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Bounded q-variation Functions in Spaces with Indefinite Metric

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Abstract. In this paper we establish the definition of bounded q-variation functions on Krein spaces (definition 7) and give a complete characterisation by comparing it with bounded q-variation functions on classical Hilbert spaces and with Hilbert spaces associated to a Krein space (theorem 6). The fundamental tools of the theory of bounded q-variation functions in the formalism of Krein spaces are described (theorems 7, 8). Furthermore, we endow the spaces of bounded q-variation functions in Krein spaces with a norm, which we have called q-norm (theorem 13).

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Key Words and Phrases: Krein space, fundamental symmetry, q-variation, total q-variation, bounded q-variation, q-norm.

1. Introduction

In 1881, Jordan discovered in the work of Dirichlet the notion of function of bounded variation and proves that for this class of functions the Fourier conjecture is valid. He also shows that the function $f : [a, b] \to \mathbb{R}$ has bounded variation on $[a, b]$ if and only if f is the difference of monotone functions (nowadays this result is known as Jordan's Representation Theorem). The notion of function of bounded variation has been studied and generalized in different contexts, studying different structures and properties in spaces with research interest. For example, Chistyakov in [1–3] studied a concept of a function of bounded generalized variation in the sense of Jordan-Riesz-Orlicz for functions $f : [a, b] \to X$, where X is a normed or metric space. More recently, the notion of functions of bounded variation in spaces with indefinite metric was introduced by Ferrer, Guzmán and Naranjo in [4, 5]. Furthermore, functions of bounded variation have multiple applications in various fields, e.g. in image processing, Brokman, Burger and Gilboa in [6] presented

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an analysis of the total-variation (TV) on non-Euclidean parametrized surfaces, a natural representation of the shapes used in 3D graphics, see [7]. Among the results achieved in this research is a new way to generalize the convexity of sets from the plane to surfaces is derived by characterizing the TV eigenfunctions on surfaces. Additionally, Bugajewska, Bugajewski and Hudzik in [8] they investiged solutions of nonlinear Hammerstein and Volterra−Hammerstein integral equations in the space of functions of bounded ϕ-variation in the sense of Young. Later, Bugajewska, Bugajewski and Lewicki in [9] explored with the superposition operator as well as with solutions to non-linear integral equations in spaces of functions of generalized bounded ϕ-variation. In the field of nonlinear analysis, Xie, Liu, Li and Huang in [10] they examine the bounded variation capacity (BV capacity) and characterise the Sobolev-type inequalities associated with BV functions within a general framework of strictly local Dirichlet spaces with a doubling measure, utilising the BV capacity.

2. Preliminaries

The following is a type of spaces with indefinite metric, called Krein spaces, which are a generalisation of Hilbert spaces.

Definition 1. *[11, 12] A space $\mathcal{K}$ with an indefinite inner product $[\cdot,\cdot]$ which admits a fundamental decomposition of the form $\mathcal{K}=\mathcal{K}^+[\dot{+}]\mathcal{K}^-$ such that $(\mathcal{K}^+,[\cdot,\cdot])$ and $(\mathcal{K}^-,-[\cdot,\cdot])$ are Hilbert spaces, is called a* ***Krein space****.*

Definition 2. *Let $(\mathcal{K},[\cdot,\cdot])$ be a Krein space with fundamental decomposition $\mathcal{K}=\mathcal{K}^+[\dot{+}]\mathcal{K}^-$, then we define the operator $\mathcal{J}:(\mathcal{K},[\cdot,\cdot])\longrightarrow(\mathcal{K},[\cdot,\cdot])$*

$$\mathcal{J}k=k^+-k^-,$$

it is called the ***fundamental symmetry*** *of the Krein space $\mathcal{K}$ associated with the fundamental decomposition. From now on we will write $(\mathcal{K},[\cdot,\cdot],\mathcal{J})$ to denote the Krein space with fundamental symmetry $\mathcal{J}$ associated with the fundamental decomposition $\mathcal{K}=\mathcal{K}^+[\dot{+}]\mathcal{K}^-$.*

Remark 1. *Let $(\mathcal{K}=\mathcal{K}_+[\dot{+}]\mathcal{K}_-,[\cdot,\cdot],\mathcal{J})$ be a Krein space, and $f:[a,b]\to\mathcal{K}=\mathcal{K}_+[\dot{+}]\mathcal{K}_-$. Considering that for any t in $[a,b]$, $f(t)$ belongs to $\mathcal{K}=\mathcal{K}_+[\dot{+}]\mathcal{K}_-$, we will henceforth write the image of t under f as $f(t)=f^+(t)+f^-(t)$, for any t in $[a,b]$.*

Remark 2. *Let $(\mathcal{K}=\mathcal{K}_+[\dot{+}]\mathcal{K}_-,[\cdot,\cdot],\mathcal{J})$ be a Krein space, and $f:[a,b]\to\mathcal{K}=\mathcal{K}_+[\dot{+}]\mathcal{K}_-$. Considering that for any t in $[a,b]$, $f(t)\in\mathcal{K}$, therefore there exist $k^+\in\mathcal{K}_+$ and $k^-\in\mathcal{K}_-$ such that $f(t)=k^++k^-$, we will write $k^+=f^+(t)$ and $k^-=f^-(t)$. Therefore,*

$$(\mathcal{J}f)(t)=\mathcal{J}(f(t))=\mathcal{J}(k^++k^-)=k^+-k^-=f^+(t)-f^-(t).$$

Definition 3. *[11] Let $(\mathcal{K}=\mathcal{K}^+[\dot{+}]\mathcal{K}^-,[\cdot,\cdot])$ be a Krein space and $\mathcal{J}$ the fundamental symmetry associated to the given decomposition. We define the function $[\cdot,\cdot]_{\mathcal{J}}:\mathcal{K}\times\mathcal{K}\longrightarrow\mathbb{C}$ by*

$$[x,y]_{\mathcal{J}}=[\mathcal{J}x,y],\quad \textit{for all}\ \ x,y\ \textit{in}\ \mathcal{K}.$$

This function is a usual inner product and is called $\mathcal{J}$***-inner product****.*

Definition 4. *[11, 12] The fundamental symmetry* $\mathcal{J}$ *associated with the Krein space* $(\mathcal{K} = \mathcal{K}^+[\dot{+}]\mathcal{K}^-, [\cdot,\cdot])$ *induces a norm in* $\mathcal{K}$ *defined by:*

$$\|x\|_{\mathcal{J}} := \sqrt{[x,x]_{\mathcal{J}}}, \text{ for all } x \textit{ in } \mathcal{K},$$

this norm is called the $\mathcal{J}$***-norm*** *of* $\mathcal{K}$*. Explicitly,*

$$\|x\|_{\mathcal{J}} = ([x^+,x^+] - [x^-,x^-])^{1/2}, \text{ for all } x \textit{ in } \mathcal{K}.$$

Remark 3. *It defines for* x^+ *in* $\mathcal{K}^+$ *and* x^- *in* $\mathcal{K}^-$

$$\|x^+\|_+ = \sqrt{[x^+,x^+]}, \qquad \textit{and} \qquad \|x^-\|_- = \sqrt{-[x^-,x^-]}$$

From now on, the topology studied in Krein spaces is directly related to the $\mathcal{J}$**-norm** of $\mathcal{K}$.

