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Investigation of a Fourth-Order Nonlinear Differential Equation with Moving Singular Points of Algebraic Type

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Abstract. In this study, a fourth-order nonlinear ordinary differential equation is considered. The specificity of nonlinearity lies in the presence of moving singular points, which hinders the application of classical theory that only works in the linear case. Two research problems are addressed in this work: the theorem of existence and uniqueness of the solution, and the precise criteria for the existence of a moving singular point. These problems are solved in both the real and complex domains. The specificity of transitioning to the complex plane is demonstrated using phase spaces. The obtained results are validated through numerical experiments, confirming the reliability of the results.

2020 Mathematics Subject Classifications: 34A34, 34A12, 34M04, 34M05

Key Words and Phrases: Cauchy problem, movable singular point, analytical approximate solution, phase spaces, Puiseux series, meromorphic function

1. Introduction

Nonlinear differential equations are widely used in science and engineering, for instance, in problems of hydrodynamics [8], and continuum mechanics [15, 23].

In this study, we will focus on the Cauchy problem for a fourth-order differential equation. The equation under consideration in present research has the following form:

$$\frac{d^4w}{dz^4} + Q_0 w \left(w'\right)^2 = f\left(z\right), \tag{1}$$

This model can be viewed as a mechanical system involving an oscillator, such as a mass-spring system or a pendulum, where the oscillator's motion is influenced by external forces. The term $w\left(w'\right)^2$ represents the interaction between the displacement w and the square of the oscillator's velocity w'. It captures the effect of velocity-dependent forces, which may introduce additional nonlinear behaviors, such as dissipation or energy generation. Often, when solving many problems, nonlinear terms are neglected. However,

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as shown in studies [4–6, 9, 10, 12, 14, 17, 22, 25, 44, 48], accounting for nonlinearity in the form of the derivative squared can impact the solution of the investigated problems.

The complexity of investigating such a class of equations lies in their nonlinearity, which is a condition for the emergence of movable singular points. The presence of movable singular points of algebraic type poses a challenge for finding an analytical solution. movable singular points make nonlinear differential equations in general unsolvable in quadratures. The classification and characteristics of movable and fixed singular points are well-described in the works [11, 18].

Singular points of the integrals of differential equations, whose position does not depend on the initial data that determine these integrals, are called *fixed singular points* .

Singular points of the integrals of differential equations, whose position depends on the initial data, are called *movable singular points.*

Algebraic movable singular points include simple and multiple poles, as well as branch points of finite order. In the neighborhood of such points, a Puiseux series expansion is assumed.

Despite the presence of movable singular points, there are methods for studying such equations. The process of finding a solution and investigating its behavior can be divided into two domains: the domain of analyticity and the neighborhood of the movable singular point.

In the domain of analyticity, linearization is possible, and numerical methods can be applied to solve such problems. In [21], two algorithms are developed to check the possibility of reducing a nonlinear differential equation to a linear one using Lie algebra and point transformations. In [16], linearization is based on the method of new approximation, taking into account both local and global properties obtained through global approximation of Lie derivatives. In [43], the linearization problem was solved using a generalized linearizing transformation. In [41, 42, 45, 46], the authors used homotopy analysis methods and numerical methods.

The second method for solving this type of nonlinear differential equations is analytical. Finding an analytical solution is rare due to the complexity of the considered physical phenomenon [49]. As shown in the works [7, 13, 19, 24, 26, 38, 39, 47], these problems are solved using a specific variable substitution or special functions.

The third method is asymptotic. The main task of this method is to investigate the behavior of the solution near singular points (both moving and stationary) as well as at infinity [1, 2, 50].

The author's method for studying nonlinear differential equations is based on solving four mathematical problems: the theorem of existence and uniqueness (in the domain of analyticity and the neighborhood of the movable singular point); the influence of perturbation of initial data (movable singular point) on the structure of the analytical approximate solution; precise criteria for the existence of movable singular points; precise boundaries for the application of the analytical approximate solution. Solving these problems allows for the development of an algorithm to find a movable singular point with a predetermined accuracy and to combine the obtained results with existing numerical methods [35]. This method has already been successfully tested in solving certain classes of equations.

In the works [27–30], the described method has been used to investigate the van der Pol equation in the complex domain. In [29], an approximate analytical solution to the initial value problem in the domain of analyticity is found. The work [30] is dedicated to studying the influence of perturbations in initial data on the structure of the approximate analytical solution. In the article [28], the authors found an approximate analytical solution in the neighborhood of the movable singular point, while [27] addressed the problem of the influence of perturbations in the movable singular point on the structure of the approximate analytical solution.

In [31–36], the authors have successfully solved all research problems related to third-order nonlinear differential equations with polynomial right-hand side of both second and seventh degrees.

For other classes of equations, the same method for finding analytical approximate solutions has been explored in [3, 20, 37, 40].

For the equation (1), two problems are addressed: the classical problem of the theory of differential equations, the proof of the existence and uniqueness theorem, and the precise criteria for the existence of a moving singular point. The existence theorem is considered in the complex domain. The solution is sought in the form of a Puiseux series. Estimates for the series coefficients are provided to determine the convergence region of the analytic part of the series. An analytical approximate solution is obtained, along with estimates of its error. The second problem, finding precise criteria for the existence of a moving singular point, which has been solved in both real and complex domains due to the different approaches to solving these problems.

2. Main results

2.1. The theorem of existence and uniqueness. Modification of the Cauchy–Kovalevskaya theorem.

Let's consider the following Cauchy problem:

$$w^{(4)} + Q_0 w \left(w'\right)^2 = f\left(z\right), \tag{2}$$

$$\begin{cases} w\left(z_0\right) = w_0, \\ w'\left(z_0\right) = w_0', \\ w''\left(z_0\right) = w_0'', \\ w'''\left(z_0\right) = w_0''', \end{cases} \tag{3}$$

where $w_0', w_0'', w_0''', Q_0 \in \mathbb{C}$.

