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Upper Bound of Radio Span for Shadow-path Network and Its Mathematical Modeling

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Abstract. Motivated by the frequency assignment problem, we investigate radio labeling of graphs. In graph theory and discrete mathematics, radio labeling of graphs has received significant attention as it is of immense importance for numerous applications to a wide range of areas such as circuit and sensor network, signal processing, design, frequency assignment in mobile communication systems, etc. An assignment of labels satisfying specific constraints to the edges, vertices, or both of graph G is known as graph labeling. Radio labeling of a graph G is a technique of labeling vertices of G by non-negative integers. Hence, radio labeling problem presents an efficient graph modeling for the frequency assignment problem. In radio labeling of a graph G, the maximum label used for labeling vertices is called the span of that radio labeling. The minimum span from all radio labelings of G is known as the radio number of G. That radio number reflects the efficient usage of the available frequencies in frequency assignment for a network modeled by the graph G. This paper contributes the mathematical proof (Theorems 1-4), an integer linear programming model for finding the upper bound of radio number of shadow graphs. It also introduces an application of radio labeling of graphs in cryptography. Additionally, a computational study has been conducted wherein it demonstrates that our results (Theorems 1-4) outperform both the mathematical model, and the results published in the literature.

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Key Words and Phrases: Frequency(channel) assignment problem, Radio labeling, Integer linear programing, Radio number, Cryptography.

An overview of the notations that will be applied throughout the paper, listed in Table 1.

Table 1: Table of Notations.

Parameter	Description
$V(G)$	The set of vertices of the graph G.
$E(G)$	The set of edges of the graph G.
$d(u,v)$	The distance between two vertices u, v in a graph.
$diam(G)$	The diameter of G.
P_n	Path of n vertices.
C_n	Cycle with n vertices.
$rn(G)$	The radio number of G.
FAP	Frequency assignment problem
$D_2(P_n)$	The shadow path graph
ILPM	Integer Linear Programming Model

1. Introduction

The communication in wireless networks depends on the frequencies (channels) allotted to them. For an effective communication, the frequencies should assigned to transmitters in such a way that avoids the interference. Here, the interference is closely related to the geographical position of the transmitting stations. The interference is increasing whenever the stations are geographical close. To avoid such interference, we have to assign frequencies to stations such that difference between the assigned channels must be large enough. The process of assigning a limited number of available frequencies to transmitters in radio networks with avoiding interference is known as the *frequency assignment problem* (FAP). Wireless networks appear in a variety of services such as T.V. and radio broadcasting, Engineering, Economics, military services, and many other. Due to increasing popularity of wireless services and the limited available frequencies, FAP has a lot of interest from the scientific and the business communities. As the optimizing usage of the available frequencies means higher traffic capacity, more bandwidth and bigger coverage for the existing radio networks. As a result, many researchers and a wide variety of models have investigated FAPs and solution techniques have been proposed. The labeling technique in graph theory has played an important role in solving FAP; thereby the time and cost will be saved. For modeling FAP in graph theory, an *interference graph* $G = (V(G), E(G))$is constructed where $V(G)$ and $E(G)$ are the set of vertices and the set of edges of G respectively. This graph represents the interference between transmitters. Each vertex belong to $V(G)$ represents a unique transmitter. In the event of the broadcasting of the two transmitters may interfere, then their corresponding vertices from $V(G)$ are joined by an edge. Positive integers are used as labels for the frequency channels. Therefore, FAP [1] is equivalent to the vertex coloring (labeling) issue of the graph G with certain labeling constraints. In [2] author examined a comprehensive overview of graph labeling. Griggs and Yeh [3] presented the first graph labeling technique, known as $L(2,1)$ labeling or distance two labeling to address the frequency assignment problem. The graph labeling $L(2,1)$ of a simple graph G is a non-negative real-valued function $L : V\ (G) \to [0,\ \infty)$ where $V\,(G)$ is the vertex set of the graph G. Whenever u and v are two vertices in G such that the distance between u and v is 1, then

$\mid L(u)\ -\ L(v)\mid \geq\ 2$, and whenever the vertices u and v are of distance 2 from each other, then $\mid L(u)\ -\ L(v)\mid \geq\ 1|$.

Another graph labeling technique called *radio Labeling* was introduced by Chartrand et al. [4]. The radio labeling problem of a connected graph $G = (V\,(G)\,, E\,(G))$is denoted as follows. Let $u, v \in V(G)$. The distance $d\,(u, v)$ denote the length of the shortest path between u, v. The diameter $diam\,(G)$ of G is defined as the maximum distance between any two vertices in G. Thus,

$diam\,(G) = \max\{d\,(u, v) : u, v \in V(G)\ \}$. A *radio labeling* of G is an injective function f from $V\,(G)$ to $N = \{0, 1, 2, 3, \cdots\}$, satisfying the following constraints

$$|f\,(u) - f\,(v)| \geq diam\,(G) + 1 - d\,(u, v).\ \textit{For all}\ u,\ v \in V\,(G)\,.$$

The integer $f\,(u)$ is said to be the color (label) of u under f and, the *span* of f is denoted by

$$\operatorname{span}(f) = \max\{|f\,(u)\ - f\,(v)|\ :\ u,\ v\ \in\ V\ \ (G)\}\,.$$

The radio number of G, or $(rn\,(G))$, is defined as

$$rn\,(G) = \min\{\operatorname{span}(f)\}$$

for all radio labeling f of G. The objective of radio labeling problem of G is finding the value of $rn\,(G)$.

2. Related Work

From viewpoint of complexity, the problem of getting the radio number of a given graph is NP complete [5]. Saha and Panigrahi [6] presented an algorithm which finds the upper bounds of rn (G) for a given graph G. In [7] Badr and Moussa suggested a developing algorithm for one presented in [6]. Besides that they gave an approach for solving the radio labeling problem using mathematical modeling. Recently radio labeling of graphs and finding the exact or the upper bounds of radio number of graphs has a lot of attention. In [8] an assignment of radio numbers to triangular networks and rhombic honeycomb networks is discussed. The study of the radio labeling of fuzzy graphs is studied in [9]. In addition, the study of the radio labeling for many different types of graphs has been the subject of intellectual research by several authors [10–23].

Many authors have found the radio labeling and radio labeling of path graphs, the middle graph of the path, a strong product of the path graph, the cross product of the path, and the corona product of the path sub division of paths,etc. This study aims to fill the research gaps by determining the radio number for the shadow of path graph. In this paper, we are concerned with finding an upper bound for radio number rn(G) of networks modeled by the shadow of path graph. We present a theoretical approach for getting an upper bound of radio number of the shadow of path graph. Moreover, modeling the problem of radio labeling of the shadow path graph using integer linear programing is proposed. Besides that, an experimental study is carried out in order to demonstrate our approach's effectiveness in comparison to previously published results. Overall, our investigation demonstrates that, in terms of the running time and radio numbers' upper bound, the suggested results perform better than the earlier findings.

The rest of the paper is structured as follows. Section 3 introduces our theoretical approach for finding an upper bound of radio number of the shadow of path graph. The modeling of radio labeling of shadow of path graph using integer linear programing is presented in section 4. Section 5, gives the computational study for comparing our theoretical approach, integer programing model and previous algorithms known in the literature. The application of radio labeling technique in cryptography is introduced in Section 6. The conclusion of work is presented in section 7.

3. Theoretical approach for upper bound of radio number of shadow networks

Hereafter, we are interested in getting an upper bound of radio number of shadow networks.

Definition 1. *Let P_n be the path with n vertices. The shadow graph $D_2(P_n)$ is constructed by taking two copies of P_n. Join each vertex u in the first path to the neighbors of the corresponding vertex v in the second path as shown in Figure 1.*

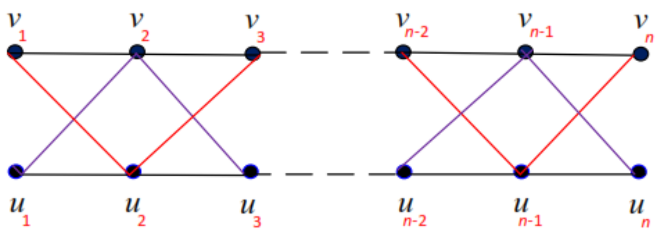

Figure 1: The graph $D_2(P_n)$.