Theorem 1. *[4] Let* $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-, [\cdot,\cdot], \mathcal{J})$ *be a Krein space, then*

$$\|x\|_{\mathcal{J}} \leq \|x^+\|_+ + \|x^-\|_- \textit{ for all } x = x^+ + x^- \textit{ in } \mathcal{K}.$$

Theorem 2. *[11, 12] Let* $(\mathcal{K}, [\cdot,\cdot])$ *be a Krein space and let*

$$\mathcal{K} = \mathcal{K}_1^+[\dot{+}]\mathcal{K}_1^-, \; \mathcal{K} = \mathcal{K}_2^+[\dot{+}]\mathcal{K}_2^-,$$

be two fundamental decompositions. If $\mathcal{J}_1$ *and* $\mathcal{J}_2$ *are the respective fundamental symmetries it follows that* $\|.\|_{\mathcal{J}_1}$ *and* $\|.\|_{\mathcal{J}_2}$ *are equivalent norms.*

Example 1. *Let us consider the vector space* $\mathbb{C}^2$*, with sum, usual product and the mapping* $[\cdot,\cdot] : \mathbb{C}^2 \times \mathbb{C}^2 \longrightarrow \mathbb{C}$ *given by:*

$$[(x_1,y_1),(x_2,y_2)] = x_1\overline{x_2} - y_1\overline{y_2}. \tag{1}$$

The space $(\mathbb{C}^2, [\cdot,\cdot])$ *is a Krein space with fundamental decomposition* $\mathbb{C}^2 = \mathcal{K}^+[\dot{+}]\mathcal{K}^-$*, where* $\mathcal{K}^+ = \{(x,0) : \; x \in \mathbb{C}\}$ *and* $\mathcal{K}^- = \{(0,y) : \; y \in \mathbb{C}\}$*, with fundamental symmetry*

$$\mathcal{J}((x,y)) = \mathcal{J}((x,0) + (0,y)) = (x,-y) \;\; \textit{that determines the } \mathcal{J}\textit{-norm } \; \|\cdot\|_{\mathcal{J}} \textit{ given by}$$

$$\|(x,y)\|_{\mathcal{J}} = [\mathcal{J}(x,y),(x,y)]^{1/2} = (x \cdot \overline{x} - (-y) \cdot \overline{y})^{1/2} = \sqrt{|x|^2 + |y|^2}.$$

In addition, the norms $\|\cdot\|_+ : \mathcal{K}^+ \to \mathbb{R}^+ \cup \{0\}$ *and* $\|\cdot\|_- : \mathcal{K}^- \to \mathbb{R}^+ \cup \{0\}$ *are given by:*

$$\|(x,0)\|_+ = \sqrt{[(x,0),(x,0)]} = \sqrt{|x|^2} = |x|$$

$$\|(0,y)\|_- = \sqrt{[(0,y),(0,y)]} = \sqrt{-y(-\overline{y})} = \sqrt{|y|^2} = |y|$$

3. Functions of bounded variation in Krein spaces

This section presents the fundamental theory of bounded variation functions in Krein spaces, which were introduced by Ferrer, Naranjo and Guzmán [4].

Definition 5. *[4] Let $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-, [\cdot,\cdot])$ be a Krein space and let $f : [a,b] \to \mathcal{K}$ defined in $[a,b]$, we will say that f is strongly of bounded variation in $[a,b]$ on $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-)$ if*

$$V_a^b(f,(\mathcal{K},[\cdot,\cdot])) = \sup_{P\in\mathcal{P}[a,b]}\left\{\sum_{i=1}^{n}(\|f^+(t_i) - f^+(t_{i-1})\|_+ + \|f^-(t_i) - f^-(t_{i-1})\|_-)\right\}$$

is finite. The set of all functions strongly of bounded variation in $[a,b]$ on $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-, [\cdot,\cdot])$ is denoted as follows $BV([a,b],\mathcal{K},[\cdot,\cdot])$

Definition 6. *[4] (**Positive and negative variations of functions in Krein spaces**) Let $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-, [\cdot,\cdot], \mathcal{J})$ be a Krein space, and $f : [a,b] \to \mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-$, with $f(t) = f^+(t) + f^-(t)$, for all t in the interval $[a,b]$. The positive and negative variation of f on $[a,b]$ with respect to $(\mathcal{K}_+,[\cdot,\cdot])$ and $(\mathcal{K}_-,-[\cdot,\cdot])$ respectively, are defined by:*

$$\overset{+}{V}{}_a^b(f,(\mathcal{K}_+,[\cdot,\cdot])) = \sup_{P\in\mathcal{P}[a,b]}\left\{\sum_{i=1}^{n}\|f^+(t_i) - f^+(t_{i-1})\|_+\right\}$$

and

$$\overset{-}{V}{}_a^b(f,(\mathcal{K}_-,-[\cdot,\cdot])) = \sup_{P\in\mathcal{P}[a,b]}\left\{\sum_{i=1}^{n}\|f^-(t_i) - f^-(t_{i-1})\|_-\right\}.$$

Theorem 3. *[4] Let $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-, [\cdot,\cdot], \mathcal{J})$ be a Krein space, $f_+ : [a,b] \to \mathcal{K}_+$ and $f_- : [a,b] \to \mathcal{K}_-$ strongly of bounded variation in the Hilbert spaces $(\mathcal{K}_+,[\cdot,\cdot])$ and $(\mathcal{K}_-,-[\cdot,\cdot])$ respectively, then $f : [a,b] \to \mathcal{K}$ defined as follows $f(t) = f_+(t) + f_-(t)$ is strongly of bounded variation in the Hilbert space $(\mathcal{K},[\cdot,\cdot]_{\mathcal{J}})$.*

Theorem 4. *[4] Let $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-, [\cdot,\cdot])$ be a Krein space, if $f : [a,b] \to \mathcal{K}$ is strongly of bounded variation in $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-, [\cdot,\cdot])$, then $\mathcal{J}f$ is strongly of bounded variation in $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-, [\cdot,\cdot])$ and $(\mathcal{K},[\cdot,\cdot]_{\mathcal{J}})$.*

Theorem 5. *[4] Let $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-, [\cdot,\cdot])$ be a Krein space and let $f : [a,b] \to \mathcal{K}$ a strongly of bounded variation function in $[a,b]$ on $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-)$, suppose that c in (a,b), then f is strongly of bounded variation in $[a,c]$ and in $[c,b]$ on $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-)$. In this case we have*

$$V_a^b(f,(\mathcal{K},[\cdot,\cdot])) = V_a^c(f,\mathcal{K}) + V_c^b(f,\mathcal{K})$$

4. Functions of bounded q-variation in Hilbert spaces associated with a Krein space

Given a Krein space $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-, [\cdot,\cdot], \mathcal{J})$ and $f : [a,b] \to \mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-$, for t in $[a,b]$, as $f(t)$ in $\mathcal{K}$, we can write $f(t) = f^+(t) + f^-(t)$, where $f^+(t)$ in $\mathcal{K}^+$ and $f^-(t)$ in

$\mathcal{K}^-$. Now, since $(\mathcal{K}_+, [\cdot,\cdot])$ and $(\mathcal{K}_-, -[\cdot,\cdot])$ are Hilbert spaces, the $q-$variations of f on $[a,b]$ denoted by $\overset{q+}{\mathrm{V}}{}_a^b(f,(\mathcal{K}_+,[\cdot,\cdot]))$ and $\overset{q-}{\mathrm{V}}{}_a^b(f,(\mathcal{K}_-,[\cdot,\cdot]))$ are given as follows:

$$\overset{q+}{\mathrm{V}}{}_a^b(f,(\mathcal{K}_+,[\cdot,\cdot])) = \sup_{P\in\mathcal{P}[a,b]} \left\{ \left(\sum_{i=1}^{n} \|f^+(t_i) - f^+(t_{i-1})\|_+^q \right)^{1/q} \right\}$$

and

$$\overset{q-}{\mathrm{V}}{}_a^b(f,(\mathcal{K}_-,-[\cdot,\cdot])) = \sup_{P\in\mathcal{P}[a,b]} \left\{ \left(\sum_{i=1}^{n} \|f^-(t_i) - f^-(t_{i-1})\|_-^q \right)^{1/q} \right\}.$$

Where the supreme is taken over the entire set of partitions $\{t_0, t_1, t_2, \ldots, t_n\}$ of the interval $[a,b]$. We will call these $q-$variations positive and negative $q-$variations respectively.