Theorem 1. *Let z^* be a moving singular point of the Cauchy problem (2) — (3), and let the function $f\left(z\right)$ be holomorphic in the domain $\left|z^* - z\right| < \rho_1$, then the solution $w\left(z\right)$ is representable in the form of a meromorphic function:*

$$w\left(z\right) = \left(z^* - z\right)^{-1} \sum_{n \geq 0} A_n \left(z^* - z\right)^n, \tag{4}$$

in the domain $|z - z^*| < \rho_2$. *Here*

$$\rho_2 = \max\left\{\rho_1, \frac{1}{\sqrt[5]{|\gamma|}}\right\},$$

where

$$|\gamma| = \max\left\{\sup_n\left(\left|\frac{f^{(n)}(z^*)}{n!}\right|\right), |w_0|, |w_0'|, |w_0''|, |w_0'''|\right\}.$$

Proof.

We will seek the solution in the form of a generalized power series:

$$w(z) = \sum_{n\geq 0} A_n (z^* - z)^{n+r}. \tag{5}$$

Let's substitute the formula (5) into the equation (2), and obtain:

$$\sum_{n\geq 0} A_n (n+r)(n+r-1)(n+r-2)(n+r-3)(z - z^*)^{n+r-4} + Q_0\left(\sum_{n\geq 0} A_n (z^* - z)^{n+r}\right)\left(\sum_{n\geq 0} A_n (n+r)(z^* - z)^{n+r-1}\right)^2 = \sum_{n\geq 0} D_n (z^* - z)^n.$$

After cubing the first term on the right-hand side, we get:

$$\sum_{n\geq 0} A_n (n+r)(n+r-1)(n+r-2)(n+r-3)(z^* - z)^{n+r-4} = -Q_0\left(\sum_{n\geq 0} A_n (z^* - z)^{n+r}\right)\left(\sum_{n\geq 0} \tilde{A}_n^* (z^* - z)^{n+2r-2}\right) + \sum_{n\geq 0} D_n (z^* - z)^n,$$

$$\sum_{n\geq 0} A_n (n+r)(n+r-1)(n+r-2)(n+r-3)(z^* - z)^{n+r-4} = -Q_0 \sum_{n\geq 0} C_n^* (z^* - z)^{n+3r-2} + \sum_{n\geq 0} D_n (z^* - z)^n,$$

where

$$\begin{cases} A_n^* = \sum_{i=0}^{n} A_i A_{n-i}, \\ C_n^* = \sum_{i=0}^{n} \tilde{A}_i^* A_{n-i}, \\ \tilde{A}_n^* = \sum_{i=0}^{n} B_i B_{n-i}, \\ B_n = A_n (n+r). \end{cases}$$

The left and right sides are identically equal, from which the following conditions follow:

$$\begin{cases} n+r-4 = n+3r-2 \\ A_n (n+r)(n+r-1)(n+r-2)(n+r-3) = -Q_0 C_n^*, \text{ if } n = 0,1,2,3,4 \\ A_n (n+r)(n+r-1)(n+r-2)(n+r-3) = -Q_0 C_n^* + D_{n-5}, \text{ if } n \geq 5 \end{cases} \tag{6}$$

From the first equality, it follows that $r = -1$. As shown in [11], if $r \in \mathbb{Q}_-$, then z^* is a movable singular point for solving the Cauchy problem (2) — (3).

The second and third equalities are recurrent relations that allow for the unique determination of the coefficients of the series (2.1).

Writing the initial values for n, we obtain the first few recurrent relations:

$$\begin{cases} A_0 = \sqrt{-\frac{24}{Q_0}} \\ A_1 = A_2 = A_3 = A_4 = 0 \\ A_5 = \frac{D_0}{192} \\ A_6 = \frac{D_1}{336} \\ A_7 = \frac{D_2}{1152} \\ \cdots\cdots\cdots \end{cases}$$

Taking into account the regularity, it is evident that:
$A_n = \frac{D_{n-5}}{C} - A_0 \cdot F\left(D_0 D_{n-10}, D_1 D_{n-9}, \ldots, D_l D_k\right), \; l+k = n-10, \; C = const, \; n \geq 5.$

Since the series is formal, it is necessary to find its convergence domain. To do this, we will use the modified majorant method used in the Cauchy–Kovalevskaya theorem. This method is based on the estimation of the coefficients A_n on the basis of which the majority series is constructed.

Theorem 2. *Assume that all conditions of Theorem 1 be satisfied; then the estimate for the coefficients of the series expansion* (2.1)*, for sufficiently large values of* n*, takes the following form:*

$$|A_n| \leq \frac{|\gamma|^{\left[\frac{n}{5}\right]} \left(|Q_0 A_0| \left|\left[\frac{n-9}{2}\right] + 1\right| + 1\right)}{|(n-1)(n-2)(n-3)(n-4) + 24(2n-3)|}, \tag{7}$$

where $[\alpha]$ *denotes the integer part of the number.*

Proof. From the system (2.1), it follows that for sufficiently large values of the index, the following equality can be used:

$$A_n\left(n-1\right)\left(n-2\right)\left(n-3\right)\left(n-4\right)=-Q_0C_n^*+D_{n-5}. \tag{8}$$

Let's expand the right-hand side:

$$-Q_0\sum_{i=0}^{n}\tilde{A}_i^*A_{n-i}+D_{n-5}=-Q_0\sum_{i=0}^{n}\left(\sum_{j=0}^{i}B_jB_{i-j}\right)A_{n-i}+D_{n-5}$$

$$=-Q_0\sum_{i=0}^{n}\left(\sum_{j=0}^{i}A_j\left(j-1\right)A_{i-j}\left(i-j-1\right)\right)A_{n-i}+D_{n-5}$$

$$=-Q_0\left(\sum_{i=0}^{n}\left(-\left(i-1\right)A_0A_i+\ldots-\left(i-1\right)A_iA_0\right)A_{n-i}\right)+D_{n-5}$$

$$=-Q_0\left(-A_nA_0^2-2\left(n-1\right)A_nA_0^2+P\left(A_1,\ldots,A_{n-1}\right)\right)+D_{n-5}.$$

