator $A(.)$ and defined as below:

$$A\{f(t)\}=\frac{1}{v}\int_0^{\infty}e^{-vt}f(t)\,dt\quad \text{where } t\geq 0 \tag{16}$$

- SEE (Sadiq, Emad and Eman) Transform: SEE transform was introduced by Sadiq, Emad A. Kuffi, and Eman M [31] and is denoted using $S(.)$ and is defined as below;

$$S\{f(t)\} = \frac{1}{v^n}\int_0^\infty e^{-vt} f(t)\,dt \quad \text{where } \Re(v) \geq 0, n \in \mathbb{Z},\ t \geq 0 \tag{17}$$

- Sadik Transform: The Sadik transform [44], denoted by S_a, is defined as follows;

$$S_a\{f(t)\} = \frac{1}{v^\beta}\int_0^\infty e^{-vt} f(t)\,dt \quad \text{where } v \in \mathbb{C}, \alpha \in \mathbb{R}\setminus\{0\}, \beta \in \mathbb{R}. \tag{18}$$

Furthermore, most of these transforms we found in the form of the extended hypergeometric function [21, 35, 41, 50] $F_p^{(\alpha,\beta)}(a,b,c;z)$; $|z| < 1$; $\min\{\Re(\alpha), \Re(\beta)\} > 0$; $\Re(c) > \Re(b) > 0$; $\Re(p) \geq 0$.

Furthermore, we present graphical representations of these transforms, which provide valuable insights into their versatile applications across fields that require complex, memory-dependent models. The EkG M-L function, an extension of the classical Mittag-Leffler function, is particularly useful in fractional calculus [3], where it plays a crucial role in describing systems governed by non-integer order differential equations. The distinct growth patterns displayed in the graphs reveal unique properties, such as power-law decay and rapid divergence, which can be tuned by adjusting parameters. This flexibility allows the EkG M-L function to effectively model anomalous diffusion processes [27, 42], commonly observed in physics, hydrology, and environmental science, where particle movement deviates from classical diffusion. The graphs illustrate how parameter variations influence growth, highlighting the function's adaptability. In materials science, for example, this adaptability supports accurate modeling of viscoelastic materials, where traditional exponential functions fail to capture time-dependent stress-strain relationships. The EkG M-L function also finds applications in control systems with memory effects, such as in biomedical engineering [6], where it aids in the design of controllers for systems exhibiting long-term transient behavior. In financial modeling [33], the function's ability to describe non-exponential waiting times and heavy-tailed distributions is valuable for modeling market volatility and credit risk, particularly in markets exhibiting self-similarity. Additionally, in epidemiology [25]and population dynamics, the extended M-L function is useful for capturing delayed effects, such as incubation and recovery periods. Through these graphical representations, researchers can more easily identify the optimal parameter configurations for modeling real-world phenomena that exhibit memory and non-linearity, thereby enhancing the accuracy and reliability of simulations across diverse applications.

2. Important Integral transforms and their graphical representations:

Theorem 1. ***The following Euler-Beta Transform of the EkG M-L function in equation 12 holds true:***

$$B\{E_{(k,l,m)}^{(\rho,\sigma,c)}(xz^l; m, b)\} = \Gamma_k(b) E_{(k,l,m+b)}^{(\rho,\sigma,c)}(x; p) \tag{19}$$

where $\Re(p), \Re(b), \Re(k), \Re(l)$ ***and*** $\Re(m) > 0$.