Remark 4. *Note that*

(i) $\overset{q+}{V}{}_a^b(f,(\mathcal{K}_+,[\cdot,\cdot])) \geq 0$, $\overset{q-}{V}{}_a^b(f,(\mathcal{K}_-,-[\cdot,\cdot])) \geq 0$ *and* $\overset{q}{V}{}_a^b(f,(\mathcal{K},[\cdot,\cdot])) \geq 0$.

(ii) Since $(\mathcal{K}_+, [\cdot,\cdot])$ *and* $(\mathcal{K}_-, -[\cdot,\cdot])$ *are Hilbert space. Given* $\alpha \in \mathbb{C}$ *and* f, g *functions functions of bounded q-variation, for [13] it is satisfied that:*

$$\begin{aligned}
i)\quad & \overset{q+}{V}{}_a^b(\alpha f,(\mathcal{K}_+,[\cdot,\cdot])) = |\alpha| \overset{q+}{V}{}_a^b(f,(\mathcal{K}_+,[\cdot,\cdot])) \\
ii)\quad & \overset{q+}{V}{}_a^b((f+g),(\mathcal{K}_+,[\cdot,\cdot])) = \overset{q+}{V}{}_a^b(f,(\mathcal{K}_+,[\cdot,\cdot])) + \overset{q+}{V}{}_a^b(g,(\mathcal{K}_+,[\cdot,\cdot])) \\
iii)\quad & \overset{q-}{V}{}_a^b(\alpha f,(\mathcal{K}_-,[\cdot,\cdot])) = |\alpha| \overset{q-}{V}{}_a^b(f,(\mathcal{K}_-,[\cdot,\cdot])) \\
iv)\quad & \overset{q-}{V}{}_a^b((f+g),(\mathcal{K}_-,[\cdot,\cdot])) = \overset{q-}{V}{}_a^b(f,(\mathcal{K}_-,[\cdot,\cdot])) + \overset{q-}{V}{}_a^b(g,(\mathcal{K}_-,[\cdot,\cdot]))
\end{aligned}$$

Example 2. *Consider example 1 and the function* $f : [2,3] \to \mathbb{C}^2$ *defined by* $f(t) = (ti, -ti) = (ti, 0) + (0, -ti)$. *Así,* $f^+(t) = (ti, 0)$, $f^+(t) = (ti, 0)$ *and* $f^-(t) = (0, -ti)$. *Then the positive and negative* $q-$*variations are given by:*

$$\overset{q+}{V}{}_a^b(f,(\mathcal{K}_+,[\cdot,\cdot])) = \sup_{P\in\mathcal{P}[a,b]} \left\{ \left(\sum_{j=1}^{n} (t_j - t_{j-1})^q \right)^{1/q} \right\} = \overset{q-}{V}{}_a^b(f,(\mathcal{K}_-,[\cdot,\cdot]))$$

The existence of functions of bounded $q-$variation on Hilbert spaces associated to a Krein space led us to think about the definition of functions of bounded $q-$variation on Krein spaces studied in the next section.

5. Functions of bounded q-variation in Krein spaces

In this section we will introduce in Krein spaces the concepts of total q-variation and give the notion of a bounded q-variation function, notions that generalise the results given in [4] and [13]

Definition 7. *Let $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-, [\cdot,\cdot])$ be a Krein space, $f : [a,b] \to \mathcal{K}$ and $q \geq 1$.at the number*

$$\overset{q}{V}{}_a^b(f,(\mathcal{K},[\cdot,\cdot])) = \sup_{P\in\mathcal{P}[a,b]}\left\{\left(\sum_{i=1}^{n}(\|f^+(t_i)-f^+(t_{i-1})\|_+ + \|f^-(t_i)-f^-(t_{i-1})\|_-)^q\right)^{1/q}\right\}$$

where the supremum is taken over the set of all partitions $P = \{t_0, t_1, \ldots, t_n\}$ of the interval $[a,b]$, we will call total q-variation of f in $[a,b]$ on $\mathcal{K}$.

Moreover, if there is a constant $M > 0$ such that $\overset{q}{V}{}_a^b(f,(\mathcal{K},[\cdot,\cdot])) \leq M$ we will say that f is of bounded q-variation in $[a,b]$ on $\mathcal{K}$.

Remark 5. *The set of all functions of bounded q-variation in $[a,b]$ on $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-, [\cdot,\cdot])$ will be denoted as follows $B\overset{q}{V}{}_a^b(\mathcal{K},[\cdot,\cdot])$.*

(i) Note that when $q = 1$ we are dealing with definition 4.3 given in [4].

(ii) In the case of a Hilbert space, we have that $\|f^-(t_i) - f^-(t_{i-1})\|_- = 0$ and so $\overset{q}{V}{}_a^b(f,\mathcal{K},[\cdot,\cdot]) = \sup_{P\in\mathcal{P}[a,b]}\left\{\left(\sum_{i=1}^n \|f^+(t_i)-f^+(t_{i-1})\|_+^q\right)^{1/q}\right\} = V_q(f)$ which is the definition of a function of q-variation on a Hilbert space in the Weiner sense [13].

(iii) If $f \in B\overset{q}{V}{}_a^b(\mathcal{K},[\cdot,\cdot])$, then $f \in B\overset{q}{V}{}_a^b(\mathcal{K}_+,[\cdot,\cdot])$ and $f \in B\overset{q}{V}{}_a^b(\mathcal{K}_-,-[\cdot,\cdot])$, since that $\|f^+(t_i) - f^+(t_{i-1})\|_+,\ \|f^-(t_i) - f^-(t_{i-1})\|_- \leq \|f^+(t_i) - f^+(t_{i-1})\|_+ + \|f^-(t_i) - f^-(t_{i-1})\|_-$

The observation3 is of utmost relevance, since it guarantees that every bounded q-variation function on Krein space $\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-$ is of bounded q-variation on the associated Hilbert spaces $\mathcal{K}_+$ and $\mathcal{K}_-$.

Remark 6. *Note that for the associated Hilbert space $(\mathcal{K},[\cdot,\cdot]_\mathcal{J})$ the total q-variation of f is:*

$$\overset{q}{V}{}_a^b(f,(\mathcal{K},[\cdot,\cdot]_\mathcal{J})) = \sup_{P\in\mathcal{P}[a,b]}\left\{\left(\sum_{i=1}^{n}\|f(t_i)-f(t_{i-1})\|_\mathcal{J}^q\right)^{1/q}\right\}$$

Next, we show an example of a bounded q-variation function in a Krein space.

Example 3. *Consider example 1 and the function $f : [2,3] \to \mathbb{C}^2$ defined by $f(t) = (ti, -ti)$. Let us see that f is a function of bounded 2-variation.*
In fact, let $P = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[2,3]$. Then,

$$\|f^+(t_j) - f^+(t_{j-1})\|_+ = \|(t_j i - t_{j-1} i, 0)\|_+ = |t_j i - t_{j-1} i| = |t_j - t_{j-1}| \cdot |i| = t_j - t_{j-1}$$

$$\|f^-(j_i) - f^-(t_{j-1})\|_- = \|(0, t_{j-1} i - t_j i)\|_- = |t_{j-1} i - t_j i| = |t_{j-1} - t_j| \cdot |i| = t_j - t_{j-1}.$$

Thus,

$$(\|f^+(t_j) - f^+(t_{j-1})\|_+ + \|f^-(t_j) - f^-(t_{j-1})\|_-)^2 = (2(t_j - t_{j-1}))^2 = 4(t_j - t_{j-1})^2$$

Now, since $t \in [2,3]$, then $0 \le t_j - t_{j-1} \le 1$. Then, $(t_j - t_{j-1})^2 \le t_j - t_{j-1}$, so

$$0 \ \le \ \sum_{j=1}^{n} 4(t_j - t_{j-1})^2 \le \sum_{j=1}^{n} 4(t_j - t_{j-1}), \quad \textit{it follows that}$$

$$\left(\sum_{j=1}^{n} 4(t_j - t_{j-1})^2\right)^{1/2} \le \left(\sum_{j=1}^{n} 4(t_j - t_{j-1})\right)^{1/2} \le 2\,(t_n - t_0)^{1/2} = 2(3-2)^{1/2} = 2$$

Therefore, we have to $\overset{2}{V}{}_2^3(f, (\mathbb{C}^2, [\cdot,\cdot]))$ is bounded, thus it follows that f is of bounded 2-variation function.