In $D_2(P_n)$, the set of vertices is $V(D_2(P_n)) = \{u_i,\ v_i : 1 \le i \le n\}$ and the set of edges is defined as:

$$E(D_2(P_n)) = \{u_iu_{i+1}, v_iv_{i+1}\ v_i : 1 \leq i \leq n-1\} \cup \{u_iv_{i+1}, v_iu_{i+1}\ v_i : 1 \leq i \leq n-1\}.$$

For convenience, we divided all vertices of $D_2(P_n)$ into two subsets V_u and V_v. Where $V_u = \{u_i : 1 \leq i \leq n\}$ and $V_v = \{v_i : 1 \leq i \leq n\}$. Hence, the graph $D_2(P_n)$ has a set of $2n$ vertices; that is $|V_u| = |V_v| = n$ and the diameter of $D_2(P_n)$ equals $n-1, n > 2$.

In the following, we investigate the radio number of $D_2(P_n)$.

Lemma 1. *For a positive integer* $n = 2, rn(D_2(P_n)) \leq 4$.

Proof. Let $(f(u_1), f(u_2), f(v_1), f(v_2)) = (0, 3, 1, 4)$.

Lemma 2. *For a positive integer* $n = 3, rn(D_2(P_n)) \leq 6$.

Proof. Let $(f(u_1), f(u_2), f(u_3), f(v_1), f(v_2), f(v_3)) = (0, 5, 1, 2, 6, 3)$.

Lemma 3. *For a positive integer* $n = 4, rn(D_2(P_n)) \leq 12$.

Proof. For $n = 4$, let
$(f(u_1), f(u_2), f(u_3), f(u_4), f(v_1), f(v_2), f(v_3), f(v_4)) = (0, 5, 10, 1, 2, 7, 12, 3)$.

Lemma 4. *For a positive integer* $n = 5, rn(D_2(P_n)) \leq 22$.

Proof. For $n = 5$ Let

$$\begin{aligned}&(f(u_1), f(u_2), f(u_3), f(u_4), f(u_5), f(v_1), f(v_2), f(v_3), f(v_4), f(v_5)) = \\ &= (6, 13, 0, 16, 7, 9, 19, 3, 22, 10)\end{aligned}$$

In the next theorems, we find upper bound of $rn(D_2(P_n))$ for $n \geq 6$.

Theorem 1. *Let* $n \geq 6$ *be a positive integer such that* $n \equiv 2(mod4)$. *Then*

$$rn(D_2(P_n)) \leq n^2 - \frac{n}{2}.$$

Proof. For $k \geq 1, n \equiv 2(mod4)$ such that $n = 4k+2$, let $f : V(D_2(P_n)) \to N$ define as follows:

$$f(u_i) = \begin{cases} (2n+3)(i-1) + 2(i-1)(i-2), & \text{if } \underline{1 \leq i \leq k+1} \\ f(u_{k+1}) - (2n-1) - (3n-5)(j-1) + 2(j-1)(j-2), & \\ \quad \text{if } \underline{k+2 \leq i \leq 2k+1} \text{ where } 1 \leq j \leq k, \text{ and } i = k+1+j & \\ f(u_{k+1+j}) + (6k+1) - 2(j-1), & \text{if } \underline{2k+2 \leq i \leq 3k+1}, \\ \quad \text{where } 1 \leq j \leq k, \text{ and }, \ i = 3k+2-j & \\ f(u_j) + 2j - 1, & \text{if } \underline{3k+2 \leq i \leq 4k+2 = n}, \\ \quad \text{where } 1 \leq j \leq k+1, \text{ and } i = 4k+3-j & \end{cases}$$

And for the vertices belongs to V_v,

$$f(v_i) = \begin{cases} (n+2) + (2n+7)(i-1) + 2(i-1)(i-2), \quad \text{if} \quad \underline{1 \leq i \leq k}, \\ f(v_k) + (2n+1) + (n+1)(j-1) + 2(j-1)(j-2), \\ \quad \text{if} \quad \underline{k+1 \leq i \leq 2k+1}, \quad \text{where} \quad 1 \leq j \leq k+1, \text{and} i = k+j \\ f(v_{k+j}) + (2k+1) + 2(j-1), \quad \text{if} \quad \underline{2k+2 \leq i \leq 3k+2}, \\ \quad \text{where} \quad 1 \leq j \leq k+1, \quad \text{and} \quad i = 3k+3-j \\ f(v_j) - 2j + 1, \text{if} \quad \underline{3k+3 \leq i \leq 4k+2 = n}, \\ \quad \text{where} \quad 1 \leq j \leq k, \quad \text{and} \quad i = 4k+3-j \end{cases}$$

Now we claim to prove that the function f given above is a radio labelling of $D_2(P_n)$ Consequently, It is required that f verifies the following condition

$$|f(x) - f(y)| \geq n - d(x, y)$$

for any two distinct vertices $x, y \in V(D_2(P_n))$.

case 1 $x, y \in V_u$. If $\{x, y\} = \{u_1, u_n\}$ then

$$\begin{aligned} |f(x) - f(y)| &= |f(u_1) - f(u_n)| \\ &= |f(u_1) - [f(u_1) + 2 - 1]| = 1 \\ &\geq diam(D_2(P_n)) + 1 - d(x, y), \text{ where } \quad d(x, y) = n - 1. \end{aligned}$$

case 2 If $\{x, y\} = \{u_1, u_{k+1}\}$ then

$$\begin{aligned} |f(x) - f(y)| &= |f(u_1) - f(u_{k+1})| \\ &= |f(u_1) - [(2n+3)(k) + 2k(k-1)]| \\ &= |-[(2n+3)(k) + 2k(k-1)]| \\ &= (2n+3)(k) + 2k(k-1) \geq n - k, \\ &\geq diam(D_2(P_n)) + 1 - d(x, y), \text{ where } \quad d(x, y) = k \end{aligned}$$

case 3 If $\{x, y\} = \{u_1, u_{2k+1}\}$ then $d(x, y) = 2k$

$$\begin{aligned} |f(x) - f(y)| &= |f(u_1) - f(u_{2k+1})| \\ &= |f(u_1) - [f(u_{k+1}) - (2n-1) - (3n-5)(k-1) + 2(k-1)(k-2)]| \\ &= |f(u_{k+1}) - (2n-1) - (3n-5)(k-1) + 2(k-1)(k-2)| \\ &= |(2n+3)(k) + 2k(k-1) - (2n-1) - (3n-5)(k-1) + 2(k-1)(k-2)| \\ &\geq n - 2k \end{aligned}$$

case 4 If $\{x, y\} = \{u_1, u_{3k+1}\}$ then $d(x, y) = 3k$

$$\begin{aligned} |f(x) - f(y)| &= |f(u_1) - f(u_{3k+1})| \\ &= |f(u_1) - [f(u_{k+2}) + (6k+1)]| \\ &= |f(u_{k+2}) + (6k+1)| \end{aligned}$$

$$
\begin{aligned}
&= |f(u_{k+1}) - (2n-1) + (6k+1)| \\
&= |(2n+3)(k) + 2k(k-1) - (2n-1) + (6k+1)| \\
&\geq n - 3k
\end{aligned}
$$

case 5 $x, y \in V_v$. If $\{x, y\} = \{v_1, v_n\}$ then

$$
\begin{aligned}
|f(x) - f(y)| &= |f(v_1) - f(v_n)| \\
&= |f(v_1) - [f(v_1) - 1]| = 1 \\
&\geq diam(D_2(P_n)) + 1 - d(x, y), \text{ where } \quad d(x, y) = n - 1.
\end{aligned}
$$

case 6 If $\{x, y\} = \{v_1, v_k\}$ then

$$
\begin{aligned}
|f(x) - f(y)| &= |f(v_1) - f(v_k)| \\
&= |(n+2) - [(n+2) + (2n+7)(k-1) + 2(k-1)(k-2)]| \\
&= |(2n+7)(k-1) + 2(k-1)(k-2)| \\
&\geq n, \text{ where } \quad n = 4k+2 \\
&\geq diam(D_2(P_n)) + 1 - d(x, y)
\end{aligned}
$$

case 7 If $\{x, y\} = \{v_1, v_{2k+1}\}$ then

$$
\begin{aligned}
|f(x) - f(y)| &= |f(v_1) - f(v_{2k+1})| \\
&= |(n+2) - [(n+2) + (2n+7)(k-1) + 2(k-1)(k-2)+ \\
&+ (2n+1) + (n+1)(k) + 2(k)(k-1)]| \\
&= |(2n+7)(k-1) + 2(k-1)(k-2) + (2n+1) + (n+1)(k) + 2(k)(k-1)| \\
&\geq n, \text{ where } \quad n = 4k+2, d(x, y) \geq 1 \\
&\geq diam(D_2(P_n)) + 1 - d(x, y)
\end{aligned}
$$

case 8 If $\{x, y\} = \{v_1, v_{3k+2}\}$ then

$$
\begin{aligned}
|f(x) - f(y)| &= |f(v_1) - f(v_{3k+2})| \\
&= |f(v_1) - [f(v_{k+1}) + (2k+1)]| \\
&= |f(v_1) - [f(v_k) + (2n+1) + (2k+1)]| \\
&= |(n+2) - [(n+2) + (2n+7)(k-1) + 2(k-1)(k-2)+ \\
&\quad + (2n+1) + (2k+1)]| \\
&= |(2n+7)(k-1) + 2(k-1)(k-2) + (2n+1) + (2k+1)| \\
&\geq n - (3k+1), \text{ where } \quad n = 4k+2, d(x, y) = 3k+1 \\
&\geq diam(D_2(P_n)) + 1 - d(x, y)
\end{aligned}
$$

case 9 $x \in V_u$ **and** $y \in V_v$
Let $x = u_i$ and $y = v_l$, where $1 \leq i, l \leq n$
In this case we have several subcases:

<u>Subcase 9.1</u> If $i = 1$, $j = k$ then

$$\begin{aligned}|f(x) - f(y)| &= |f(u_1) - f(v_k)| \\ &= |0 - [(n+2) + (2n+7)(k-1) + 2(k-1)(k-2)]| \\ &= |(n+2) + (2n+7)(k-1) + 2(k-1)(k-2)| \\ &\geq n - (k-1), \text{ where } \quad , d(x,y) = k-1 \\ &\geq diam(D_2(P_n)) + 1 - d(x,y).\end{aligned}$$

Otherwise, $|f(x) - f(y)| \geq n$.

<u>Subcase 9.2</u> , $1 \leq i \leq k+1$ **and** $k+1 \leq l \leq 2k+1$

$$\begin{aligned}|f(x) - f(y)| &= |f(u_i) - f(v_l)| \\ &= |f(u_i) - f(v_{k+j})|, \text{ where } \quad 1 \leq j \leq k \\ &= |(2n+3)(i-1) + 2(i-1)(i-2) - [f(v_k) + (2n+1) + (n+1)(j-1) + \\ &\quad + 2(j-1)(j-2)]| \\ &= |(2n+3)(i-1) + 2(i-1)(i-2) - [(n+2) + (2n+7)(k-1) + \\ &+ 2(k-1)(k-2) + (2n+1) + (n+1)(j-1) + 2(j-1)(j-2)]|.\end{aligned}$$

If $i = 1$ **,** $l = k+1$ **i.e.** $j = 1$ then

$$\begin{aligned}|f(x) - f(y)| &= |f(u_i) - f(v_l)| \\ &= |f(u_i) - f(v_{k+j})|, \text{ where } \quad 1 \leq j \leq k \\ &= |0 - [f(v_k) + (2n+1) + (n+1)(j-1) + 2(j-1)(j-2)]| \\ &= |-[(n+2) + (2n+7)(k-1) + 2(k-1)(k-2) + (2n+1)| \\ &= |(n+2) + (2n+7)(k-1) + 2(k-1)(k-2) + (2n+1)| \\ &\geq n, \text{ where } \quad n = 4k+2, d(x,y) \geq 1 \\ &\geq diam(D_2(P_n)) + 1 - d(x,y).\end{aligned}$$

<u>Subcase 9.3.</u> Suppose, $1 \leq i \leq k+1$ and $2k+2 \leq l \leq 3k+2$ If $i = 1$ **,** $l = 3k+2$ **i.e.** $j = 1$ then

$$\begin{aligned}|f(x) - f(y)| &= |f(u_1) - f(v_{3k+2})| \\ &= |f(u_i) - f(v_{n-k})| \\ &= |0 - f(v_{k+1}) + 2k + 1| \\ &= |(n+2) + (2n+7)(k-1) + 2(k-1)(k-2) + 2k + 1| \\ &\geq n, \text{ where } \quad n = 4k+2, d(x,y) \geq 1 \\ &\geq diam(D_2(P_n)) + 1 - d(x,y).\end{aligned}$$

Otherwise, $|f(x) - f(y)| \geq n$.

For more illustration, Figure 2 shows the radio labeling of $D_2(P_{14})$ according to Theorem 1.

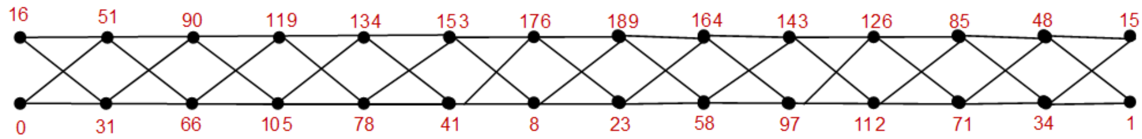

Figure 2: Radio labeling of $D_2(P_14)$ following Theorem 1.

Theorem 2. *Let $n > 6$ be a positive integer such that $n \equiv 3(mod4)$. Then*

$$rn(D_2(P_n)) \leq n^2 - 3.$$

Proof. Let $k \geq 1, n \equiv 3(mod4)$ such that $n = 4k+3$, we define $f : V(D_2(P_n)) \to N$ as follows:

$$f(u_i) = \begin{cases} (2n+3)(i-1) + 2(i-1)(i-2), & \text{if } \underline{1 \leq i \leq k+1} \\ f(u_{k+1}) - (2n-2) - (3n-6)(j-1) + 2(j-1)(j-2), & \\ \quad \text{if } \underline{k+2 \leq i \leq 2k+1}, \quad \text{where } 1 \leq j \leq k, \quad \text{and} \quad i = k+1+j & \\ n^2 - n - 1, \quad \text{if } \underline{i = 2k+2} & \\ f(u_{k+1+j}) + (6k+2) - 2(j-1), \quad \text{if } \underline{2k+3 \leq i \leq 3k+2} & \\ \quad \text{where } 1 \leq j \leq k, \quad \text{and} \quad i = 3k+3-j) & \\ f(u_j) + 2j - 1, \quad \text{if } \underline{3k+3 \leq i \leq 4k+3 = n} & \\ \quad \text{where } 1 \leq j \leq k+1, \quad \text{and} \quad i = 4k+4-j & \end{cases}$$

And for the vertices belongs to V_v,

$$f(v_i) = \begin{cases} (n+3) + (2n+7)(i-1) + 2(i-1)(i-2), & \text{if } \underline{1 \leq i \leq k} \\ f(v_k) + (2n-1) + n(j-1) + 2(j-1)(j-2), & \\ \quad \text{if } \underline{k+1 \leq i \leq 2k+1}, \text{where } 1 \leq j \leq k+1, \quad \text{and} \quad i = k+j & \\ n^2 - 3, \quad \text{if } \underline{i = 2k+2} & \\ f(v_{k+j}) + (2k+1) + 2(j-1), \quad \text{if } \underline{2k+3 \leq i \leq 3k+3} & \\ \quad \text{where } 1 \leq j \leq k+1, \quad \text{and} \quad i = 3k+4-j & \\ f(v_{n+1-j}) = f(v_j) - 2j, \quad \text{if } \underline{3k+4 \leq i \leq 4k+3 = n} & \\ \quad \text{where } 1 \leq j \leq k, \quad \text{and} \quad i = 4k+4-j & \end{cases}$$

Now we claim to prove that the function f given above is a radio labelling of $D_2(P_n)$ Consequently, It is required that f verifies the following condition with a span equal to the desired number. So, we need only to check it for $n \geq 6$, $n = 4k+3$ where $k \geq 1$. It remains to verify that

$$|f(x) - f(y)| \geq diam(D_2(P_n)) + 1 - d(x, y)$$

for any two distinct vertices x, y of $D_2(P_n)$. We distinguish several cases:

case 1 $x, y \in V_u$. If $\{x, y\} = \{u_1, u_n\}$ then

$$\begin{aligned} |f(x) - f(y)| &= |f(u_1) - f(u_n)| \\ &= |f(u_1) - [f(u_1) + 2 - 1]| = 1 \\ &\geq diam(D_2(P_n)) + 1 - d(x, y), \text{ where } \quad d(x, y) = n - 1. \end{aligned}$$

case 2 If $\{x, y\} = \{u_1, u_{k+1}\}$ then

$$\begin{aligned}
|f(x) - f(y)| &= |f(u_1) - f(u_{k+1})| \\
&= |f(u_1) - [(2n+3)(k) + 2k(k-1)]| \\
&= |-[(2n+3)(k) + 2k(k-1)]| \\
&= (2n+3)(k) + 2k(k-1) \geq n \\
&\geq diam(D_2(P_n)) + 1 - d(x, y)
\end{aligned}$$