Proof. **Using the definition of the Euler-Beta Transform in equation 13, we can express equation 12 as follows:**
$B\{E^{(\rho,\sigma,c)}_{(k,l,m)}(xz^l; m, b)\}$
$= \int_0^1 z^{m-1}(1-z)^{b-1} E^{(\rho,\sigma,c)}_{(k,l,m)}(xz^l; p)\, dz$
$= \int_0^1 z^{m-1}(1-z)^{b-1} \sum_{n=0}^{\infty} \frac{B_k(\rho+n\sigma k, c-\rho; p)}{B_k(\rho, c-\rho)} \frac{(c)_{(n\sigma,k)}}{\Gamma_k(nl+m)} \frac{x^n}{n!} z^l n\, dz$
After interchanging the order of integration and summation, we can easily say that the above equation uniformly converges and we can get the below:
$= \sum_{n=0}^{\infty} \frac{B_k(\rho+n\sigma k, c-\rho; p)}{B_k(\rho, c-\rho)} \frac{(c)_{(n\sigma,k)}}{\Gamma_k(nl+m)} \frac{x^n}{n!} \int_0^1 z^{m+ln-1}(1-z)^{b-1}\, dz$
$= \sum_{n=0}^{\infty} \frac{B_k(\rho+n\sigma k, c-\rho; p)}{B_k(\rho, c-\rho)} \frac{(c)_{(n\sigma,k)}}{\Gamma_k(nl+m)} \frac{x^n}{n!} \left(\frac{\Gamma_k(nl+m)\Gamma_k(b)}{\Gamma_k(nl+m+b)}\right)$
$= \Gamma_k(b) \sum_{n=0}^{\infty} \frac{B_k(\rho+n\sigma k, c-\rho; p)}{B_k(\rho, c-\rho)} \frac{(c)_{(n\sigma,k)} x^n}{n!\Gamma_k(nl+m+b)}$
$= \Gamma_k(b) E^{(\rho,\sigma,c)}_{(k,l,m+b)}(x; p)$

Theorem 2. ***The following Laplace Transform of the EkG M-L function in equation 12 holds true:***

$$L\{z^{m-1} E^{(\rho,\sigma,c)}_{(k,l,m)}(xz^l; p)\} = \frac{1}{p^m} F_p(1, \rho; 1, \frac{x}{p^l}) \tag{20}$$

where $\Re(p), \Re(b), \Re(k), \Re(l)$ ***and*** $\Re(m) > 0$.

Proof. **Using the definition of the Laplace Transform in equation 14, we can express equation 12 as follows:**
$L\{z^{m-1} E^{(\rho,\sigma,c)}_{(k,l,m)}(xz^l; p)\}$
$= \int_0^1 z^{m-1} e^{-pz} E^{(\rho,\sigma,c)}_{(k,l,m)}(xz^l; p)\, dz$
$= \int_0^1 z^{m-1} e^{-pz} \sum_{n=0}^{\infty} \frac{B_k(\rho+n\sigma k, c-\rho; p)}{B_k(\rho, c-\rho)} \frac{(c)_{(n\sigma,k)}}{\Gamma_k(nl+m)} \frac{x^n}{n!} z^{ln}\, dz$
After interchanging the order of integration and summation, we can easily say that the above equation uniformly converges and we can get the below:
$= \sum_{n=0}^{\infty} \frac{B_k(\rho+n\sigma k, c-\rho; p)}{B_k(\rho, c-\rho)} \frac{(c)_{(n\sigma,k)}}{\Gamma_k(nl+m)} \frac{x^n}{n!} \int_0^1 z^{nl+m-1} e^{-pz}\, dz$
$= \sum_{n=0}^{\infty} \frac{B_k(\rho+n\sigma k, c-\rho; p)}{B_k(\rho, c-\rho)} \frac{(c)_{(n\sigma,k)}}{\Gamma_k(nl+m)} \frac{x^n}{n!} \left(\frac{\Gamma_k(nl+m)}{p^{nl+m}}\right)$
$= \sum_{n=0}^{\infty} \frac{B_k(\rho+n\sigma k, c-\rho; p)}{B_k(\rho, c-\rho)} \frac{(c)_{(n\sigma,k)} x^n}{n!\, p^{nl+m}}$
$= \frac{1}{p^m} \sum_{n=0}^{\infty} \frac{B_k(\rho+n\sigma k, c-\rho; p)}{B_k(\rho, c-\rho)} (c)_{(n\sigma,k)} \frac{(x/p^l)^n}{n!}$
By using equation 11, we can get the needed result.

Remark 1. ***When*** $p = 0$***, then the above result will be***
$\int_0^1 z^{m-1} e^{-pz} E^{(\rho,\sigma,c)}_{(k,l,m)}(xz^l; p)\, dz = \frac{1}{p^m}\left(1 - \frac{x}{p^l}\right)^{-\rho}$.

Theorem 3. ***The following Mohand Transform of the EkG M-L function in equation 12 holds true:***

$$M\{z^{m-1} E^{(\rho,\sigma,c)}_{(k,l,m)}(xz^l; p)\} = p^{2-m} F_p(1, \rho; 1, \frac{x}{p^l}) \tag{21}$$

where $\Re(p), \Re(b), \Re(k), \Re(l)$ ***and*** $\Re(m) > 0$.