In definition 7 we introduce the concept of a bounded q-variation function in Krein spaces. In the following, we show that this concept is broader than the one given in [13].

Theorem 6. *Let $(\mathcal{K} = \mathcal{K}_+ [\dot{+}] \mathcal{K}_-, [\cdot,\cdot])$ be a Krein space and $f : [a,b] \to \mathcal{K}$ of bounded q-variation function in $(\mathcal{K} = \mathcal{K}_+ [\dot{+}] \mathcal{K}_-, [\cdot,\cdot])$, then f is of bounded q-variation function in the Hilbert space $(\mathcal{K}, [\cdot,\cdot]_{\mathcal{J}})$.*

Proof. If f is of bounded q-variation in $[a,b]$ on $(\mathcal{K} = \mathcal{K}_+ [\dot{+}] \mathcal{K}_-, [\cdot,\cdot])$, then exists $M > 0$ such that:

$$\overset{q}{\mathrm{V}}{}_a^b(f, (\mathcal{K}, [\cdot,\cdot])) = \sup_{P \in \mathcal{P}[a,b]} \left\{ \Big(\sum_{i=1}^{n} (\|f^+(t_i) - f^+(t_{i-1})\|_+ + \|f^-(t_i) - f^-(t_{i-1})\|_-)^q \Big)^{1/q} \right\} \le M$$

By Theorem 1 we have that:

$$\|f(t_i) - f(t_{i-1})\|_{\mathcal{J}} \le \|f_+(t_i) - f_+(t_{i-1})\|_+ + \|f_-(t_i) - f_-(t_{i-1})\|_-$$

Therefore, for $q \ge 1$, it is satisfied that :

$$\|f(t_i) - f(t_{i-1})\|_{\mathcal{J}}^q \le (\|f_+(t_i) - f_+(t_{i-1})\|_+ + \|f_-(t_i) - f_-(t_{i-1})\|_-)^q$$

whence:

$$\overset{q}{\mathrm{V}}{}_a^b(f, (\mathcal{K}, [\cdot,\cdot]_{\mathcal{J}})) \le M$$

Therefore, f is of bounded q-variation in $[a, b]$ on the Hilbert space $(\mathcal{K}, [\cdot, \cdot]_{\mathcal{J}})$.

The following result is very significant in this research, as it shows the robustness of the definition 7 introduced in this paper, more precisely it shows that the bounded q-variation functions on a Krein space are independent of the fundamental decomposition of the space. This finding is remarkable as it reinforces the robustness of these functions in different decomposition structures, this not only challenges traditional conceptions, but also opens new perspectives in functional analysis, providing a solid basis for future research and applications in economics, quantum mechanics, signal processing[14–17] where the bounded q-variation plays a very important role.

Theorem 7. *Let* $(\mathcal{K}, [\cdot, \cdot])$ *be a Krein space with decompositions* $(\mathcal{K} = \mathcal{K}_{1_+}[\dot{+}]\mathcal{K}_{1_-}, \mathcal{J}_1)$, $(\mathcal{K} = \mathcal{K}_{2_+}[\dot{+}]\mathcal{K}_{2_-}, \mathcal{J}_2)$ *and* $f : [a, b] \to \mathcal{K}$ *of bounded* q*-variation function in* $[a, b]$ *on* $(\mathcal{K} = \mathcal{K}_{1_+}[\dot{+}]\mathcal{K}_{1_-})$, *then* f *is of bounded* q*-variation in* $[a, b]$ *on* $(\mathcal{K} = \mathcal{K}_{2_+}[\dot{+}]\mathcal{K}_{2_-})$.

Proof. If f is of bounded q-variation in $[a, b]$ on $(\mathcal{K} = \mathcal{K}_{1_+}[\dot{+}]\mathcal{K}_{1_-}, [\cdot, \cdot])$, then exists $M > 0$ such that:

$$\overset{q}{\mathrm{V}}{}_a^b(f, (\mathcal{K}, [\cdot, \cdot])) = \sup_{P\in\mathcal{P}[a,b]} \Big\{ \Big(\sum_{i=1}^{n} (\|f^+(t_i) - f^+(t_{i-1})\|_{1+} + \|f^-(t_i) - f^-(t_{i-1})\|_{1-})^q \Big)^{1/q} \Big\} \leq M$$

Moreover, as $\|\cdot\|_{\mathcal{J}_1}$ and$\|\cdot\|_{\mathcal{J}_2}$ are equivalent norms [12], then there are $a, b > 0$ such that $a\|\cdot\|_{\mathcal{J}_2} \leq \|\cdot\|_{\mathcal{J}_1} \leq b\|\cdot\|_{\mathcal{J}_2}$. Therefore,

$$\begin{aligned}(a\|f(t_i) - f(t_{i-1})\|_{\mathcal{J}_2})^q &\leq \|f(t_i) - f(t_{i-1})\|_{\mathcal{J}_1}^q \\ &\leq (\|f^+(t_i) - f^+(t_{i-1})\|_{1+} + \|f^-(t_i) - f^-(t_{i-1})\|_{1-})^q\end{aligned}$$

The last inequality is obtained thanks to theorem 1. Later,

$$\textstyle\sum_{i=1}^{n}(a\|f(t_i) - f(t_{i-1})\|_{\mathcal{J}_2})^q \leq \sum_{i=1}^{n}(\|f^+(t_i) - f^+(t_{i-1})\|_{1+} + \|f^-(t_i) - f^-(t_{i-1})\|_{1-})^q$$

$$(a^q \textstyle\sum_{i=1}^{n}(\|f(t_i) - f(t_{i-1})\|_{\mathcal{J}_2})^q)^{1/q} \leq$$

$$(\textstyle\sum_{i=1}^{n}(\|f^+(t_i) - f^+(t_{i-1})\|_{1+} + \|f^-(t_i) - f^-(t_{i-1})\|_{1-})^q)^{1/q}$$

$$a(\textstyle\sum_{i=1}^{n}(\|f(t_i) - f(t_{i-1})\|_{\mathcal{J}_2})^q)^{1/q} \leq$$

$$(\textstyle\sum_{i=1}^{n}(\|f^+(t_i) - f^+(t_{i-1})\|_{1+} + \|f^-(t_i) - f^-(t_{i-1})\|_{1-})^q)^{1/q} \leq M$$

It follows that:

$$\begin{aligned}\overset{q}{\mathrm{V}}{}_a^b(f, (\mathcal{K}, [\cdot, \cdot]_{\mathcal{J}_2})) &= \sup_{P\in\mathcal{P}[a,b]} \{(\sum_{i=1}^{n}(\|f^+(t_i) - f^+(t_{i-1})\|_{2+} + \|f^-(t_i) - f^-(t_{i-1})\|_{2-})^q)^{1/q}\} \\ &\leq M/a\end{aligned}$$

This implies that$f$ is of bounded q-variation in $[a, b]$ on $(\mathcal{K} = \mathcal{K}_{2_+}[\dot{+}]\mathcal{K}_{2_-}, [\cdot, \cdot]_{\mathcal{J}_2})$.