Considering the obtained equality, equation (8) can be expressed as:

$$A_n\left(n-1\right)\left(n-2\right)\left(n-3\right)\left(n-4\right)$$
$$=-Q_0\left(-A_nA_0^2-2\left(n-1\right)A_nA_0^2+P\left(A_0,\ldots,A_{n-1}\right)\right)+D_{n-5},$$

$$A_n\left(\left(n-1\right)\left(n-2\right)\left(n-3\right)\left(n-4\right)+24\left(2n-3\right)\right)$$
$$=-Q_0\cdot P\left(A_0,\ldots,A_{n-1}\right)+D_{n-5},$$

$$A_n=\frac{D_{n-5}-Q_0\cdot P\left(A_0,\ldots,A_{n-1}\right)}{\left(n-1\right)\left(n-2\right)\left(n-3\right)\left(n-4\right)+24\left(2n-3\right)}.$$

Considering the form of the function $P\left(A_0,\ldots,A_{n-1}\right)$ and the initial values A_i, as well as the condition of analyticity of the function $f\left(z\right)$, which implies the boundedness of the coefficients $\left|D_n\right|=\left|\frac{f^{(n)}(x_0)}{n!}\right|\leq\theta_n$, we obtain the following estimate for the coefficients:

$$\left|A_n\right|=\left|\frac{D_{n-5}-Q_0\cdot P\left(A_0,\ldots,A_{n-1}\right)}{\left(n-1\right)\left(n-2\right)\left(n-3\right)\left(n-4\right)+24\left(2n-3\right)}\right|\leq$$

$$\leq\frac{\left|D_{n-5}\right|+\left|Q_0\cdot P\left(A_0,\ldots,A_{n-1}\right)\right|}{\left|\left(n-1\right)\left(n-2\right)\left(n-3\right)\left(n-4\right)+24\left(2n-3\right)\right|}\leq$$

$$\leq \frac{|D_{n-5}| + \left| Q_0 A_0 \sum\limits_{\substack{k+l=n-9 \\ k,l \in \mathbb{N} \cup \{0\}}} D_k D_l \right|}{|(n-1)(n-2)(n-3)(n-4) + 24(2n-3)|} \leq$$

$$\leq \frac{|\gamma|^{\left[\frac{n}{5}\right]} + \left| Q_0 A_0 \gamma^{\left[\frac{n}{5}\right]} \right| \left| \left[\frac{n-9}{2}\right] + 1 \right|}{|(n-1)(n-2)(n-3)(n-4) + 24(2n-3)|} =$$

$$= \frac{|\gamma|^{\left[\frac{n}{5}\right]} \left(|Q_0 A_0| \left| \left[\frac{n-9}{2}\right] + 1 \right| + 1 \right)}{|(n-1)(n-2)(n-3)(n-4) + 24(2n-3)|}.$$

Let's find the radius of convergence of the analytic part of the series (2.1), taking into account the obtained estimate (7):

$$|z^* - z|^5 < \frac{1}{|\gamma|} \Rightarrow |z^* - z| < \frac{1}{\sqrt[5]{|\gamma|}}.$$

Taking into account the existing estimates for the series coefficients, we can write the analytical approximate solution:

$$w_N(z) = (z^* - z)^{-1} \sum_{n=0}^{N} A_n (z^* - z)^n, \tag{9}$$

in the domain $|z^* - z| < \rho_2$.

Next, let's proceed to estimate the error of the analytical approximate solution.

Theorem 3. *Assume that all conditions of Theorem 1 and Theorem 2 be satisfied, then the analytical approximate solution* (9) *of the Cauchy problem* (2) — (3)*, for sufficiently large values of* N*, has the following error estimates:*

$$\Delta w_N \leq \sum_{k=0}^{4} \frac{|\gamma|^{N+1+k} \left(|Q_0 A_0| \left| \left[\frac{5(N+1)+k-9}{2}\right] + 1 \right| + 1 \right) |z^* - z|^{N+k}}{\left| \prod_{i=1}^{4} (5(N+1)+k-i) + 24((2(5(N+1)+k)-3)) \right|} \times \frac{1}{1 - |\gamma \cdot (z^* - z)|^5}. \tag{10}$$

Proof. Let's use the triangle inequality:

$$\Delta w_N = \left| \sum_{n \geq 0} A_n (z^* - z)^{n-1} - \sum_{n=0}^{N} A_n (z^* - z)^{n-1} \right| = \left| \sum_{n \geq N+1} A_n (z^* - z)^{n-1} \right|$$

Considering that $\left|\sum a_i\right| \leq \sum |a_i|$, $\forall a_i \in \mathbb{C}$, we have:

$$\Delta w_N = \left|\sum_{n\geq N+1} A_n \left(z^* - z\right)^{n-1}\right| \leq \sum_{n\geq N+1} |A_n| \left|z^* - z\right|^{n-1}$$

$$\leq \sum_{n\geq N+1} \frac{|\gamma|^{\left[\frac{n}{5}\right]}\left(|Q_0 A_0| \left|\left[\frac{n-9}{2}\right]+1\right|+1\right)}{\left|(n-1)(n-2)(n-3)(n-4)+24(2n-3)\right|} \left|z^* - z\right|^{n-1}$$

$$\leq \sum_{n\geq N+1} \sum_{k=0}^{4} \frac{|\gamma|^{n+k}\left(|Q_0 A_0| \left|\left[\frac{5n+k-9}{2}\right]+1\right|+1\right)}{\left|\prod_{i=1}^{4}(5n+k-i)+24(2(5n+k)-3)\right|} \left|z^* - z\right|^{n+k-1}$$

$$\leq \sum_{k=0}^{4} \frac{|\gamma|^{(N+1)+k}\left(|Q_0 A_0| \left|\left[\frac{5(N+1)+k-9}{2}\right]+1\right|+1\right) \left|z^* - z\right|^{5(N+1)+k-1}}{\left|\prod_{i=1}^{4}(5(N+1)+k-i)+24(2(5(N+1)+k)-3)\right|}$$

$$\times \frac{1}{1-\left|\gamma \cdot (z^* - z)\right|^5}.$$

2.2. Numerical experiment

Let's consider the Cauchy problem (2) — (3) with a specific example, when $Q_0 = -4$ and $f(z) = sin(z)$ with given initial conditions:

$$w^{(4)} - 4w\left(w'\right)^2 = \sin z \tag{11}$$

$$\begin{cases} w(0) = 0.2 \\ w'(0) = 0.3 \\ w''(0) = 0 \\ w'''(0) = 0 \end{cases} \tag{12}$$