case 3 If $\{x, y\} = \{u_1, u_{2k+1}\}$ then

$$\begin{aligned}
|f(x) - f(y)| &= |f(u_1) - f(u_{2k+1})| \\
&= |f(u_1) - [f(u_{k+1}) - (2n-2) - (3n-6)(k-1) + 2(k-1)(k-2)]| \\
&= |f(u_{k+1}) - (2n-2) - (3n-6)(k-1) + 2(k-1)(k-2)| \\
&= |(2n+3)(k) + 2k(k-1) - (2n-2) - (3n-6)(k-1) + 2(k-1)(k-2)| \\
&\geq n - 2k \\
&\geq diam(D_2(P_n)) + 1 - d(x, y)
\end{aligned}$$

case 4 If $\{x, y\} = \{u_1, u_{2k+2}\}$ then

$$\begin{aligned}
|f(x) - f(y)| &= |f(u_1) - f(u_{2k+2})| \\
&= |f(u_1) - [n^2 - n - 1]| \\
&= |n^2 - n - 1| \\
&\geq n \\
&\geq diam(D_2(P_n)) + 1 - d(x, y)
\end{aligned}$$

case 5 If $\{x, y\} = \{u_1, u_{3k+2}\}$ then

$$\begin{aligned}
|f(x) - f(y)| &= |f(u_1) - f(u_{3k+2})| \\
&= |f(u_1) - [f(u_{k+2}) + (6k+2)]| \\
&= |f(u_{k+2}) + (6k+2)| \\
&= |(n^2 - n - 1) + (6k+2)| \\
&\geq n \\
&\geq diam(D_2(P_n)) + 1 - d(x, y)
\end{aligned}$$

case 6 $x, y \in V_v$.
If $\{x, y\} = \{v_1, v_n\}$ then

$$\begin{aligned}
|f(x) - f(y)| &= |f(v_1) - f(v_n)| \\
&= |f(v_1) - [f(v_1) - 2]| = 2 \\
&\geq diam(D_2(P_n)) + 1 - d(x, y), \text{ where } \quad d(x, y) = n - 1.
\end{aligned}$$

case 7 If $\{x, y\} = \{v_1, v_k\}$ then

$$\begin{aligned}|f(x) - f(y)| &= |f(v_1) - f(v_k)|\\ &= |(n+3) - [(n+3) + (2n+7)(k-1) + 2(k-1)(k-2)]|\\ &= |(2n+7)(k-1) + 2(k-1)(k-2)|\\ &\geq n, \text{ where } \quad n = 4k+3, d(x,y) \geq 1\\ &\geq diam(D_2(P_n)) + 1 - d(x,y)\end{aligned}$$

case 8 If $\{x, y\} = \{v_1, v_{2k+1}\}$ then

$$\begin{aligned}|f(x) - f(y)| &= |f(v_1) - f(v_{2k+1})|\\ &= |(n+3) - [(n+3) + (2n+7)(k-1) + 2(k-1)(k-2)+\\ &\quad + (2n-1) + (n)(k) + 2(k)(k-1)]|\\ &= |(2n+7)(k-1) + 2(k-1)(k-2) + (2n-1) + (n)(k) + 2(k)(k-1)|\\ &\geq n, \text{ where } \quad n = 4k+3, d(x,y) \geq 1\\ &\geq diam(D_2(P_n)) + 1 - d(x,y)\end{aligned}$$

case 9 If $\{x, y\} = \{v_1, v_{2k+2}\}$ then

$$\begin{aligned}|f(x) - f(y)| &= |f(v_1) - f(v_{2k+2})|\\ &= |f(v_1) - [n^2 - 3]|\\ &= |n^2 - 3|\\ &\geq n\\ &\geq diam(D_2(P_n)) + 1 - d(x,y)\end{aligned}$$

case 10 If $\{x, y\} = \{v_1, v_{3k+3}\}$ then

$$\begin{aligned}|f(x) - f(y)| &= |f(v_1) - f(v_{3k+3})|\\ &= |f(v_1) - [f(v_{k+1}) + (2k+1)]|\\ &= |f(v_1) - [f(v_k) + (2n-1) + (2k+1)]|\\ &= |(n+2) - [(n+3) + (2n+7)(k-1) + 2(k-1)(k-2) + (2n+1)+\\ &\quad + (2k+1)]|\\ &= |(2n+7)(k-1) + 2(k-1)(k-2) + (2n+1) + (2k+1) + 1|\\ &\geq n\\ &\geq diam(D_2(P_n)) + 1 - d(x,y)\end{aligned}$$

case 11 $x \in V_u$ **and** $y \in V_v$

Let $x = u_i$ and $y = v_l$, where $1 \leq i, l \leq n$

In this case we have several subcases:

Subcase 11.1 If $i = l$, $1 \leq i, l \leq k$ then

$$|f(x) - f(y)| = |f(u_i) - f(v_i)|, \quad 1 \leq i \leq k$$

$$\begin{aligned}
&= |(2n+3)(i-1) + 2(i-1)(i-2) - [(n+3) + (2n+7)(i-1)+ \\
&\quad + 2(i-1)(i-2)]| \\
&= |-[(n+3) + (4)(i-1)]| \\
&= |(n+3) + (4)(i-1)|, \quad 1 \le i \le k \\
&\ge n \\
&\ge diam(D_2(P_n)) + 1 - d(x,y).
\end{aligned}$$

Subcase 11.2 If $i = l = 2k+2$

$$\begin{aligned}
|f(x) - f(y)| &= |f(u_i) - f(v_i)|, \quad i = 2k+2 \\
&= |n^2 - n - 1 - [n^2 - 3]| \\
&= |-[(n-2)]| \\
&= |(n-2)|, \text{where} \quad d(x,y) = 2 \\
&\ge diam(D_2(P_n)) + 1 - d(x,y).
\end{aligned}$$

Subcase 11.3 If $i = l$, $3k+4 \le i, l \le 4k+3 = n$ then

$$\begin{aligned}
|f(x) - f(y)| &= |f(u_i) - f(v_i)|, \quad 3k+4 \le i, l \le 4k+3 = n \\
&= |f(u_j) + 2j - 1 - [f(v_j) - 2j]|, \quad 1 \le j \le k \\
&= |f(u_j) - f(v_j) + 4j - 1|, \quad 1 \le j \le k \\
&\ge diam(D_2(P_n)) + 1 - d(x,y).
\end{aligned}$$

Otherwise, $|f(x) - f(y)| \ge n$.

Subcase 11.4 Assume $i \ne l$. Let, $1 \le i \le k+1$ and $k+1 \le l \le 2k+1$

$$\begin{aligned}
|f(x) - f(y)| &= |f(u_i) - f(v_l)| \\
&= |f(u_i) - f(v_{k+j})|, \text{ where } \quad 1 \le j \le k \\
&= |(2n+3)(i-1) + 2(i-1)(i-2) - [f(v_k) + (2n-1)+ \\
&\quad + (n)(j-1) + 2(j-1)(j-2)]|, \\
&= |(2n+3)(i-1) + 2(i-1)(i-2) - [(n+3) + (2n+7)(k-1)+ \\
&\quad + 2(k-1)(k-2) + (2n-1) + (n)(j-1) + 2(j-1)(j-2)]|
\end{aligned}$$

If $i = 1$, $l = 2k+1$ i.e. $j = k+1$ then

$$\begin{aligned}
|f(x) - f(y)| &= |f(u_i) - f(v_l)| \\
&= |f(u_1) - f(v_{2k+1})| \\
&= |0 - [f(v_k) + (2n-1) + (n)(j-1) + 2(j-1)(j-2)]| \\
&= |-(n+3) + (2n+7)(k-1) + 2(k-1)(k-2)+ \\
&= |(n+3) + (2n+7)(k-1) + 2(k-1)(k-2)+ \\
&\quad + (2n-1) + (n)(k) + 2(k)(k-1)|
\end{aligned}$$

$$
\begin{aligned}
&\geq n, \text{ where } \quad n = 4k+3, d(x,y) \geq 1 \\
&\geq diam(D_2(P_n)) + 1 - d(x,y).
\end{aligned}
$$

Otherwise, $|f(x) - f(y)| \geq n$.

If $i = 4k+3$ and $l = 2k+2$ then

$$
\begin{aligned}
|f(x) - f(y)| &= |f(u_i) - f(v_l)| \\
&= |f(u_{4k+3}) - f(v_{2k+2})| \\
&= |f(u_1) + 1 - f(v_{2k+2})| \\
&= |0 + 1 - [n^2 - 3]| \\
&= |-[n^2 - 3 - 1]| \\
&= |n^2 - 4| \\
&\geq n, \text{where} \quad n = 4k+3, k \geq 1, d(x,y) \geq 1 \\
&\geq diam(D_2(P_n)) + 1 - d(x,y).
\end{aligned}
$$

Otherwise, $|f(x) - f(y)| \geq n$. The radio condition is $|f(x) - f(y)| \geq n$ satisfied by every pair of vertices

Figure 3. , illustrates the proof of this case, presents the labeling $D_2(P_{15})$ According Theorem 2.