Theorem 8. *Let* $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-, [\cdot,\cdot])$ *be a Krein space and* $f : [a,b] \to \mathcal{K}$ *of bounded* q*-variation function in* $[a,b]$ *on* $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-, [\cdot,\cdot])$*, then* f *is bounded.*

Proof. Suppose that f is of bounded q-variation on $[a,b]$, then there exists $M > 0$ such that $\overset{q}{\mathrm{V}}{}_a^b(f,(\mathcal{K},[\cdot,\cdot]_{\mathcal{J}})) \le M$ for all partition P of $[a,b]$. Let be $t \in (a,b)$ and consider the partition $P = \{a,t,b\}$, then,

$$\left(\sum_{i=1}^{2}(\|f^+(t_i) - f^+(t_{i-1})\|_+ + \|f^-(t_i) - f^-(t_{i-1})\|_-)^q\right)^{1/q} =$$

$$\left((\|f^+(t) - f^+(a)\|_+ + \|f^-(t) - f^-(a)\|_-)^q + \left(\|f^+(b) - f^+(t)\|_+ + \|f^-(b) - f^-(t)\|_-\right)^q\right)^{1/q}$$

$$\le \sup_{P\in\mathcal{P}[a,b]}\left\{\left(\sum_{i=1}^{n}(\|f^+(t_i) - f^+(t_{i-1})\|_+ + \|f^-(t_i) - f^-(t_{i-1})\|_-)^q\right)^{1/q}\right\} \le M$$

Thus,

$$((\|f^+(t)-f^+(a)\|_+ + \|f^-(t)-f^-(a)\|_-)^q + (\|f^+(b)-f^+(t)\|_+ + \|f^-(b)-f^-(t)\|_-)^q)^{1/q} \le M$$

Therefore,

$$(\|f^+(t)-f^+(a)\|_+ + \|f^-(t)-f^-(a)\|_-)^q + \left(\|f^+(b) - f^+(t)\|_+ + \|f^-(b) - f^-(t)\|_-\right)^q \le M^q$$

Whence,

$$(\|f^+(t) - f^+(a)\|_+ + \|f^-(t) - f^-(a)\|_-)^q\ ,\ (\|f^+(b) - f^+(t)\|_+ + \|f^-(b) - f^-(t)\|_-)^q \le M^q$$

Since $(\|f^+(t) - f^+(a)\|_+ + \|f^-(t) - f^-(a)\|_-)^q \le M^q$, then,

$$\|f^+(t) - f^+(a)\|_+ + \|f^-(t) - f^-(a)\|_- \le M$$

Then, using the theorem 1, it follows that:

$$\|f(t) - f(a)\|_{\mathcal{J}} \le \|f^+(t) - f^+(a)\|_+ + \|f^-(t) - f^-(a)\|_- \le M$$

Since, $\|f(t)\|_{\mathcal{J}} - \|f(a)\|_{\mathcal{J}} \le \|f(t) - f(a)\|_{\mathcal{J}}$,then

$$\|f(t)\|_{\mathcal{J}} \le M + \|f(a)\|_{\mathcal{J}} \le M + \|f(a)\|_{\mathcal{J}} + \|f(b)\|_{\mathcal{J}} = d.$$

Therefore for all $t \in [a,b]$, it follows that $\|f(t)\|_{\mathcal{J}} \le d$. This implies that f is bounded in $[a,b]$.

We show below that the reciprocal of 8 is not true.

Example 4. *Consider the Krein space given in example 1 and the function* $f : [\sqrt{3}, 4] \to \mathbb{C}^2$ *defined by*

$$f(t) = \begin{cases} (i,i), & \text{if } t \text{ is rational}, \ t \in [\sqrt{3},4], \\ \\ (0,0), & \text{if } t \text{ is irrational}, \ t \in [\sqrt{3},4]. \end{cases}$$

Let's see that f is bounded on $[\sqrt{3}, 4]$. In fact,

$$\|f(t)\|_{\mathcal{J}} = \begin{cases} \sqrt{2}, & \text{if } t \text{ is rational}, \ t \in [\sqrt{3},4], \\ \\ 0, & \text{if } t \text{ is irrational}, \ t \in [\sqrt{3},4]. \end{cases}$$

Therefore, $\|f(t)\|_{\mathcal{J}} \leq \sqrt{2}$ for all $t \in [\sqrt{3}, 4]$. Thus f is bounded.
Now, let's see that f is not of $q-$bounded variation. Let $t_0 = \sqrt{3}$, as between any two reals there is a rational number and an irrational number, we can choose t_1 as a rational number between $\sqrt{3}$ and 4, t_2 as an irrational number between t_1 and 4, t_3 as a rational number between t_2 and 4, and so on t_{2i} would be an irrational number between t_{2i-1} and 4, t_{2i+1} would be a rational number between t_{2i} and 4, finally we choose $t_n = 4$. Then,

$$f^+(t) = (i, 0) \quad \text{and} \quad f^-(t) = (0, i)$$

Furthermore,

$$\begin{aligned} \overset{2}{V}{}^{4}_{\sqrt{3}}(f, (\mathbb{C}^2, [\cdot,\cdot])) &\geq \left(\sum_{j=1}^{n} \left(\|f^+(t_j) - f^+(t_{j-1})\|_+ + \|f^-(t_j) - f^-(t_{j-1})\|_- \right)^2 \right)^{1/2} \\ &= \Big((\| -(i,i)\|_+ + \|(i,i)\|_-)^2 + \cdots + (\| -(i,i)\|_+ + \|(i,i)\|_-)^2 \Big)^{1/2} \\ &= \Big((\|(i,i)\|_+ + \|(i,i)\|_-)^2 + \cdots + (\|(i,i)\|_+ + \|(i,i)\|_-)^2 \Big)^{1/2} \\ &\geq \left(\|(i,i)\|^2_{\mathcal{J}} + \cdots + \|(i,i)\|^2_{\mathcal{J}} \right)^{1/2} \\ &= \Big((\sqrt{2})^2 + \cdots + (\sqrt{2})^2 \Big)^{1/2} = (2 + \cdots + 2)^{1/2} = \sqrt{2n} \end{aligned}$$

Note that a partition of the interval $[\sqrt{3}, 4]$ was constructed, starting at $\sqrt{3}$, then alternating between rational and irrational numbers until it ends at 4, for which $\overset{2}{V}{}^{4}_{\sqrt{3}}(f, (\mathbb{C}^2, [\cdot,\cdot]))$ is not finite, thus f is not of bounded 2-variation in $(\mathbb{C}^2, [\cdot,\cdot])$.

Theorem 9. *Let $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-, [\cdot,\cdot])$ be a Krein space, $f : [a, b] \to \mathcal{K}$, is of bounded q-variation in $[a, b]$ on $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-)$, then f is of bounded q-variation in $[a, b]$ on the Hilbert spaces $(\mathcal{K}^+, [\cdot,\cdot])$ and $(\mathcal{K}^-, -[\cdot,\cdot])$.*

Proof. Consider the partition $P = \{a, t_1, t_2, \cdots, t_{i-1}, t_i, \cdots, t_{n-1}, b\} \in \mathcal{P}[a, b]$ y $q \geq 1$.

The proof is a consequence of:
$\|f^-(t_i) - f^-(t_{i-1})\|^q_-, \|f^+(t_i) - f^+(t_{i-1})\|^q_+ \leq (\|f^+(t_i) - f^+(t_{i-1})\|_+ + \|f^-(t_i) - f^-(t_{i-1})\|_-)^q$

Theorem 10. *Let* $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-, [\cdot,\cdot])$ *be a Krein space and* $f : [a,b] \to \mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-$ *a function,* $f \in B\overset{q}{V}([a,b],\mathcal{K})$*, then* $\overset{q+}{V}{}_a^b(f,(\mathcal{K}_+,[\cdot,\cdot])) = 0 = \overset{q-}{V}{}_a^b(f,(\mathcal{K}_-,-[\cdot,\cdot]))$ *if and only if* f *is constant in* $[a,b]$ *with respect to* $(\mathcal{K} = \mathcal{K}_+[+]\mathcal{K}_-)$.

Proof.