The coefficients of the series expansion (2.1) for the solution to the Cauchy problem (11) — (12) take the following form:

$$\begin{cases} A_0 = \sqrt{6} \\ A_1 = A_2 = A_3 = A_4 = 0 \\ A_5 = \frac{1}{192} \\ A_6 = -\frac{1}{2016} \\ A_7 = \frac{1}{40320} \\ \cdots\cdots\cdots \end{cases}$$

The estimate for the coefficients of the series expansion (2.1) according to Theorem 2 for the Cauchy problem (11) — (12) is given by:

$$|A_n| \le \frac{4\sqrt{6}\left|\left[\frac{n-9}{2}\right]+1\right|+1}{|(n-1)(n-2)(n-3)(n-4)+24(2n-3)|}.$$

Next, according to Theorem 3, we determine the error estimate of the analytical approximate solution:

$$\Delta w_N \le \sum_{k=0}^{4} \frac{\left(4\sqrt{6}\left|\left[\frac{5(N+1)+k-9}{2}\right]+1\right|+1\right)|z^*-z|^{5(N+1)+k-1}}{\left|\prod_{i=1}^{4}(5(N+1)+k-i)+24(2(5(N+1))+k-3)\right|}\frac{1}{1-|z^*-z|^5}.$$

The value of the moving singular point is $z^* = 3.513$, and the convergence radius is $\rho = 1$. For a solution error of $\varepsilon \approx 10^{-7}$ at the point $z_1 = 3.2$, according to Theorem 3, it is sufficient to take $N = 7$. In this case, the analytical approximate solution is given by:

$$w_7(z) = \sqrt{6}(3.513-z)^{-1} + \frac{1}{192}(3.513-z)^4 - \frac{1}{2016}(3.513-z)^5 + \frac{1}{40320}(3.513-z)^6.$$

Below is a table of characteristics and a graph comparing the solution of the Cauchy problem (11) — (12) obtained using the analytical approximate method proposed by the authors and the solution obtained numerically using the Runge-Kutta method.

Table 1. Numerical characteristics of the analytical approximate solution.

z_1	$w_7(z_1)$	Δ_1
3.2	7.6125622	10^{-7}

2.3. Exact criteria for the existence of a moving singular point in the real and complex domains

This section is dedicated to criteria for the existence of a moving singular point. Both point and interval criteria are considered. Point criteria only guarantee the existence of a moving singular point, while interval criteria allow determining their location. To define such criteria in the complex plane, a transition to phase spaces is necessary.

Before formulating the theorems, it is necessary to transform the Cauchy problem (2) — (3) into the inverse Cauchy problem by a change of variable $w(z) = \frac{1}{u(z)}$:

$$w' = -u^{-2}u',$$

$$w'' = 2u^{-3}(u')^2 - u^{-2}u'',$$

$$w''' = -6(u')^3u^{-4} + 6u'u''u^{-3} - u'''u^{-2},$$

$$w^{(4)} = 24(u')^4u^{-5} - 36(u')^2u''u^{-4} + 6(u'')^2u^{-3} + 3u'u'''u^{-3} - u^{(4)}u^{-2}.$$

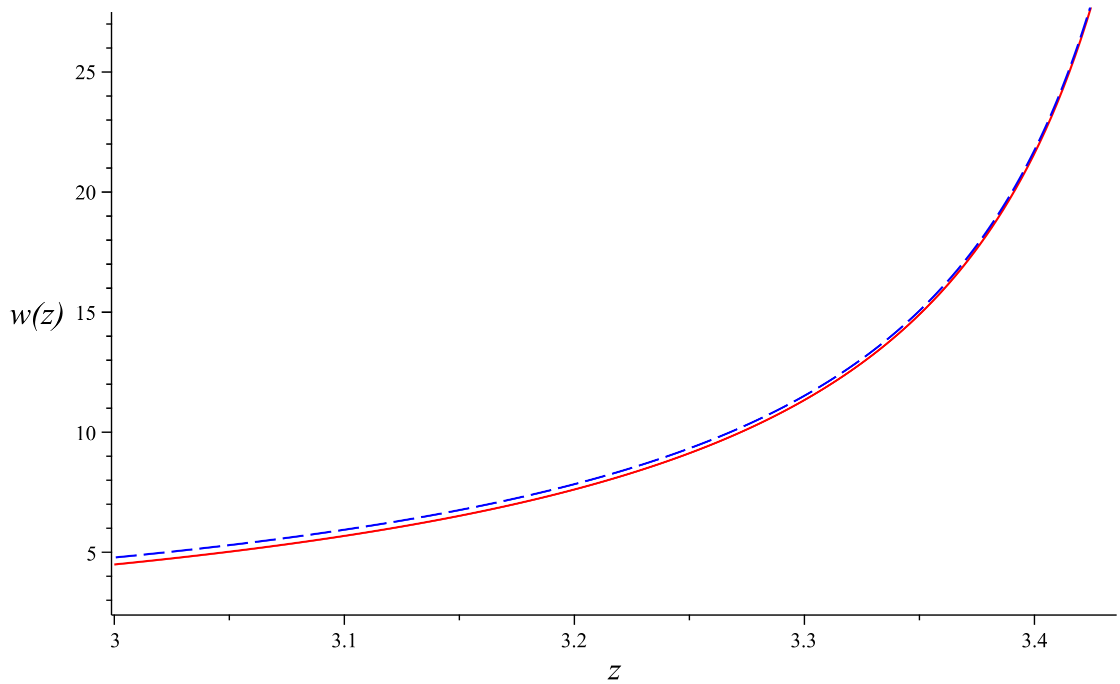

Figure 1: The solution to the Cauchy problem (11) — (12) the analytical approximate method (dashed blue line) and the Runge-Kutta method (solid red line)

Thus, the inverse Cauchy problem will have the form:

$$-u^{(4)}u^3+33u'u'''u^2+6\left(u''\right)^2u^2-36\left(u'\right)^2u''u \\ +24\left(u'\right)^4-Q_0u'u^2-u^5f\left(z\right)=0 \quad (13)$$

$$\begin{cases} u\left(z_0\right)=u_0, \\ u'\left(z_0\right)=u_0', \\ u''\left(z_0\right)=u_0'', \\ u'''\left(z_0\right)=u_0'''. \end{cases} \quad (14)$$

Since (13) — (14) is the inverse Cauchy problem, the solution $u\left(z\right)$ will take a regular value at the point z^*, namely, it will be equal to zero.