Proof. **Using the definition of the Mohand Transform in equation 15, we can express equation 12 as follows:**
$M\{z^{m-1}E_{(k,l,m)}^{(\rho,\sigma,c)}(xz^l;p)\}$
$= p^2\int_0^1 z^{m-1}e^{-pz}E_{(k,l,m)}^{(\rho,\sigma,c)}(xz^l;p)\,dz$
$= p^2\int_0^1 z^{m-1}e^{-pz}\sum_{n=0}^{\infty}\frac{B_k(\rho+n\sigma k,c-\rho;p)}{B_k(\rho,c-\rho)}\frac{(c)_{(n\sigma,k)}}{\Gamma_k(nl+m)}\frac{x^n}{n!}z^{ln}\,dz$
After interchanging the order of integration and summation, we can easily say that the above equation uniformly converges and we can get the below:
$= p^2\sum_{n=0}^{\infty}\frac{B_k(\rho+n\sigma k,c-\rho;p)}{B_k(\rho,c-\rho)}\frac{(c)_{(n\sigma,k)}}{\Gamma_k(nl+m)}\frac{x^n}{n!}\int_0^1 z^{ln+m-1}e^{-pz}\,dz$
$= p^2\sum_{n=0}^{\infty}\frac{B_k(\rho+n\sigma k,c-\rho;p)}{B_k(\rho,c-\rho)}(c)_{(n\sigma,k)}\frac{x^n}{n!p^{(nl+m)}}$
$= p^{2-m}\sum_{n=0}^{\infty}\frac{B_k(\rho+n\sigma k,c-\rho;p)}{B_k(\rho,c-\rho)}(c)_{(n\sigma,k)}\frac{\left(\frac{x}{p^l}\right)^n}{n!}$
By using equation 11, we can get the needed result.

Theorem 4. ***The following Aboodh Transform of the EkG M-L function in equation 12 holds true:***

$$A\{z^{m-1}e^{-pz}E_{(k,l,m)}^{(\rho,\sigma,c)}(xz^l;p)\} = \frac{1}{p^{m+1}}F_p(1,\rho;1,\frac{x}{p^l}) \tag{22}$$

where $\Re(p), \Re(b), \Re(k), \Re(l)$ ***and*** $\Re(m) > 0$.

Proof. **Using the definition of the Aboodh Transform in equation 16, we can express equation 12 as follows:**
$A\{z^{m-1}e^{-pz}E_{(k,l,m)}^{(\rho,\sigma,c)}(xz^l;p)\}$
$= \frac{1}{p}\int_0^1 z^{m-1}e^{-pz}E_{(k,l,m)}^{(\rho,\sigma,c)}(xz^l;p)\,dz$
$= \frac{1}{p}\int_0^1 z^{m-1}e^{-pz}\sum_{n=0}^{\infty}\frac{B_k(\rho+n\sigma k,c-\rho;p)}{B_k(\rho,c-\rho)}\frac{(c)_{(n\sigma,k)}}{\Gamma_k(nl+m)}\frac{x^n}{n!}z^{ln}\,dz$
After interchanging the order of integration and summation, we can easily say that the above equation uniformly converges and we can get the below:
$= \frac{1}{p}\sum_{n=0}^{\infty}\frac{B_k(\rho+n\sigma k,c-\rho;p)}{B_k(\rho,c-\rho)}\frac{(c)_{(n\sigma,k)}}{\Gamma_k(nl+m)}\frac{x^n}{n!}\int_0^1 z^{ln+m-1}e^{-pz}\,dz$
$= \frac{1}{p}\sum_{n=0}^{\infty}\frac{B_k(\rho+n\sigma k,c-\rho;p)}{B_k(\rho,c-\rho)}(c)_{(n\sigma,k)}\frac{x^n}{n!p^{(nl+m)}}$
$= \frac{1}{p^{m+1}}\sum_{n=0}^{\infty}\frac{B_k(\rho+n\sigma k,c-\rho;p)}{B_k(\rho,c-\rho)}(c)_{(n\sigma,k)}\frac{\left(\frac{x}{p^l}\right)^n}{n!}$
By using equation 11, we can get the needed result.

Theorem 5. ***The following SEE transform of the EkG M-L function in equation 12 holds true:***

$$S\{z^{m-1}e^{-pz}E_{(k,l,m)}^{(\rho,\sigma,c)}(xz^l;p)\} = \frac{1}{p^{2m}}F_p(1,\rho;1,\frac{x}{p^l}) \tag{23}$$

where $\Re(p), \Re(b), \Re(k), \Re(l)$ ***and*** $\Re(m) > 0$.