($\to$) Suppose that $\overset{q+}{\mathrm{V}}{}_a^b(f,(\mathcal{K}_+,[\cdot,\cdot])) = 0 = \overset{q-}{\mathrm{V}}{}_a^b(f,(\mathcal{K}_-,-[\cdot,\cdot]))$, that is,

$$\overset{q+}{\mathrm{V}}{}_a^b(f,(\mathcal{K}_+,[\cdot,\cdot])) = \sup_{P\in\mathcal{P}[a,b]} \left\{ \left(\sum_{i=1}^{n} \|f^+(t_i) - f^+(t_{i-1})\|_+^q \right)^{1/q} \right\} = 0$$

and

$$\overset{q-}{\mathrm{V}}{}_a^b(f,(\mathcal{K}_-,-[\cdot,\cdot])) = \sup_{P\in\mathcal{P}[a,b]} \left\{ \left(\sum_{i=1}^{n} \|f^-(t_i) - f^-(t_{i-1})\|_-^q \right)^{1/q} \right\} = 0.$$

Then,

$$\left(\sum_{i=1}^{n} \|f^+(t_i) - f^+(t_{i-1})\|_+^q \right)^{1/q} = 0 \text{ and } \left(\sum_{i=1}^{n} \|f^-(t_i) - f^-(t_{i-1})\|_-^q \right)^{1/q} = 0.$$

Let $x \in [a,b]$. Then, in particular, for the partition $P = \{a,x,b\}$, we have that $\big(\|f^+(x) - f^+(a)\|_+^q + \|f^+(b) - f^+(x)\|_+^q\big)^{1/q} = 0$ and
$\big(\|f^-(x) - f^-(a)\|_-^q + \|f^-(b) - f^-(x)\|_-^q\big)^{1/q} = 0$.

Whence,
$\|f^+(x)-f^+(a)\|_+^q+\|f^+(b)-f^+(x)\|_+^q = 0$, $\|f^-(x)-f^-(a)\|_-^q+\|f^-(b)-f^-(x)\|_-^q = 0$.
Thus,

$\|f^+(x) - f^+(a)\|_+^q = 0, \|f^+(b) - f^+(x)\|_+^q = 0, \quad \|f^-(x) - f^-(a)\|_-^q = 0,$
$\|f^-(b) - f^-(x)\|_-^q = 0$. Therefore,

$\big(f^+(x)-f^+(a) = \mathbf{0},\ f^+(b)-f^+(x) = \mathbf{0}\big)$ and $\big(f^-(x)-f^-(a) = \mathbf{0}, f^-(b)-f^-(x) = \mathbf{0}\big)$.

Next,

$$\big(f^+(x) = f^+(a), f^+(b) = f^+(x)\big) \text{ and } \big(f^-(x) = f^-(a), f^-(b) = f^-(x)\big)$$

Follows,

$$f^+(x) = f^+(a) = f^+(b) \text{ and } f^-(x) = f^-(a) = f^-(b).$$

Therefore,

$$f(x) = f^+(x) + f^-(x) = f^+(a) + f^-(a) = f^+(b) + f^-(b).$$

Thus, f is constant in the interval $[a,b]$.

($\leftarrow$) Suppose that f is constant in $[a,b]$, then exists $c \in \mathcal{K}$ such that for any $x \in [a,b]$, $f(x) = c$. Since $c \in \mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-$, there are $c^+ \in \mathcal{K}_+$ and $c^- \in \mathcal{K}_-$ such that $c = c^+ + c^-$. Furthermore, as $f(x) = f^+(x) + f^-(x)$, it follows that:

$$f^+(x) + f^-(x) = c^+ + c^-, \text{ whence } f^+(x) - c^+ = c^- - f^-(x) \in \mathcal{K}_+ \cap \mathcal{K}_- = \{\mathbf{0}\}$$

Thus,

$$f^+(x) - c^+ = \mathbf{0} \text{ and } c^- - f^-(x) = \mathbf{0}, \text{ therefore, } f^+(x) = c^+ \text{ and } c^- = f^-(x),$$

Thus,

$$\begin{aligned}\overset{q_+}{\mathrm{V}}{}_a^b(f,(\mathcal{K}_+,[\cdot,\cdot])) &= \sup_{P\in\mathcal{P}[a,b]}\left\{\left(\sum_{i=1}^n \|f^+(t_i) - f^+(t_{i-1})\|_+^q\right)^{1/q}\right\}\\ &= \sup_{P\in\mathcal{P}[a,b]}\left\{\left(\sum_{i=1}^n \|c^+ - c^+\|_-^q\right)^{1/q}\right\} = \sup\{0\} = 0\end{aligned}$$

and

$$\begin{aligned}\overset{q_-}{\mathrm{V}}{}_a^b(f,(\mathcal{K}_-,-[\cdot,\cdot])) &= \sup_{P\in\mathcal{P}[a,b]}\left\{\left(\sum_{i=1}^n \|f^-(t_i) - f^-(t_{i-1})\|_-^q\right)^{1/q}\right\}\\ &= \sup_{P\in\mathcal{P}[a,b]}\left\{\left(\sum_{i=1}^n \|c^- - c^-\|_-^q\right)^{1/q}\right\} = \sup\{0\} = 0\end{aligned}$$

5.1. Algebra of bounded q-variation functions in Krein spaces

Theorem 11. *Let $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-, [\cdot,\cdot])$ be a Krein space, if $f, g : [a,b] \to \mathcal{K}$ be of bounded q-variation functions in $[a,b]$ on $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-)$ and α a scalar, then αf and $f+g$ are also of bounded q-variation functions in $[a,b]$ on $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-)$.*

Proof. If f and g are functions of bounded q-variation in $[a,b]$ on $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-, [\cdot,\cdot])$, then exists $L, M > 0$ such that

$$\overset{q}{\mathrm{V}}{}_a^b(f,(\mathcal{K},[\cdot,\cdot])) \leq L \quad \text{and} \quad \overset{q}{\mathrm{V}}{}_a^b(g,(\mathcal{K},[\cdot,\cdot])) \leq M$$

(i) If α is a scalar,considering part two (i) and (iii) of the remark 4, for all partition $P \in \mathcal{P}[a,b]$, it follows that:

$$\overset{q}{\mathrm{V}}{}_a^b(\alpha f, (\mathcal{K}, [\cdot,\cdot]))$$
$$= \sup_{P\in\mathcal{P}[a,b]} \{(\sum_{i=1}^{n} \left(\|\alpha f^+(t_i) - \alpha f^+(t_{i-1})\|_+ + \|\alpha f^-(t_i) - \alpha f^-(t_{i-1})\|_-\right)^q)^{1/q}\}$$
$$= \sup_{P\in\mathcal{P}[a,b]} \left\{ \left(\sum_{i=1}^{n} \left(\|\alpha\big(f^+(t_i) - f^+(t_{i-1})\big)\|_+ + \|\alpha(f^-(t_i) - f^-(t_{i-1}))\|_-\right)^q\right)^{1/q} \right\}$$
$$= \sup_{P\in\mathcal{P}[a,b]} \left\{ \left(\sum_{i=1}^{n} \left(|\alpha|\|f^+(t_i) - f^+(t_{i-1})\|_+ + |\alpha|\|f^-(t_i) - f^-(t_{i-1})\|_-\right)^q\right)^{1/q} \right\}$$
$$= \sup_{P\in\mathcal{P}[a,b]} \left\{ \left(|\alpha|^q \sum_{i=1}^{n} \left(\|f^+(t_i) - f^+(t_{i-1})\|_+ + \|f^-(t_i) - f^-(t_{i-1})\|_-\right)^q\right)^{1/q} \right\}$$
$$= |\alpha| \sup_{P\in\mathcal{P}[a,b]} \left\{ \left(\sum_{i=1}^{n} \left(\|f^+(t_i) - f^+(t_{i-1})\|_+ + \|f^-(t_i) - f^-(t_{i-1})\|_-\right)^q\right)^{1/q} \right\}$$
$$= |\alpha|\overset{q}{\mathrm{V}}{}_a^b(f, (\mathcal{K}, [\cdot,\cdot])) \leq |\alpha| L.$$

Therefore, $\overset{q}{\mathrm{V}}{}_a^b(\alpha f, (\mathcal{K}, [\cdot,\cdot]))$ is finite, which implies that αf is of bounded q-variation in $[a, b]$ on $(\mathcal{K} = \mathcal{K}_+[+]\mathcal{K}_-, [\cdot,\cdot])$.