Let's proceed to formulate the criteria for the existence of a moving singular point. To begin with, let's formulate them for the real domain.

The following lemma helps to take into account the specifics of the regularization method when constructing an algorithm for finding a moving singular point.

Lemma 1. *Let the function be $u\left(z\right)$ does not change the sign on some segment $[a,b]$. Then in order for $w\left(z\right)$ to reach a local maximum at point $c\in(a,b)$, it is necessary and sufficient that at this point $u\left(z\right)$ had a local minimum.*

Proof. The proof obviously follows from classical analysis, a necessary and sufficient condition for a local extremum.

Theorem 4. *If z^* is a movable singular point of the Cauchy problem* (2) — (3)*, and the function $w(z)$ defined on the half-interval $[z_0; z^*)$. Then there is a number γ, such that the function $w(z)$ on the half-interval $[\gamma; z^*)$ has the following property:*

$$\left[\begin{array}{l} w(z), w'(z), w''(z), w'''(z) > 0, \\ w(z), w'(z), w''(z), w'''(z) < 0. \end{array}\right.$$

Proof.

Consider formula (2.1) obtained in Theorem 1. According to the existence theorem $\exists \xi : \xi \in [z_0; z^*)$ such that the analytic part of (2.1) converges in the area $[\xi; z^*)$, then we get:

$$w(z) = \sqrt{-\frac{24}{Q_0}}(z^* - z)^{-1} + \frac{D_0}{192}(z^* - z)^4 + \frac{D_1}{336}(z^* - z)^5 + \dots$$

Since

$$\frac{1}{\sqrt{Q_0}}(z^* - z)^{-1} \overset{z \to z^* - 0}{\to} +\infty,$$

and

$$\left(\frac{D_0}{192}(z^* - z)^4 + \frac{D_1}{336}(z^* - z)^5 + \dots\right) \overset{z \to z^* - 0}{\to} 0,$$

there is such a point $\xi_1 \geq \xi : \forall z \in [\xi_1; z^*)$ the condition $w > 0$ is fulfilled.

Similarly, in the case of the derivative:

$$w'(z) = \sqrt{-\frac{24}{Q_0}}(z^* - z)^{-2} + \frac{D_0}{48}(z^* - z)^3 + \frac{5D_1}{336}(z^* - z)^4 + \dots$$

Since

$$\sqrt{-\frac{24}{Q_0}}(z^* - z)^{-2} \overset{z \to z^* - 0}{\to} +\infty,$$

and

$$\left(\frac{D_0}{48}(z^* - z)^3 + \frac{5D_1}{336}(z^* - z)^4 + \dots\right) \to 0,$$

there is such a point $\xi_1 \geq \xi : \forall z \in [\xi_1; z^*)$ the condition $w' > 0$ is fulfilled. Similarly for w'', w'''.

Theorem 5. *Let $z(u)$ be the inverse function to the solution of the inverse Cauchy problem* (13) — (14)*, and the following conditions hold: $z(0) = z^*$, $z'(0) = -\sqrt{-\frac{Q_0}{24}}$, $z''(0) = z'''(0) = 0$, then z^* is a moving singular point of the Cauchy problem* (2) — (3)*. This condition is necessary and sufficient.*

Proof. **Necessity.** Let z^* be a moving singular point of the Cauchy problem (2) — (3), then the previously proven theorems 1, 2, 3 hold. With the substitution $w(z) = \frac{1}{u(z)}$, it is evident that $u(z^*) = 0$. Using this substitution to transition to the inverse equation, we get:

$$u(z) = \frac{1}{(z^* - z)^{-1} \sum_{n\geq 0} A_n (z^* - z)^n} = \sum_{n\geq 0} \tilde{A}_n (z^* - z)^{n+1}, \tag{15}$$

where $\tilde{A}_0 = \frac{1}{A_0}$, $\tilde{A}_1 = \tilde{A}_2 = \tilde{A}_3 = \tilde{A}_4 = 0$.

Based on the theorem on the inversion of series [11], we obtain the following equality:

$$z^* - z(u) = \sum_{n\geq 0} B_n u^{n+1},\ B_0 = \frac{1}{A_0},\ B_1 = 0,\ B_2 = 0. \tag{16}$$

From equality (16), it is evident that $z(0) = z^*$. Further, by differentiating (16), we get:

$$z'(u) = -\left(B_0 u + B_3 u^4 + \ldots\right)' = -B_0 - 4B_3 u^3 - \ldots \tag{17}$$

From (17), it follows that $z'(0) = -B_0 = -\frac{1}{A_0} = -\sqrt{-\frac{Q_0}{24}}$. Differentiating (17), we obtain:

$$z''(u) = \left(-B_0 - 4B_3 u^3 - \ldots\right)' = -12 B_3 u^2 - \ldots \tag{18}$$

From (18), we get that $z''(0) = 0$. Following a similar algorithm as before, we obtain $z'''(0) = 0$.

Sufficiency. Let $z(u)$ be the inverse function to the solution of the inverse Cauchy problem (13) — (14), and the following conditions hold: $z(0) = z^*$, $z'(0) = -\sqrt{-\frac{Q_0}{24}}$, $z''(0) = z'''(0) = 0$. Let's prove that z^* is a moving singular point of the Cauchy problem (2) — (3).

Since $u(z)$ is an analytic function in the vicinity of the point z^*, its inverse is also analytic in this domain and can be expanded into a Taylor series:

$$z(u) = \sum_{n\geq 0} D_n u^n \tag{19}$$

Given that $z(0) = z^*$ and the existing expansion (19), we obtain $D_0 = z^*$. Further, by differentiating equality (19), we get:

$$z'(u) = \sum_{n\geq 1} n D_n u^{n-1},$$

thus, taking into account the condition $z'(0) = -\sqrt{-\frac{Q_0}{24}}$, we obtain $D_1 = -\sqrt{-\frac{Q_0}{24}}$.