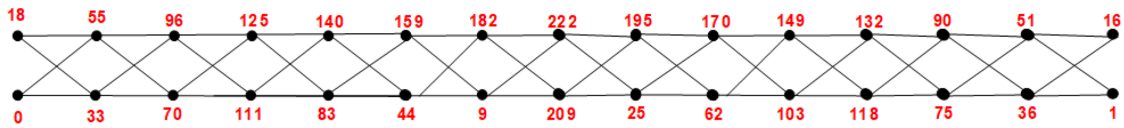

Figure 3: Radio labeling of $D_2(P_{15})$ following Theorem 2.

Theorem 3. *Let $n > 6$ be a positive integer such that $n \equiv 0(mod 4)$. Then*

$$rn(D_2(P_n)) \leq n^2 - \frac{n}{2}.$$

Proof. For any positive integer $n \equiv 0(mod4)$, such that $n = 4k, k \geq 2$, let $f : V(D_2(P_n)) \to N$ define as follows:

$$
f(u_i) = \begin{cases}
(2n+3)(i-1) + 2(i-1)(i-2), \quad \text{if} \quad \underline{1 \leq i \leq k} \\
f(u_k) + 2n - 3 \quad \text{if} \quad \underline{i = k+1} \\
f(u_k) - 2n + 2 - (3n-7)(j-1) + 2(j-1)(j-2), \\
\quad \text{if} \quad \underline{k+2 \leq i \leq 2k}, \quad \text{where} \quad 1 \leq j \leq k-1, \quad \text{and} \quad i = k+1+j \\
f(u_{k+1+j}) + (6k-3) - 2(j-1), \quad \text{if} \quad \underline{2k+1 \leq i \leq 3k-1} \\
\quad \text{where} \quad 1 \leq j \leq k-1, \quad \text{and} \quad i = 3k - J \\
f(u_j) + 2j - 1, \quad \text{then} \quad \underline{3k \leq i \leq 4k = n} \\
\quad \text{where} \quad 1 \leq j \leq k+1 \quad \text{and} \quad i = 4k+1-j
\end{cases}
$$

And for the vertices belongs to V_v,

$$f(v_i) = \begin{cases} (n+2) + (2n+7)(i-1) + 2(i-1)(i-2), \quad \text{if} \quad \underline{1 \leq i \leq k-1} \\ f(v_{k-1}) + 2n - 1 \quad \text{if} \quad \underline{i = k} \\ f(v_k) + (2n+1) + (n+3)(j-1) + 2(j-1)(j-2), \\ \quad \text{if} \quad \underline{k \leq i \leq 2k} \quad \text{where} \quad 1 \leq j \leq k, \text{ and } \quad i = k+j \\ f(v_{k-1+j}) + (2k-1) + 2(j-1), \quad \text{if} \quad \underline{2k+1 \leq i \leq 3k+1} \\ \quad \text{where} \quad 1 \leq j \leq k+1 \text{and} \quad i = 3k+2-j \\ f(v_j) - (2j-1), \quad \text{if} \quad \underline{3k+2 \leq i \leq 4k = n} \\ \quad \text{where} \quad 1 \leq j \leq k-1, \text{and} \quad i = 4k+1-j \end{cases}$$

Hereafter, we show that the function f satisfies the radio labeling condition for any two distinct vertices x, y of $D_2(P_n)$ for $n > 6$, $n = 4k$ where $k \geq 2$.

case 1 $x, y \in V_u$. If $\{x, y\} = \{u_1, u_n\}$ then

$$\begin{aligned} |f(x) - f(y)| &= |f(u_1) - f(u_n)| \\ &= |f(u_1) - [f(u_1) + 2 - 1]| = 1 \\ &\geq diam(D_2(P_n)) + 1 - d(x, y), \text{ where } \quad d(x, y) = n - 1. \end{aligned}$$

case 2 If $\{x, y\} = \{u_1, u_k\}$ then

$$\begin{aligned} |f(x) - f(y)| &= |f(u_1) - f(u_k)| \\ &= |f(u_1) - [(2n+3)(k-1) + 2(k-1)(k-2)]| \\ &= | - [(2n+3)(k-1) + 2(k-1)(k-2)]| \\ &= (2n+3)(k-1) + 2(k-1)(k-2) \geq n, \text{ where } \quad n = 4k \\ &\geq diam(D_2(P_n)) + 1 - d(x, y) \end{aligned}$$

case 3 If $\{x, y\} = \{u_1, u_{k+1}\}$ then

$$\begin{aligned} |f(x) - f(y)| &= |f(u_1) - f(u_{k+1})| \\ &= |f(u_1) - [f(u_k) + 2n - 3]| \\ &= |[f(u_k) + 2n - 3]| \\ &= |(2n+3)(k-1) + 2(k-1)(k-2) + 2n - 3| \\ &\geq n, \text{ where } \quad n = 4k, d(x, y) \geq 1 \\ &\geq diam(D_2(P_n)) + 1 - d(x, y) \end{aligned}$$

case 4 If $\{x, y\} = \{u_1, u_{2k}\}$ then

$$\begin{aligned} |f(x) - f(y)| &= |f(u_1) - f(u_{2k})| \\ &= |f(u_1) - [f(u_k) - 2n + 2 - (3n-7)(k-2) + 2(k-2)(k-3)]| \\ &= |f(u_k) - 2n + 2 - (3n-7)(k-2) + 2(k-2)(k-3)| \\ &= |(2n+3)(k-1) + 2(k-1)(k-2) - 2n + 2 - (3n-7)(k-2)+ \end{aligned}$$

$$+ 2(k-2)(k-3)|$$
$$\geq n, \text{ where } \quad n = 4k, d(x,y) \geq 1$$
$$\geq diam(D_2(P_n)) + 1 - d(x,y)$$

case 5 If $\{x,y\} = \{u_1, u_{3k+1}\}$ then

$$\begin{aligned}|f(x) - f(y)| &= |f(u_1) - f(u_{3k-1})|\\ &= |f(u_1) - [f(u_{k+2}) + (6k-3)]|\\ &= |f(u_{k+2}) + (6k-3)|\\ &= |f(u_k) - 2n + 2 + (6k-3)|\\ &= |(2n+3)(k-1) + 2(k-1)(k-2) - 2n + 2 + (6k-3)|\\ &\geq n, \text{ where } \quad n = 4k, d(x,y) \geq 1\\ &\geq diam(D_2(P_n)) + 1 - d(x,y)\end{aligned}$$

case 6 $x, y \in V_v$. If $\{x,y\} = \{v_1, v_n\}$ then

$$\begin{aligned}|f(x) - f(y)| &= |f(v_1) - f(v_n)|\\ &= |f(v_1) - [f(v_1) - 1]| = 1\\ &\geq diam(D_2(P_n)) + 1 - d(x,y), \text{ where } \quad d(x,y) = n-1.\end{aligned}$$

case 7 If $\{x,y\} = \{v_1, v_{k-1}\}$ then

$$\begin{aligned}|f(x) - f(y)| &= |f(v_1) - f(v_{k-1})|\\ &= |(n+2) - [(n+2) + (2n+7)(k-2) + 2(k-2)(k-3)]|\\ &= |(2n+7)(k-2) + 2(k-2)(k-3)|\\ &\geq n, \text{ where } \quad n = 4k\\ &\geq diam(D_2(P_n)) + 1 - d(x,y).\end{aligned}$$

case 8 If $\{x,y\} = \{v_1, v_{2k-1}\}$ then

$$\begin{aligned}|f(x) - f(y)| &= |f(v_1) - f(v_{2k-1})|\\ &= |(n+2) - [(n+2) + (2n+7)(k-2) + 2(k-2)(k-3)+\\ &+ (2n+1) + (n+1)(k-1) + 2(k-1)(k-2)]|\\ &= |(2n+7)(k-1) + 2(k-2)(k-3) + (2n+1) + (n+1)(k-1)+\\ &\quad + 2(k-1)(k-2)|\\ &\geq n, \text{ where } \quad n = 4k, d(x,y) \geq 1\\ &\geq diam(D_2(P_n)) + 1 - d(x,y)\end{aligned}$$

case 9 If $\{x,y\} = \{v_1, v_{3k-1}\}$ then

$$|f(x) - f(y)| = |f(v_1) - f(v_{3k-1})|$$

$$\begin{aligned}
&= |f(v_1) - [f(v_k) + (2k-1)]| \\
&= |f(v_1) - [f(v_{k-1}) + (2n+1) + (2k-1)]| \\
&= |(n+2) - [(n+2) + (2n+7)(k-2) + 2(k-2)(k-3) + (2n+1)+ \\
&\quad + (2k-1)]| \\
&= |(2n+7)(k-2) + 2(k-2)(k-3) + (2n+1) + (2k-1)| \\
&\geq n, \text{ where } \quad n = 4k, d(x,y) \geq 1 \\
&\geq diam(D_2(P_n)) + 1 - d(x,y)
\end{aligned}$$

case 10 $x \in V_u$ **and** $y \in V_v$

Let $x = u_i$ and $y = v_l$, where $1 \leq i, l \leq n$

In this case we have several subcases:

Subcase 10.1 If $i = 1$, $j = k-1$ then

$$\begin{aligned}
|f(x) - f(y)| &= |f(u_1) - f(v_{k-1})| \\
&= |0 - [(n+2) + (2n+7)(k-2) + 2(k-2)(k-3)]| \\
&= |(n+2) + (2n+7)(k-2) + 2(k-2)(k-3)| \\
&\geq n, \text{ where } \quad n = 4k, d(x,y) \geq 1 \\
&\geq diam(D_2(P_n)) + 1 - d(x,y).
\end{aligned}$$

Otherwise, $|f(x) - f(y)| \geq n$.