(ii) Using (ii) and (iv) in part two of remark 4 it follows that for any partition $P \in \mathcal{P}[a, b]$, it is satisfied that:

$$\left(\sum_{i=1}^{n} \left(\|(f+g)^+(t_i) - (f+g)^+(t_{i-1})\|_+ + \|(f+g)^-(t_i) - (f+g)^-(t_{i-1})\|_-\right)^q\right)^{1/q} =$$
$$(\sum_{i=1}^{n}(\|f^+(t_i) - f^+(t_{i-1}) + g^+(t_i) - g^+(t_{i-1})\|_+ + \|f^-(t_i) - f^-(t_{i-1}) + g^-(t_i) - g^-(t_{i-1})\|_-)^q)^{1/q}$$
$$\leq \left(\sum_{i=1}^{n} \left(\left(\|f^+(t_i) - f^+(t_{i-1})\|_+ + \|f^-(t_i) - f^-(t_{i-1}\|_-\right) + \right.\right.$$
$$\left.\left.\left(\|g^+(t_i) - g^+(t_{i-1})\|_+ + \|g^-(t_i) - g^-(t_{i-1})\|_-\right)\right)^q\right)^{1/q}$$

Then by the Minkowski inequality [7], it follows that:

$$\left(\sum_{i=1}^{n} \left(\|(f+g)^+(t_i) - (f+g)^+(t_{i-1})\|_+ + \|(f+g)^-(t_i) - (f+g)^-(t_{i-1})\|_-\right)^q\right)^{1/q}$$

$$\leq \left(\sum_{i=1}^{n} \Big(\|f^+(t_i) - f^+(t_{i-1})\|_+ + \|f^-(t_i) - f^-(t_{i-1})\|_- \Big)^q \right)^{1/q} + \left(\sum_{i=1}^{n} \Big(\|g^+(t_i) - g^+(t_{i-1})\|_+ + \|g^-(t_i) - g^-(t_{i-1})\|_- \Big)^q \right)^{1/q}$$

Whence,

$$\overset{q}{\mathrm{V}}{}_a^b((f+g), (\mathcal{K}, [\cdot,\cdot])) \leq \overset{q}{\mathrm{V}}{}_a^b(f, (\mathcal{K}, [\cdot,\cdot])) + \overset{q}{\mathrm{V}}{}_a^b(g, (\mathcal{K}, [\cdot,\cdot])) \leq L + M.$$

Therefore, $f+g$ is of bounded q-variation in $[a,b]$ on the Krein space $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-, [\cdot,\cdot])$.

Theorem 12. *Let $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-, [\cdot,\cdot])$ be a Krein space and $f : [a,b] \to \mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-$ a function, if $f \in B\overset{q}{V}{}_a^b(\mathcal{K}, [\cdot,\cdot])$, then $f \in B\overset{p}{V}{}_a^b(\mathcal{K}, [\cdot,\cdot])$ para $p > q$.*

Proof. Suppose that $f \in \overset{q}{V}{}_a^b(\mathcal{K}, [\cdot,\cdot])$, then exists $M > 0$ such that $\overset{q}{\mathrm{V}}{}_a^b(f, (\mathcal{K}, [\cdot,\cdot])) \leq M$

Now, for $p > q$, let us consider two cases:

i) $\|f^+(t_i) - f^+(t_{i-1})\|_+ + \|f^-(t_i) - f^-(t_{i-1})\|_- \geq 1$. Then,

$(\|f^+(t_i) - f^+(t_{i-1})\|_+ + \|f^-(t_i) - f^-(t_{i-1})\|_-)^q \leq (\|f^+(t_i) - f^+(t_{i-1})\|_+ + \|f^-(t_i) - f^-(t_{i-1})\|_-)^p$. Later,

$\sum_{i=1}^n (\|f^+(t_i) - f^+(t_{i-1})\|_+ + \|f^-(t_i) - f^-(t_{i-1})\|_-)^q \leq \sum_{i=1}^n (\|f^+(t_i) - f^+(t_{i-1})\|_+ + \|f^-(t_i) - f^-(t_{i-1})\|_-)^p$

Since $p > q$, then $\dfrac{1}{p} < \dfrac{1}{q}$ and thus:

$$\begin{aligned}
\overset{p}{\mathrm{V}}{}_a^b(f, (\mathcal{K}, [\cdot,\cdot])) &= \sup_{P \in \mathcal{P}[a,b]} \left\{ \left(\sum_{i=1}^{n} \big(\|f^+(t_i) - f^+(t_{i-1})\|_+ + \|f^-(t_i) - f^-(t_{i-1})\|_- \big)^p \right)^{1/p} \right\} \\
&\leq \sup_{P \in \mathcal{P}[a,b]} \left\{ \left(\sum_{i=1}^{n} \big(\|f^+(t_i) - f^+(t_{i-1})\|_+ + \|f^-(t_i) - f^-(t_{i-1})\|_- \big)^q \right)^{1/q} \right\} \\
&\leq M
\end{aligned}$$

ii) $0 \leq \|f^+(t_i) - f^+(t_{i-1})\|_+ + \|f^-(t_i) - f^-(t_{i-1})\|_- < 1$. Then

$(\|f^+(t_i) - f^+(t_{i-1})\|_+ + \|f^-(t_i) - f^-(t_{i-1})\|_-)^p \leq (\|f^+(t_i) - f^+(t_{i-1})\|_+ + \|f^-(t_i) - f^-(t_{i-1})\|_-)^q$. Whence,

$\sum_{i=1}^{n}(\|f^+(t_i) - f^+(t_{i-1})\|_+ + \|f^-(t_i) - f^-(t_{i-1})\|_-)^p \leq \sum_{i=1}^{n}(\|f^+(t_i) - f^+(t_{i-1})\|_+ + \|f^-(t_i) - f^-(t_{i-1})\|_-)^q$

and therefore, $\left(\sum_{i=1}^{n}\left(\|f^+(t_i) - f^+(t_{i-1})\|_+ + \|f^-(t_i) - f^-(t_{i-1})\|_-\right)^p\right)^{1/p}$

is less than or equal to $\left(\sum_{i=1}^{n}(\|f^+(t_i) - f^+(t_{i-1})\|_+ + \|f^-(t_i) - f^-(t_{i-1})\|_-)^q\right)^{1/q}$,
it follows that:

$$\overset{p}{\mathrm{V}}{}_a^b(f,(\mathcal{K},[\cdot,\cdot])) \leq \overset{q}{\mathrm{V}}{}_a^b(f,(\mathcal{K},[\cdot,\cdot]))$$

From which it follows for cases *i)* y *ii)* that $\overset{p}{\mathrm{V}}{}_a^b(f,(\mathcal{K},[\cdot,\cdot]))$ is finite, therefore
$f \in B\overset{p}{V}{}_a^b(\mathcal{K},[\cdot,\cdot])$.

Next, the set of bounded $q-$variation functions on Krein space, que which we introduce in 7 will be given a norm, which we will call $q-$norm and denote by $\|\cdot\|_q$

Theorem 13. *Let* $(\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-, [\cdot,\cdot])$ *be a Krein space, the function*

$$||\cdot||_q : B\overset{q}{V}{}_a^b(\mathcal{K},[\cdot,\cdot]) \to \mathbb{R}$$

defined by

$$\|f\|_q = \|f^+(x)\|_+ + \|f^-(x)\|_- + \overset{q_+}{V}{}_a^b(f,(\mathcal{K}_+,[\cdot,\cdot])) + \overset{q_-}{V}{}_a^b(f,(\mathcal{K}_-,-[\cdot,\cdot])),$$

is a norm in $B\overset{q}{V}{}_a^b(\mathcal{K},[\cdot,\cdot])$.