Similarly, we find $D_2 = D_3 = 0$. Thus, equality (19) takes the form:

$$z(u) = z^* - \sqrt{-\frac{Q_0}{24}} u + D_4 u^4 + \ldots,$$

$$z^* - z(u) = \sqrt{-\frac{Q_0}{24}u - D_4 u^4 + \dots} \tag{20}$$

Based on the theorem on the inversion of series [11], we obtain:

$$u(z) = \sqrt{-\frac{Q_0}{24}}(z^* - z) - \tilde{D}_4 (z^* - z)^4 + \dots \tag{21}$$

Taking into account the relationship between the solutions of the Cauchy problems (2) — (3) and (13) — (14), we have:

$$w(z) = \frac{1}{u(z)} = \frac{1}{\sqrt{-\frac{Q_0}{24}}(z^* - z) + \dots}$$
$$= \sqrt{-\frac{24}{Q_0}}(z^* - z)^{-1} + C_1 (z^* - z) + C_2 (z^* - z)^3 + \dots,$$

Thus, z^* is a moving singular point of algebraic type of the Cauchy problem (2) — (3).

Theorem 6. *The fact that z^* is a moving singular point of the function $w(z)$ is equivalent to the existence of a certain neighborhood of this point in which the function $u(z)$ is continuous and has different signs at the endpoints of this interval.*

Proof. **Necessity.** Given that $u(z)$ is the inverse function, then $u(z^*) = 0$, and the function is continuous in some neighborhood of this point. Considering that $u(z) = \sqrt{-\frac{Q_0}{24}}(z^* - z) + o(z^* - z)$, when crossing the moving singular point, the function changes its sign. Thus, there exists an interval on the ends of which the function takes values of different signs.

Sufficiency. Due to the continuity of the function $u(z)$ and the different signs at the ends of a certain interval, according to the Bolzano-Cauchy theorem $\exists \xi : u(\xi) = 0$. Taking into account the relation $w(z) = \frac{1}{u(z)}$, we obtain that ξ is a moving singular point of the solution to the Cauchy problem (2) — (3).

Next, let's proceed to the formulation of the exact criteria for the existence of a moving singular point in the complex domain. For the complex domain, these criteria are related to the specifics of transitioning to phase spaces.

Let's express the solution to the inverse Cauchy problem (13) — (14) as $u(z) = P(x, y) + iQ(x, y)$ where the functions $P(x, y)$ and $Q(x, y)$ are characterized, respectively, by the phase spaces $\Phi_1(x, y, P(x, y))$ and $\Phi_2(x, y, Q(x, y))$.

Let's use the terminology introduced in the work [32] to define correct and incorrect lines to facilitate the statement of the following theorems.

Theorem 7. *For z^* to be a moving singular point of algebraic type of the solution $w(z)$ of the Cauchy problem (2) — (3), it is necessary and sufficient that for $Re(u(z))$ and $Im(u(z))$, where the function $u(z)$ is the solution to the inverse Cauchy problem (5) — (6), in some region G, which is the neighborhood of the regular point $z^*(x^*, y^*)$ of the function $u(z)$, the phase spaces Φ_1 and Φ_2 satisfy the following conditions:*

(i) $P(x,y)$ *and* $Q(x,y)$ *are continuous with respect to their arguments;*

(ii) when crossing the point $z^*(x^*,y^*)$*, moving along the regular line* l *in the direction of the axes* Ox *and* Oy*, the functions* $P(x,y)$ *and* $Q(x,y)$ *change signs, where* $l:\{z^* \in l \subset G,\ l \in (\Phi_1 \cup \Phi_2)\}$.

Proof. **Necessity.** According to the theorem statement, we have that z^* is a moving singular point of $y(z)$ of the Cauchy problem (2) — (3). Let's demonstrate that in this case $Re(w(z))$ and $Im(w(z))$ satisfy Theorem 7.

Since Theorem 1 holds, the principal part of the series representing the solution to the Cauchy problem (2) — (3) takes the form $w(z) = O\left({A_0}/{z^* - z}\right)$, Therefore, for the inverse solution in the region G, we can assert that $u(z) = o\left({z^* - z}/{A_0}\right)$. In this situation, we have the following:

$$sgn(P(x,y)) = sgn((x^* - x)) \tag{22}$$

$$sgn(Q(x,y)) = sgn((y^* - y)). \tag{23}$$

Analysis of the analytic part $u(z)$: the sign of the function $P(x,y)$ is determined by the sign of x, while the sign of the function $Q(x,y)$ is determined by the sign of y.

Without loss of generality, let's assume that the moving singular point z^* is located in the first quadrant of the phase plane. As a correct line, we can consider a segment of the circle $|z| = |z^*|$ in the region G. Moving along this circle in the direction of the correct line l, we observe that for points $z \in l : \arg z < \arg z^* \Rightarrow \begin{cases} x > x^* \\ y < y^* \end{cases}$ and for points $z \in l : \arg z > \arg z^* \Rightarrow \begin{cases} x < x^* \\ y > y^* \end{cases}$, therefore, the imaginary and real parts of the function $u(z)$ are continuous functions with respect to the arguments, and they change their sign when passing through the point z^*. Since, without loss of generality, only the first quadrant was considered, the necessity is similarly proven in the other quadrants.

Theorem 8. *In order for* z^* *to be a moving singular point of the function* $w(z)$*, which is a solution to the Cauchy problem* (2) — (3)*, it is necessary and sufficient for the imaginary and real parts of* $u(z)$ *in some sufficiently small neighborhood* G *of the point* z^* *in the phase spaces* Φ_1 *and* Φ_2*, to be continuous functions with respect to their arguments and to change their signs when passing through the point* $z^*(x^*,y^*)$*, moving sequentially along certain incorrect lines* $l_1, l_2 : \{z^* \in l_1 \subset G, z^* \in l_2 \subset G, l_1 \in \Phi_1, l_2 \in \Phi_2\}$.

Proof. **Necessity.** Similar to Theorem 7, we have the principal part of the series, which is the solution to the Cauchy problem (2) — (3): $w(z) = O\left({A_0}/{z^* - z}\right)$, and the analytic part of the inverse function $u(z) = o\left({z^* - z}/{A_0}\right)$.