Proof. **Using the definition of the SEE transform in equation 17, we can express equation 12 as follows:**

$S\{z^{m-1}e^{-pz}E^{(\rho,\sigma,c)}_{(k,l,m)}(xz^l;p)\}$

$= \frac{1}{p^m}\int_0^1 z^{m-1}e^{-pz}E^{(\rho,\sigma,c)}_{(k,l,m)}(xz^l;p)\,dz$

$= \frac{1}{p^m}\int_0^1 z^{m-1}e^{-pz}\sum_{n=0}^{\infty}\frac{B_k(\rho+n\sigma k,c-\rho;p)}{B_k(\rho,c-\rho)}\frac{(c)_{(n\sigma,k)}}{\Gamma_k(nl+m)}\frac{x^n}{n!}z^{ln}\,dz$

After interchanging the order of integration and summation, we can easily say that the above equation uniformly converges and we can get the below:

$= \frac{1}{p^m}\sum_{n=0}^{\infty}\frac{B_k(\rho+n\sigma k,c-\rho;p)}{B_k(\rho,c-\rho)}\frac{(c)_{(n\sigma,k)}}{\Gamma_k(nl+m)}\frac{x^n}{n!}\int_0^1 z^{ln+m-1}e^{-pz}\,dz$

$= \frac{1}{p^m}\sum_{n=0}^{\infty}\frac{B_k(\rho+n\sigma k,c-\rho;p)}{B_k(\rho,c-\rho)}(c)_{(n\sigma,k)}\frac{x^n}{n!p^{(nl+m)}}$

$= \frac{1}{p^{2m}}\sum_{n=0}^{\infty}\frac{B_k(\rho+n\sigma k,c-\rho;p)}{B_k(\rho,c-\rho)}(c)_{(n\sigma,k)}\frac{\left(\frac{x}{p^l}\right)^n}{n!}$

By using equation 11, we can get the needed result.

Theorem 6. ***The following Sadik transform of the EkG M-L function in equation 12 holds true:***

$$S_a\left(z^{m-1}e^{-pz}E^{\rho,\sigma,c}_{k,l,m}(xz^l;p)\right) = \frac{1}{p^{\beta+m}}F_p\left(1,\rho;1,\frac{x}{p^l}\right) \tag{24}$$

where $\Re(p),\Re(b),\Re(k),\Re(l),$ ***and*** $\Re(m) > 0.$

Proof. **Using the definition of the Sadik transform in equation 18, we can express equation 12 as follows:**

$_a\left(z^{m-1}e^{-pz}E^{\rho,\sigma,c}_{k,l,m}\left(xz^l;p\right)\right)$

$= \frac{1}{p^\beta}\int_0^1 z^{m-1}e^{-pz}E^{\rho,\sigma,c}_{k,l,m}\left(xz^l;p\right)\,dz$

$= \frac{1}{p^\beta}\int_0^1 z^{m-1}e^{-pz}\sum_{n=0}^{\infty}\frac{B_k(\rho+n\sigma k,c-\rho;p)}{B_k(\rho,c-\rho)}\frac{(c)_{n\sigma,k}}{\Gamma_k(nl+m)}\frac{x^n}{n!}z^{ln}\,dz$

After interchanging the order of integration and summation, we can easily say that the above equation uniformly converges and we can get the below:

$= \frac{1}{p^\beta}\sum_{n=0}^{\infty}\frac{B_k(\rho+n\sigma k,c-\rho;p)}{B_k(\rho,c-\rho)}\frac{(c)_{n\sigma,k}}{\Gamma_k(nl+m)}\frac{x^n}{n!}\int_0^1 z^{ln+m-1}e^{-pz}\,dz$

$= \frac{1}{p^\beta}\sum_{n=0}^{\infty}\frac{B_k(\rho+n\sigma k,c-\rho;p)}{B_k(\rho,c-\rho)}(c)_{n\sigma,k}\frac{x^n}{n!p^{nl+m}}$

$= \frac{1}{p^{\beta+m}}\sum_{n=0}^{\infty}\frac{B_k(\rho+n\sigma k,c-\rho;p)}{B_k(\rho,c-\rho)}(c)_{n\sigma,k}\frac{\left(\frac{x}{p^l}\right)^n}{n!}$

By using equation 11, we can get the needed result.