Subcase 10.2. Suppose, $1 \leq i \leq k$ and $k \leq l \leq 2k-1$

$$\begin{aligned}
|f(x) - f(y)| &= |f(u_i) - f(v_l)| \\
&= |f(u_i) - f(v_{k-1+j})|, \text{ where } \quad 1 \leq j \leq k-1 \\
&= |(2n+3)(i-1) + 2(i-1)(i-2) - [f(v_{k-1}) + (2n+1) + (n+1)(j-1)+ \\
&\quad + 2(j-1)(j-2)]| \\
&= |(2n+3)(i-1) + 2(i-1)(i-2) - [(n+2) + (2n+7)(k-2)+ \\
&\quad + 2(k-2)(k-3) + (2n+1) + (n+1)(j-1) + 2(j-1)(j-2)]|.
\end{aligned}$$

Subcase 10.3. If $i = 1$, $l = k$ i.e. $j = 1$ then

$$\begin{aligned}
|f(x) - f(y)| &= |f(u_i) - f(v_l)| \\
&= |f(u_i) - f(v_{k-1+j})|, \text{ where } \quad 1 \leq j \leq k-1 \\
&= |0 - [f(v_{k-1}) + (2n+1) + (n+1)(j-1) + 2(j-1)(j-2)]| \\
&= | - (n+2) + (2n+7)(k-2) + 2(k-2)(k-3) + (2n+1)| \\
&= |(n+2) + (2n+7)(k-2) + 2(k-2)(k-3) + (2n+1)| \\
&\geq n, \text{ where } \quad n = 4k, d(x,y) \geq 1 \\
&\geq diam(D_2(P_n)) + 1 - d(x,y).
\end{aligned}$$

Subcase 10.4. Suppose, $1 \leq i \leq k$ and $2k \leq l \leq 3k-1$

$$|f(x) - f(y)| = |f(u_i) - f(v_l)|$$

$$= |f(u_i) - f(v_{n-(k+j-2)})|, \text{ where } \quad 1 \le j \le k.$$

Subcase 10.5. If $i = 1$, $l = 3k - 1$ i.e. $j = 1$ then

$$\begin{aligned} |f(x) - f(y)| &= |f(u_1) - f(v_{3k-1})| \\ &= |f(u_i) - f(v_{n-k-1})| \\ &= |0 - f(v_k) + 2k - 1| \\ &= |(n+2) + (2n+7)(k-2) + 2(k-2)(k-3) + 2k - 1| \\ &\ge n, \text{ where } \quad n = 4k, d(x,y) \ge 1. \end{aligned}$$

Otherwise, $|f(x) - f(y)| \ge n$. Similarly, proved if $k+2 \le i \le 2k+1$ and $k+1 \le l \le 2k+1$, for this , the radio condition is $|f(x) - f(y)| \ge n$ satisfied by every pair of vertices.

Figure 4. , illustrates the proof of this case, presents the labeling $D_2(P_{16})$ According Theorem 3.

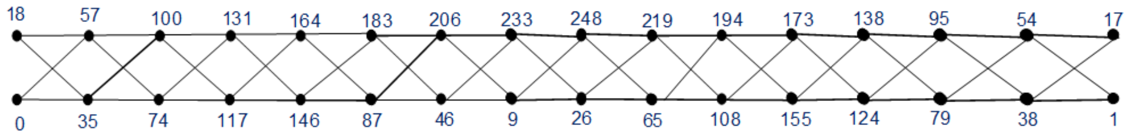

Figure 4: Radio labeling of $D_2(P_16)$ following Theorem 3.

Theorem 4. *Let $n > 6$ be a positive integer such that $n \equiv 1(mod4)$. Then*

$$rn(D_2(P_n)) \le n^2 - 2.$$

Proof. For any positive integer $n \equiv 1(mod4)$, such that $n = 4k+1, k \ge 2$, let $f : V(D_2(P_n)) \to N$ define as follows:

$$f(u_i) = \begin{cases} (2n+3)(i-1) + 2(i-1)(i-2), & \text{if } \underline{1 \le i \le k} \\ f(u_k) + 2n - 3 \quad \text{if } \underline{i = k+1} \\ f(u_k) - (2n-3) - (3n-8)(j-1) + 2(j-1)(j-2), \\ \qquad \text{if } \underline{k+2 \le i \le 2k} \quad \text{where } \quad 1 \le j \le k-1, \text{and } \quad i = k+1+j \\ n^2 - 2, \quad \text{if } \underline{i = 2k+1} \\ f(u_{k+1+j}) + (6k-2) - 2(j-1), \text{if } \underline{2k+2 \le i \le 3k}, \\ \qquad \text{where } \quad 1 \le j \le k-1, \quad \text{and } \quad i = 3k+1-j \\ f(u_j) + 2j - 1, \quad \text{if } \underline{3k+1 \le i \le 4k+1 = n} \\ \qquad \text{where } \quad 1 \le j \le k+1 \quad \text{and } \quad i = 4k+2-j \end{cases}$$

and

$$f(v_i) = \begin{cases} (n+3)+(2n+7)(i-1)+2(i-1)(i-2), \quad \text{if} \quad \underline{1 \le i \le k-1} \\ f(u_{k-1})+2n-2 \quad \text{if} \quad \underline{i=k} \\ f(v_k)+(2n)+(n+2)(j-1)+2(j-1)(j-2), \\ \quad \text{if} \quad \underline{k+1 \le i \le 2k}, \quad \text{where} \quad 1 \le j \le k, \text{and} \quad i=k+j \\ n^2-2, \quad \text{if} \quad \underline{i=2k+1} \\ f(v_{k-1+j})+(2k+1)+2(j-1), \quad \text{if} \quad \underline{2k+2 \le i \le 3k+2} \\ \quad \text{where} \quad 1 \le j \le k+1, \quad \text{and} \quad i=3k+3-j \\ f(v_j)-2j, \quad \text{if} \quad \underline{3k+3 \le i \le 4k+1=n} \\ \quad \text{where} \quad 1 \le j \le k-1, \quad \text{and} \quad i=4k+2-j \end{cases}$$

The proof is left to the reader. We omit the proof, but Figure 5, presents the labeling $D_2(P_{13})$ illustrates Theorem 4 namely, $n \equiv 1 \pmod 4$, if $n = 4k+1$, $k = 3$.

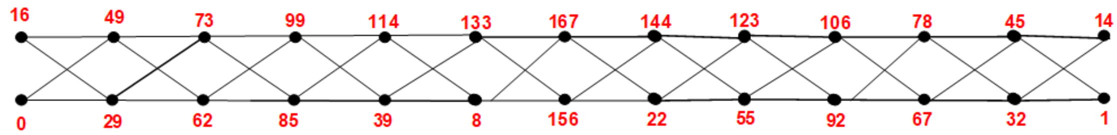

Figure 5: Radio labeling $D_2(P_{13})$ following Theorem 4.

4. Integer Linear Programming Model (ILPM)

For resolving optimization issues, mathematical programming is a crucial instrument. The references [18–20] provide specific information regarding the process of conceptualizing real-life problems as mathematical models. For a connected graph G of order n. Let $V(G) = \{x_1, x_2, \cdots, x_n\}$. We construct the distance matrix $D = [d_{ij}]$ of G where $d_{ij} = d(x_i, x_j)$ for $1 \le i,\ j \le n$. For $1 \le i \le n$, assume that y_i is the radio label of the vertex x_i. Now, we can use integer programming to present the mathematical model for the radio labeling problem. The function F is denoted as

$$F = y_1 + y_2 + \cdots + y_n$$

The claim is minimizing F subject to the $\binom{n}{2}$ constraints
$|y_i - y_j| \ge n - d(x_i, x_j)$ for $1 \le i \le n-1;\ 2 \le j \le n$ and $i < j$
where $y_1, y_2, \cdots, y_n$ are integer numbers. Therefore, $rn(G) = \max_{1 \le i \le n}\{y_i\}$.