Let's consider the line $l_1 : y = y^* = const$ — an incorrect line with respect to the Ox axis. Moving along this line, taking into account the signs of the arguments (9) and the theorem of existence and uniqueness of the solution, $u(z)$ as a function of a single variable changes sign when crossing the point z^*.

Similarly, we consider the line $l_2 : x = x^* = const$, which is an incorrect line with respect to the Oy axis, which completes the proof of the necessity.

Sufficiency. Based on the theorem, we conclude that the imaginary and real parts of the function $u(z)$ are continuous functions with respect to their arguments in some neighborhood of the point z^* in the phase spaces Φ_1 and Φ_2, and change their signs when crossing the point $z^*(x^*, y^*)$, moving sequentially along incorrect lines l_1 and l_2 in the direction of the corresponding axes $l_1, l_2 : \{z^* \in l_1 \subset G, z^* \in l_2 \subset G\,, l_1 \in \Phi_1, l_2 \in \Phi_2\}$. Utilizing this fact, we obtain that $u(z^*) = 0$, and the function $w(z) = P(x, y) + iQ(x, y)$ can be represented as:

$$u(z) = \sum_{n\geq 0} \tilde{A}_n (z^* - z)^{n+1}.$$

Taking into account the introduced substitution $u(z) = \frac{1}{w(z)}$, we get

$$w(z) = (z^* - z)^{-1} \sum_{n\geq 0} A_n (z^* - z)^n.$$

3. Conclusion

In this paper, we considered a nonlinear fourth-order differential equation with a movable singular point of algebraic type. The author proposed an analytical approximate solution method based on splitting the solution search into the area of analyticity and the neighborhood of a moving singular point. In this paper, the theorem of existence and uniqueness in the vicinity of a moving singular point was formulated and proved. The solution obtained during the proof has a simple pole at the point z^*. Using the modified majorant method, estimates for the coefficients were obtained, and as a result, the convergence domain of the solution under consideration. The second task of the study was to formulate the necessary, as well as necessary and sufficient conditions for the existence of a mobile singular point in both the real and complex domains. The main idea of solving this problem was the regularization of a moving singular point. The theoretical results were tested in a numerical experiment. The results of the numerical experiment were compared with existing numerical methods. The analytical approximate method used by the author can be applied to other classes of equations, as it was previously indicated in the introduction. This method complements the existing methods for solving nonlinear differential equations, such as asymptotic, exact methods and others. This work can be utilized to formulate an algorithm for finding a movable singular point with a specified accuracy, by combining the obtained results with numerical methods [35].

References

[1] I. Astashova. On asymptotic classification of solutions to nonlinear regular and singular third- and fourth-order differential equations with power nonlinearity. *Springer proceedings in mathematics & statistics*, page 191–203, 2016.

[2] I. Astashova, M. Bartušek, Z. Došlá, and M. Marini. Asymptotic proximity to higher order nonlinear differential equations. *Advances in Nonlinear Analysis*, pages 1598–1613, 2022.

[3] E. Az-Zo'bi, K. Al-Khaled, and A. Darweesh. Numeric-analytic solutions for nonlinear oscillators via the modified multi-stage decomposition method. *Mathematics*, 2019.

[4] F. Bernal-Vílchis, N. Hayashi, and P. Naumkin. Quadratic derivative nonlinear schrödinger equations in two space dimensions. *Nonlinear Differential Equations and Applications*, 18:329–355, 2011.

[5] G. Bonanno and B. Di Bella. A boundary value problem for fourth-order elastic beam equations. *Journal of Mathematical Analysis and Applications*, 55:1166–1176, 2008.

[6] G. Bonanno, B. Di Bella, and D. O'Regan. Non-trivial solutions for nonlinear fourth-order elastic beam equations. *Computers & Mathematics with Applications*, page 1862–1869, 2011.

[7] V. Chandrasekar, M. Senthilvelan, and M. Lakshmanan. On the complete integrability and linearization of certain second-order nonlinear ordinary differential equations. *Proceedings of The Royal Society A: Mathematical, Physical and Engineering Sciences*, page 2451–2477, 2005.

[8] W. Durand. Aerodynamic theory. *Springer*, 2013.

[9] M. Galewski. On the nonlinear elastic simply supported beam equation. *Analele Stiintifice ale Universitatii Ovidius Constanta, Seria Matematica*, page 109–120, 2011.

[10] M. Galewski and J. Smejda. A note on a fourth order discrete boundary value problem. *Opuscula Mathematica*, pages 115–123, 2012.

[11] V. Golubev. *Lectures on Analytical Theory of Differential Equations*. Gostekhizdat, Moscow, 1950.

[12] V. Grigoryan and A. Tanguay. Improved well-posedness for the quadratic derivative nonlinear wave equation in 2d. *Journal of Mathematical Analysis and Applications*, 475, 2019.

[13] T. Harkó, S. Lobo, and M. Mak. A class of exact solutions of the liénard-type ordinary nonlinear differential equation. *Journal of Engineering Mathematics*, page 193–205, 2014.

[14] H. Hirayama, S. Kinoshita, and M. Okamoto. Well-posedness for a system of quadratic derivative nonlinear schrödinger equations in almost critical spaces. *Journal of Mathematical Analysis and Applications*, 499, 2021.

[15] M. Fontelos J. Eggers. Singularities: Formation, structure, and propagation. *Cambridge University Press*, 2015.

[16] R. Jungers and P. Tabuada. Non-local linearization of nonlinear differential equations via polyflows. *2019 American Control Conference*, pages 1–6, 2019.

[17] A. Khanfer and L. Bougoffa. On the fourth-order nonlinear beam equation of a small

deflection with nonlocal conditions. *AIMS Mathematics*, page 9899–9910, 2021.

[18] N. Kudryashov. *Analitical theory of nonlinear differential equations.* Institute of Computer Research, Moskow - Igevsk, 2004.

[19] N. Kudryashov. Nonlinear differential equations with exact solutions expressed via the weierstrass function. *Zeitschrift für Naturforschung A*, pages 443–454, 2004.