2.1. Graphical representations and discussions:

By applying various transforms, such as the Euler-Beta transform, Laplace transform, Mohand transform, Aboodh transform, SEE (Sadiq, Emad, and Eman) transform, and Sadik transform, the EkG M-L function can be further generalized to different parameters with an infinite range. In the below graphs X-axis represents the independent variable and the Y-axis represents the function values.

Figure 1 was obtained after setting $p = 0.5$,$m = 0.5$,$l = 0.5$and $\rho = 0.5$ in equation 19 to

24.

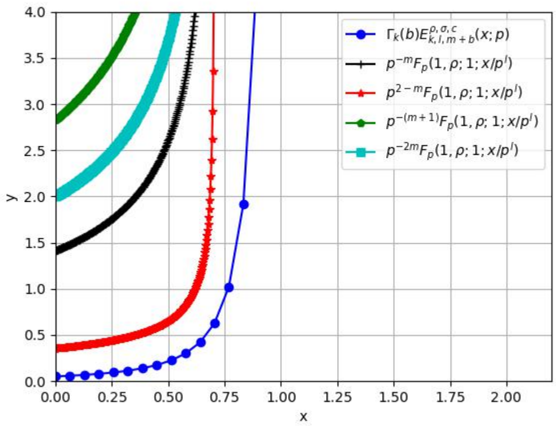

Figure 1: $p = 0.5$,$m = 0.5$,$l = 0.5$and $\rho = 0.5$.

Figure 2 was obtained after setting $p = 0.75$,$m = 0.75$,$l = 0.75$ and $\rho = 0.75$ in equation 19 to 24.

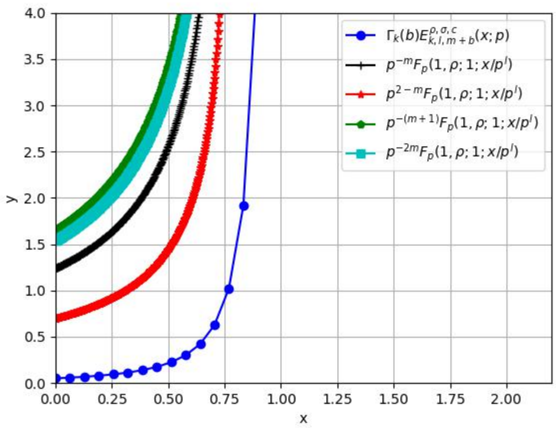

Figure 2: $p = 0.75$,$m = 0.75$,$l = 0.75$ and $\rho = 0.75$

Figure 3 was obtained after setting $p = 1.25$,$m = 1.25$,$l = 1.25$ and $\rho = 1.25$ in equation 19 to 24.

In Figure 1, the Euler-Beta Transform of the EkG M-L function shows a pronounced increase around $x = 0.75$, diverging more rapidly than the other curves. In Figure 2, while

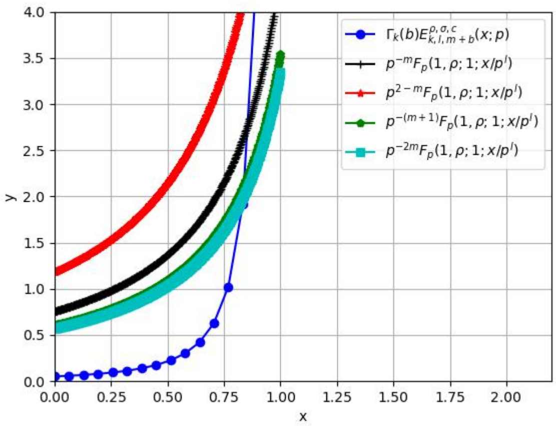

Figure 3: $p = 1.25$,$m = 1.25$,$l = 1.25$ and $\rho = 1.25$

the Euler-Beta Transform of the EkG M-L function continues to rise sharply near $x = 0.75$, the other transforms appear closer together, indicating that changes in parameter values are influencing their growth rates. This suggests that these transforms are sensitive to parameter adjustments, which can either enhance or moderate their growth. In Figure 3, the Euler-Beta Transform of the EkG M-L function demonstrates an even steeper rate of growth, starting from a lower position on the y-axis and climbing more sharply as x increases. Comparing these three figures, it is evident that the Euler-Beta Transform of the EkG M-L function consistently exhibits the steepest rise, underscoring its sensitivity to parameter values. The relative positioning of the other transforms shifts across the figures, reflecting how variations in parameters impact their growth behavior. Collectively, these graphical representations provide insights into the behavior of the different transforms under various parameter conditions, illustrating the impact of parameter modifications on each transform's growth trajectory.