Proof. Let $f, g \in B\overset{q}{V}{}_a^b(\mathcal{K},[\cdot,\cdot])$, $\lambda \in \mathbb{R}$.
If $x \in [a,b]$, then we have:

(i) $\|f^+(x)\|_+ \geq 0, \|f^-(x)\|_- \geq 0$. Also satisfied that $\overset{q_+}{\mathrm{V}}{}_a^b(f,(\mathcal{K}_+,[\cdot,\cdot])) \geq 0$ and $\overset{q_-}{\mathrm{V}}{}_a^b(f,(\mathcal{K}_-,-[\cdot,\cdot])) \geq 0$. Therefore, $\|f\|_{B\overset{q}{V}{}_a^b(\mathcal{K},[\cdot,\cdot])} \geq 0$.

(ii) If $\|f\|_{B\overset{q}{V}{}_a^b(\mathcal{K},[\cdot,\cdot])} = 0$, then

$$\overset{q_+}{V}{}_a^b(f,(\mathcal{K}_+,[\cdot,\cdot])) + \overset{q_-}{V}{}_a^b(f,(\mathcal{K}_-,-[\cdot,\cdot])) + \|f^+(x)\|_+ + \|f^-(x)\|_- = 0.$$

Since $\overset{q_+}{V_a^b}(f,(\mathcal{K}_+,[\cdot,\cdot]))$, $\overset{q_-}{V_a^b}(f,(\mathcal{K}_-,-[\cdot,\cdot]))$, $\|f^+(x)\|_+$, $\|f^-(x)\|_- \geq 0$, then

$$\overset{q_+}{V_a^b}(f,(\mathcal{K}_+,[\cdot,\cdot])) = \overset{q_-}{V_a^b}(f,(\mathcal{K}_-,-[\cdot,\cdot])) = \|f^+(x)\|_+ = \|f^-(x)\|_- = 0. \qquad (2)$$

As the positive and negative variations are zero, then by theorem 10 f is constant in $\mathcal{K}_+$ and in $\mathcal{K}_-$, so there exist $c^+ \in \mathcal{K}_+$ and $c^- \in \mathcal{K}_-$ such that:

$$f^+(x) = c^+ \text{ and } f^-(x) = c^-, \quad \forall x \in [a,b]$$

Then, using the above and equation 2 it follows that:

$$\|f^+(x)\|_+ = \|c^+\|_+ = 0 \text{ and } \|f^-(x)\|_- = \|c^-\|_- = 0$$

it implies that $c^+ = \mathbf{0}$ and $c^- = \mathbf{0}$. Therefore, $f(x) = \mathbf{0} + \mathbf{0} = \mathbf{0}$ for all $x \in [a,b]$, then f is the null function.

(iii) Since $f \in B\overset{q}{V_a^b}(\mathcal{K},[\cdot,\cdot])$ and $\lambda \in \mathbb{C}$, using (i) and (iii) of the remark 4, then

$$\begin{aligned}
&\|\lambda f\|_{B\overset{q}{V_a^b}(\mathcal{K},[\cdot,\cdot])} \\
&= \|\lambda f^+(x)\|_+ + \|\lambda f^-(x)\|_- + \overset{q_+}{V_a^b}(\lambda f,(\mathcal{K}^+,[\cdot,\cdot])) + \overset{q_-}{V_a^b}(\lambda f,(\mathcal{K}^-,-[\cdot,\cdot])) \\
&= |\lambda|\|f^+(x)\|_+ + |\lambda|\|f^-(x)\|_- + |\lambda|\overset{q_+}{V_a^b}(f,(\mathcal{K}^+,[\cdot,\cdot])) + |\lambda|\overset{q_-}{V_a^b}(f,(\mathcal{K}^-,-[\cdot,\cdot])) \\
&= |\lambda|(\|f^+(x)\|_+ + \|f^-(x)\|_- + \overset{q_+}{V_a^b}(f,(\mathcal{K}^+,[\cdot,\cdot])) + \overset{q_-}{V_a^b}(f,(\mathcal{K}^-,-[\cdot,\cdot]))) \\
&= |\lambda|\|f\|_{B\overset{q}{V_a^b}(\mathcal{K},[\cdot,\cdot])}.
\end{aligned}$$

(iv) **(Triangular inequality)**

Let $f,g \in B\overset{q}{V_a^b}(\mathcal{K},[\cdot,\cdot])$, using (ii) and (iv) of the remark 4, it follows that:

$$\begin{aligned}
&\|f+g\|_{B\overset{q}{V_a^b}(\mathcal{K},[\cdot,\cdot])} \\
&= \|(f+g)^+(x)\|_+ + \|(f+g)^-(x)\|_- + \overset{q_+}{V_a^b}(f+g,(\mathcal{K}_+,[\cdot,\cdot])) + \overset{q_-}{V_a^b}(f+g,(\mathcal{K}_-,-[\cdot,\cdot])) \\
&\leq \|f^+(x)\|_+ + \|g^+(x)\|_+ + \|f^-(x)\|_- + \|g^-(x)\|_- + \overset{q_+}{V_a^b}(f,(\mathcal{K}_+,[\cdot,\cdot])) + \\
&\overset{q_+}{V_a^b}(g,(\mathcal{K}_+,[\cdot,\cdot])) + \overset{q_-}{V_a^b}(f,(\mathcal{K}_-,-[\cdot,\cdot])) + \overset{q_-}{V_a^b}(g,(\mathcal{K}_-,-[\cdot,\cdot])) \\
&= \|f^+(x)\|_+ + \|f^-(x)\|_- + \overset{q_+}{V_a^b}(f,(\mathcal{K}_+,[\cdot,\cdot])) + \overset{q_-}{V_a^b}(f,(\mathcal{K}_-,-[\cdot,\cdot])) + \|g^+(x)\|_+ + \\
&\|g^-(x)\|_- + \overset{q_+}{V_a^b}(g,(\mathcal{K}_+,[\cdot,\cdot])) + \overset{q_-}{V_a^b}(g,(\mathcal{K}_-,-[\cdot,\cdot])) \\
&= \|f\|_{B\overset{q}{V}{}_a^b(\mathcal{K},[\cdot,\cdot])} + \|g\|_{B\overset{q}{V}{}_a^b(\mathcal{K},[\cdot,\cdot])}
\end{aligned}$$

Thus, $\|f\|_{\overset{q}{BV}{}_a^b(\mathcal{K},[\cdot,\cdot])} = \|f^+(x)\|_+ + \|f^-(x)\|_- + \overset{q_+}{V}{}_a^b(f,(\mathcal{K}_+,[\cdot,\cdot])) + \overset{q_-}{V}{}_a^b(f,(\mathcal{K}_-,-[\cdot,\cdot]))$ is a norm in $\overset{q}{BV}{}_a^b(\mathcal{K},[\cdot,\cdot])$.

6. Conclusion and future work

In this study, the concept of bounded q- variation function in Krein spaces was defined (definition 7) and exemplified (example 3) , extending the existing notion of these functions in Hilbert spaces. Classical results were extended (theorem 6, theorem 7, theorem 8, theorem 9, theorem 10, theorem 11, theorem 12, theorem 13), showing the potential that this research has for further extensions and applications. Future research could focus on extending work presented in [18] to spaces with indefinite metrics. In addition, investigate the interaction between bounded q-variational functions on spaces with indefinite metrics and fixed point theory. Knowing the importance of spaces of indefinite metric [19], in quantum mechanics [15], introducing a q-norm (theorem 13) for bounded variation functions in spaces with an indefinite metric can provide applications such as those given in which gives insight into the future impact of this work.

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Conflict of interest

The authors declare that they have no conflict of interest in this work.

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