[20] T. Leont'eva. About one generalization of exact criteria for the existence moving singular points of one class of nonlinear ordinary differential equations in the complex area. *Belgorod State Univ. Sci. Bull. Math. Phys.*, page 51–57, 2017.

[21] D. Lyakhov, V. Gerdt, and D. Michels. On the algorithmic linearizability of nonlinear ordinary differential equations. *Journal of Symbolic Computation*, page 3–22, 2020.

[22] R. Ma, J. Li, and C. Gao. Existence of positive solutions of a discrete elastic beam equation. *Discrete Dynamics in Nature and Society*, page 1–15, 2010.

[23] K. Maekawa, H. Okamura, and A. Pimanmas. Nonlinear mechanics of reinforced concrete. *Boca Raton, Fl: Crc Press*, 2019.

[24] S. Mancas and H. Rosu. Integrable dissipative nonlinear second order differential equations via factorizations and abel equations. *Physics Letters*, page 1434–1438, 2013.

[25] K. Mohsen, Y. Khalili, and R. Wieteska. Existence of two solutions for a fourth-order difference problem with p(k) exponent. *Afrika Matematika*, page 959–970, 2020.

[26] W. Nakpim. Linearization of second-order ordinary differential equations by generalized sundman transformations. *Symmetry, Integrability and Geometry: Methods and Applications*, 2010.

[27] V. Orlov. Dependence of the analytical approximate solution to the van der pol equation on the perturbation of a moving singular point in the complex domain. *Axioms*, 2023.

[28] V. Orlov. Moving singular points and the van der pol equation, as well as the uniqueness of its solution. *Mathematics*, 2023.

[29] V. Orlov and A. Chichurin. About analytical approximate solutions of the van der pol equation in the complex domain. *Fractal and fractional*, 2023.

[30] V. Orlov and A. Chichurin. The influence of the perturbation of the initial data on the analytic approximate solution of the van der pol equation in the complex domain. *Symmetry*, 2023.

[31] V. Orlov and M. Gasanov. Analytic approximate solution in the neighborhood of a moving singular point of a class of nonlinear equations. *Axioms*, 2022.

[32] V. Orlov and M. Gasanov. Exact criteria for the existence of a moving singular point in a complex domain for a nonlinear differential third-degree equation with a polynomial seventh-degree right-hand side. *Axioms*, 2022.

[33] V. Orlov and M. Gasanov. Existence and uniqueness theorem for a solution to a class of a third-order nonlinear differential equation in the domain of analyticity. *Axioms*, 2022.

[34] V. Orlov and M. Gasanov. The influence of a perturbation of a moving singular point on the structure of an analytical approximate solution of a class of third-order nonlinear differential equations in a complex domain. *Herald of the Bauman Moscow*

State Tech. Univ., Nat. Sci., page 60–76, 2022.

[35] V. Orlov and M. Gasanov. Technology for obtaining the approximate value of moving singular points for a class of nonlinear differential equations in a complex domain. *Mathematics*, 2022.

[36] V. Orlov and M. Gasanov. The maximum domain for an analytical approximate solution to a nonlinear differential equation in the neighborhood of a moving singular point. *Axioms*, 2023.

[37] V. Orlov and T. Leontieva. On the expansion of the domain for an analytical approximate solution of one class of second-order nonlinear differential equations in the complex domain. *J. Samara State Tech. Univ., Ser. Phys. Math. Sci.*, page 174–186, 2020.

[38] S. Pandey, P. Bindu, M. Senthilvelan, and M. Lakshmanan. A group theoretical identification of integrable cases of the liénard-type equation $\ddot{x} + f(x)\dot{x} + g(x) = 0$ i. equations having nonmaximal number of lie point symmetries. *Journal of Mathematical Physics*, 2009.

[39] S. Pandey, P. Bindu, M. Senthilvelan, and M. Lakshmanan. A group theoretical identification of integrable cases of the liénard-type equation $\ddot{x} + f(x)\dot{x} + g(x) = 0$ ii. equations having nonmaximal number of lie point symmetries. *Journal of Mathematical Physics*, 2009.

[40] A. Pchelova. Construction of approximate solutions for a class of first-order nonlinear differential equations in the analyticity region. *Herald of the Bauman Moscow State Tech. Univ., Nat. Sci.*, 2016.

[41] K. P. V. Preethi, H. Alotaibi, and J. Visuvasam. Analysis of amperometric biosensor utilizing synergistic substrates conversion: Akbari-ganji's method. *Mathematical Modelling and Control*, pages 350–360, 2024.

[42] R. Shanthi, T. Iswarya, J. Visuvasam, L. Rajendran, and Michael E.G. Lyons. Voltammetric and mathematical analysis of adsorption of enzymes at rotating disk electrode. *International Journal of Electrochemical Science*, 2022.

[43] E. Thailert and S. Suksern. Linearizability of nonlinear third-order ordinary differential equations by using a generalized linearizing transformation. *Journal of Applied Mathematics*, page 1–12, 2014.

[44] O. Urszula, E. Schmeidel, and M. Zdanowicz. Existence of solutions to nonlinear fourth-order beam equation. *Qualitative Theory of Dynamical Systems*, 2023.

[45] J. Visuvasam and H. Alotaibi. Analysis of von kármán swirling flows due to a porous rotating disk electrode. *Micromachines*, 2023.

[46] J. Visuvasam, A. Meena, and L. Rajendran. New analytical method for solving nonlinear equation in rotating disk electrodes for second-order ece reactions. *Journal of Electroanalytical Chemistry*, 2020.

[47] N. Vitanov. Simple equations method (sesm): An effective algorithm for obtaining exact solutions of nonlinear differential equations. *Entropy*, 2022.

[48] Q. Yao. Positive solutions of a nonlinear elastic beam equation rigidly fastened on the left and simply supported on the right. *Nonlinear Analysis: Theory, Methods & Applications*, 2008.

[49] V. Zaitsev and A. Polyanin. *Handbook of Exact Solutions for Ordinary Differential Equations*. CRC Press, New York, 2002.

[50] P. Řehák. On asymptotic relationships between two higher order dynamic equations on time scales. *Applied Mathematics Letters*, page 84–90, 2017.