5. Experimental Study

Hereafter, a set of experiments have been performed in order to compare the performance of algorithms presented in [6] with the obtained upper bounds by Theorems 1-4. Furthermore, a comparison is carried out between the findings of those theorems and the mathematical model that was presented in Section 4. In our experiments we have used a

PC with Core i7 processor with 2.8 GHz CPU and 8 GB of RAM. Our implementations are done with MATLAB R2016a and MS Windows 7 Professional system.

According to Table 2, the results presented in Lemma 4 demonstrate best performance compared to those in [6] for $n = 5$. For $n = 2, 3,$ and 4, similar outcomes are observed. Conversely, the outcomes in Lemma 1, 2, 3, and 4 outperform those in [7] across all values of n. In terms of time complexity table 2 shows that the computations of Lemmas 1, 2, 3, and 4 give superior performance compared to those in [6, 7].

Table 2: Comparison among our approach, Saha [6] and ILPM [7] for the radio number of Shadow graph with $n = 2, 3, 4, 5$.

n	$2n$	Lemma 1,2,3,4		Saha [6]		ILPM [7]	
		rn	CPU Time	rn	CPU Time	rn	CPU Time
2	4	4	$O(1)$	4	0.002396	5	0.003654
3	6	6	$O(1)$	6	0.004862	9	0.004854
4	8	12	$O(1)$	12	0.005359	19	0.019494
5	10	22	$O(1)$	23	0.005539	33	0.034559

Table 3: Comparison among Theorem 1, Saha [6] and ILPM [7] for the radio number of Shadow graph with $n = 2(mod)\ 4$.

n	$2n$	Theorem 1		Saha [6]		ILPM [7]	
		rn	CPU Time	rn	CPU Time	rn	CPU Time
6	12	33	$O(1)$	33	0.042993	51	0.036009
10	20	95	$O(1)$	95	0.043329	163	0.100475
14	28	189	$O(1)$	189	0.043968	343	0.150742
18	36	315	$O(1)$	315	0.048544	579	0.151717
22	44	473	$O(1)$	473	0.054947	883	0.162229
26	52	663	$O(1)$	663	0.082347	1251	0.182440
30	60	885	$O(1)$	885	0.083501	1683	0.190866
34	68	1139	$O(1)$	1139	0.096042	2179	0.218329
38	76	1425	$O(1)$	1425	0.096626	2739	0.222084
42	84	1743	$O(1)$	1743	0.117833	3363	0.264229
46	92	2093	$O(1)$	2093	0.118883	4051	0.278491
50	100	2475	$O(1)$	2475	0.121298	4803	0.301669
54	108	2889	$O(1)$	2889	0.128889	5619	0.420457
58	116	3335	$O(1)$	3335	0.131909	6499	0.475030
62	124	3813	$O(1)$	3813	0.148007	7443	0.477373
64	128	4064	$O(1)$	4064	0.178655	7939	0.527846
68	136	4590	$O(1)$	4590	0.198579	8979	0.635992
72	144	5148	$O(1)$	5148	0.20673	10083	0.720457
76	152	5738	$O(1)$	5738	0.213292	11251	0.875030
80	160	6360	$O(1)$	6360	0.231493	12483	0.976116

Based on the upper bound of the radio number of shadow graph with $n = 2\ (mod)\ 4$, it can be observed from Table 3 and Figure 6 that the outcomes presented in Theorem 1 coincide with those in [6]. Conversely, the results from Theorem 1 surpass the findings in [7] for all values of n. Table 3 indicates that the results from Theorem 1 have a time complexity of O(1), while those in [6] have a complexity of O(n^3). Furthermore, the results from Theorem 1 require less time compared to the results in [7].

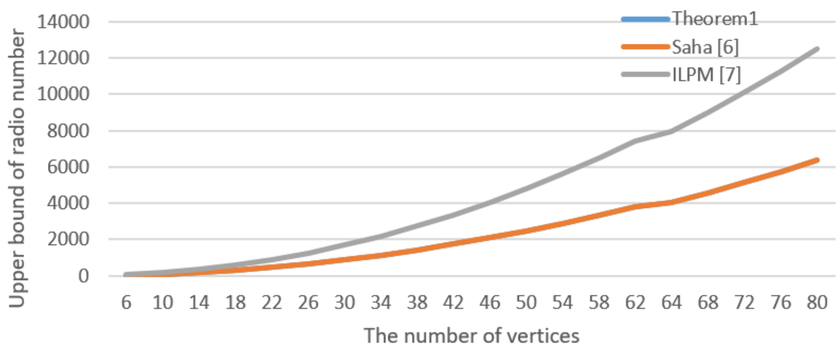

Figure 6: Comparison among Theorem 1, Saha [6] and ILPM [7] for the radio number of Shadow graph with $n = 2(mod)\ 4$.

Table 4: Comparison among Theorem 2, Saha [6] and ILPM [7] for the radio number of Shadow graph with $n = 3(mod)4$.

n	2n	Theorem 2		Saha [6]		ILPM [7]	
		rn	CPU Time	rn	CPU Time	rn	CPU Time
7	14	46	$O(1)$	46	0.043355	73	0.089586
11	22	118	$O(1)$	118	0.048763	201	0.090475
15	30	222	$O(1)$	222	0.050551	393	0.130588
19	38	358	$O(1)$	358	0.050987	649	0.161792
23	46	526	$O(1)$	526	0.051813	969	0.220337
27	54	726	$O(1)$	726	0.081522	1353	0.305778
31	62	958	$O(1)$	958	0.086948	1801	0.207713
35	70	1222	$O(1)$	1222	0.092306	2313	0.266207
39	78	1518	$O(1)$	1518	0.095106	2889	0.288857
43	86	1846	$O(1)$	1846	0.102625	3529	0.368033
47	94	2206	$O(1)$	2206	0.116411	4233	0.486650
51	102	2598	$O(1)$	2598	0.121707	5001	0.530145
55	110	3022	$O(1)$	3022	0.124297	5833	0.544023
59	118	3478	$O(1)$	3478	0.140879	6729	0.551847
63	126	3966	$O(1)$	3966	0.15269	7689	0.635992
67	134	4486	$O(1)$	4486	0.164255	8713	0.720457
71	142	5038	$O(1)$	5038	0.190907	9801	0.829255
75	150	5622	$O(1)$	5622	0.206634	10953	0.972498
79	158	6238	$O(1)$	6238	0.224507	12169	0.998945
83	166	6886	$O(1)$	6886	0.273817	13449	1.034510
87	174	7566	$O(1)$	7566	0.351691	14793	1.293168

Table 4 and Figure 7 demonstrate that the results presented in Theorem 2 align with the findings in [6], considering the upper bound of the radio number of shadow graph with $n = 3(mod)\ 4$. Additionally, it is worth noting that the results obtained in Theorem 2 outperform the results in [7] for all values of n. In terms of running time, Table 4 indicates that the results derived from Theorem 2 have a constant time complexity of O(1), whereas the results in [6] have a time complexity of O(n^3). Furthermore, the results from Theorem 2 require less time compared to the results in [7].

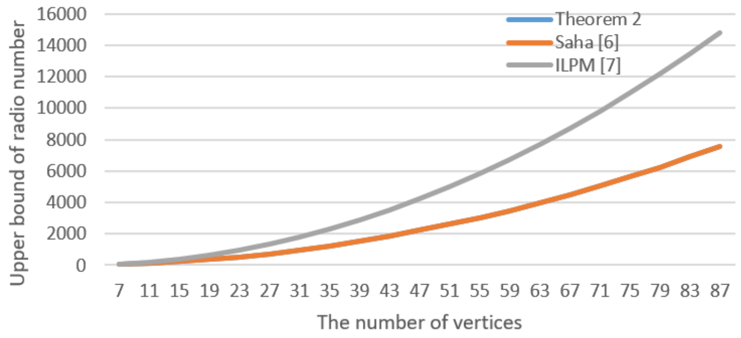

Figure 7: Comparison among Theorem 2, Saha [6] and ILPM [7] for the radio number of Shadow graph with $n = 3(mod)4$

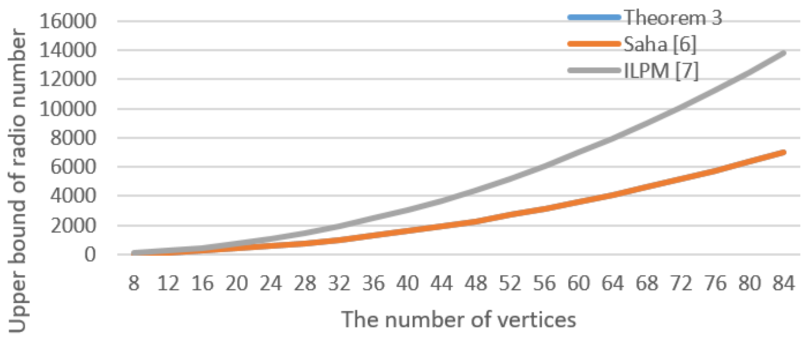

Figure 8: Comparison among Theorem 3, Saha [6] and ILPM [7] for the radio number of Shadow graph with $n = 0(mod)4$.