The graph illustrates the growth behavior of various mathematical functions, where applying different transforms or parameters results in distinct curves, each exhibiting unique growth patterns as x increases. As the value of x rises, some curves show exponential or accelerated growth beyond specific points, highlighting their divergent behavior. This suggests comparing special functions with similar structural forms but differ in parametrization or the types of transforms applied. The variations in growth demonstrate how these parameters influence the behavior of each function over time.

2.2. Practical Applications of these Integral transforms:

The EkG M-L function and its integral transformations offer substantial potential for practical applications across various fields, thanks to the flexibility of the Mittag-Leffler function family in representing complex, nonlinear, and memory-dependent systems. Key areas of application include: The EkG M-L function is especially useful in fractional calculus [4, 15], which extends classical calculus to non-integer order derivatives and integrals.

It plays a notable role in modeling anomalous diffusion processes—where particles disperse at non-linear rates—making it invaluable in fields such as physics, hydrology, and environmental science. These transformations provide a more accurate modeling approach for real-world phenomena that deviate from standard diffusion patterns, including subdiffusion in porous media and super-diffusion in turbulent environments[14]. In materials science, the integral transformations of this function can effectively describe complex, viscoelastic behaviors. Traditional exponential-based models often struggle to capture the time-dependent stress-strain relationships seen in polymers, biological tissues, and other materials with memory effects. By using these extended functions in the differential equations governing stress and strain, it becomes possible to simulate delayed elastic responses with improved accuracy, aiding in material design and testing. The Mittag-Leffler function also has applications in fractional control systems where standard PID controllers fall short [11, 22, 24, 26]. Systems with long-term memory effects or complex transient behaviors—such as those in biomedical engineering or telecommunications—benefit from controllers that utilize fractional derivatives. The function's integral transformations facilitate the analysis of systems exhibiting power-law frequency response behaviors, supporting enhanced system design in adaptive filtering and robust control. In finance, processes with non-exponential waiting times and heavy-tailed distributions—especially in modeling market volatility and credit risk—often require the EkG M-L function. This function models returns with memory effects or power-law decay more precisely than Gaussian models. Through integral transforms, researchers can represent complex return dynamics and option pricing, particularly in markets characterized by long memory or self-similarity [1, 47]. For population growth and epidemiological modeling, integral transforms of the extended Mittag-Leffler function are instrumental in accounting for factors like incubation periods, recovery times, and seasonal variations—factors that traditional exponential models cannot easily capture. These models support more accurate predictions of epidemic progress and population trends, ultimately enhancing intervention effectiveness and resource allocation. In thermal and electromagnetic wave propagation[8, 34, 39, 43, 53], the function finds applications where systems do not simply decay exponentially but exhibit more complex attenuation governed by fractional dynamics. Extended functions enable the study of wave propagation in inhomogeneous media and non-Fourier heat conduction, where temperature or electromagnetic field intensity decays non-linearly.

3. Conclusions

In conclusion, this paper emphasizes the important applications of various integral transforms in science and engineering, with a particular focus on new integral transforms of EkG M-L function. We have investigated several integral transforms, such as the Euler-Beta, Laplace, Mohand, Aboodh, SEE, and Sadik transforms. Furthermore, we aimed to create graphical representations of these transforms to deepen the understanding of their behavior and applications. The findings presented in this work contribute to the ongoing advancement of integral transforms, offering valuable insights for researchers and practitioners in the field.

A critical aspect of our research involved the creation of graphical representations of these integral transforms. These visual aids serve to enhance the understanding of their behavior and practical implications, making complex concepts more accessible to both researchers and practitioners. Through these graphical analyses, we have provided insights into how these transforms operate under various conditions and their effectiveness in solving real-world problems.

Looking ahead, future work in this area can expand on several fronts. One promising avenue is the exploration of additional novel integral transforms that may emerge from recent mathematical developments. Further research could also involve applying these transforms to a broader range of problems, particularly in fields such as signal processing, image analysis, and control systems. Additionally, the integration of computational techniques to facilitate more complex and multidimensional analyses could lead to new insights and applications. Ultimately, this work lays the groundwork for ongoing advancements in integral transforms, paving the way for further exploration and innovation in both theoretical and applied mathematics.

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