Table 5 and Figure 8 demonstrate that the results presented in Theorem 3 are identical to the results in [6], considering the upper bound of the radio number of shadow graph with $n = 0\ (mod)\ 4$. Additionally, it is worth noting that the results obtained in Theorem 3 outperform the results in [7] for all values of n. In terms of running time, Table 5 indicates that the results derived from Theorem 3 have a constant time complexity of O(1), whereas the results in [6] have a time complexity of O(n^3). Furthermore, the results from Theorem 3 require less time compared to the results in [7].

Table 5: Comparison among Theorem 3, Saha and ILPM [7] for the radio number of Shadow graph with $n = 0(mod)4$.

n	2n	Theorem 3		Saha [6]		ILPM [7]	
		rn	CPU Time	rn	CPU Time	rn	CPU Time
8	16	60	$O(1)$	60	0.036009	99	0.033053
12	24	138	$O(1)$	138	0.046854	243	0.089839
16	32	248	$O(1)$	248	0.052537	451	0.097359
20	40	390	$O(1)$	390	0.053037	723	0.102315
24	48	564	$O(1)$	564	0.054454	1059	0.124432
28	56	770	$O(1)$	770	0.082983	1459	0.171284
32	64	1008	$O(1)$	1008	0.086663	1923	0.185284
36	72	1278	$O(1)$	1278	0.099965	2451	0.218490
40	80	1580	$O(1)$	1580	0.102013	3043	0.235134
44	88	1914	$O(1)$	1914	0.123509	3699	0.254231
48	96	2280	$O(1)$	2280	0.123897	4419	0.287511
52	104	2678	$O(1)$	2678	0.134616	5203	0.451393
56	112	3108	$O(1)$	3108	0.141561	6051	0.466374
60	120	3570	$O(1)$	3570	0.149171	6963	0.477208
64	128	4064	$O(1)$	4064	0.178655	7939	0.527846
68	136	4590	$O(1)$	4590	0.198579	8979	0.635992
72	144	5148	$O(1)$	5148	0.20673	10083	0.976116
76	152	5738	$O(1)$	5738	0.213292	11251	0.983264
80	160	6360	$O(1)$	6360	0.231493	12483	0.998945
84	168	7014	$O(1)$	7014	0.252509	13779	1.034510

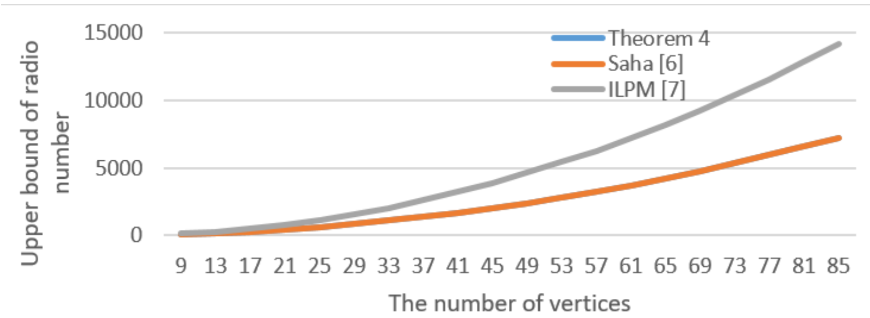

Figure 9: Comparison among Theorem 4, Saha [6] and ILPM [7] for the radio number of Shadow graph with $n = 1(mod)4$.

Table 6 and Figure 9 demonstrate that the results presented in Theorem 4 align with the results in [6], considering the upper bound of the radio number of shadow graph with $n = 1\ (mod)\ 4$. Additionally, it is worth noting that the results in Theorem 4 outperform the results in [7] for all values of n. In terms of running time, Table 6 indicates that the results in Theorem 4 have a constant time complexity of O(1), whereas the results in [6] have a time complexity of $O(n^3)$. Furthermore, the results in Theorem 4 require less time compared to the results in [7].

Table 6: Comparison among Theorem 4, Saha [6] and ILPM [7] for the radio number of Shadow graph with $n = 1(mod)4$.

n	2n	Theorem 4		Saha [6]		ILPM [7]	
		rn	CPU Time	rn	CPU Time	rn	CPU Time
9	18	79	$O(1)$	79	0.033053	129	0.091875
13	26	167	$O(1)$	167	0.046598	289	0.124002
17	34	287	$O(1)$	287	0.04728	513	0.134112
21	42	439	$O(1)$	439	0.049498	801	0.158377
25	50	623	$O(1)$	623	0.052472	1153	0.205898
29	58	839	$O(1)$	839	0.08865	1569	0.218329
33	66	1087	$O(1)$	1087	0.089514	2049	0.222084
37	74	1367	$O(1)$	1367	0.092204	2593	0.264229
41	82	1679	$O(1)$	1679	0.100206	3201	0.278491
45	90	2023	$O(1)$	2023	0.102479	3873	0.305885
49	98	2399	$O(1)$	2399	0.114788	4609	0.308508
53	106	2807	$O(1)$	2807	0.133461	5409	0.451393
57	114	3247	$O(1)$	3247	0.148361	6273	0.466374
61	122	3719	$O(1)$	3719	0.166031	7201	0.477208
65	130	4223	$O(1)$	4223	0.16886	8193	0.527846
69	138	4759	$O(1)$	4759	0.21307	9249	0.644822
73	146	5327	$O(1)$	5327	0.241707	10369	0.720457
77	154	5927	$O(1)$	5927	0.260753	11553	0.829255
81	162	6559	$O(1)$	6559	0.299115	12801	0.937023
85	170	7223	$O(1)$	7223	0.307929	14113	1.069794

6. Application of radio labeling of graph in cryptography

Recently, information technology has been widely used in various fields of life. Hence, information security becomes a vital issue. Cryptography is a significant method for information security. In cryptographic algorithms, keys used for encryption and decryption are lengthy number sequences which are created using random number generators. Such generators follow a uniform distribution that can be is easily determined by hackers, as any number has an equal chance of being generated. Using radio labeling of graphs, Keys can be generated in more faster and effective manner. The radio labeled graph is used as the cipher graph. At the receiver end the encrypted message is received in the form of edge or vertex sequence of this cipher graph. Consequently, it is difficult for an opponent to guess and hack, as the radio labeling problem is an NP-hard problem. The radio number of a graph may also be used to generate invertible matrices that serve for keys generation. In this manner, the radio labeling considered an efficient tool for cryptography as the problem of finding the radio number of a graph is an NP-complete problem. We refer the reader to [24–26] for applications of radio labeling in cryptography

7. Conclusions

Effective allocating of frequency resources in wireless networks is a challenging problem. The claim is avoiding the interference among stations and minimizing the usage of the available frequencies. Here we consider the radio graph labeling as a graph-theoretical approach to modeling the problem of frequency assignment. In such modeling an interference graph G is built. The vertices of the interference graph represent the stations. Two vertices are adjacent if their corresponding stations may interfere. Radio labeling of the interference graph is done to avoid the interference and moreover emphasizing on minimization the using of frequency resources. Then the minimum value of the maximum assigned frequency (radio label) among all radio labeling of G , called the radio number of the graph is obtained, this radio number represents the highest frequency should be used to avoid interference. In this paper, the radio labeling and the upper bound of radio number of networks as shadow of path graph are investigated. Wherein, a theoretical approach for finding such number is presented and mathematical model for finding upper bound of radio number of shadow of path graph is proposed. In order to validate the effectiveness of our approach, an experimental study is conducted, comparing our results to previous findings. The results of the study demonstrate that our proposed approach surpasses the previous results in terms of both the running time and the upper bound of the radio number. Moreover, we give an overview of the application of radio labeling in cryptography. For future work we claim that the integer linear programming model will be improved to obtain the exact radio number.

Conflicts of Interest

The authors declare no conflict of interest.

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