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Some Rigidity Theorems of Closed Geodesic Polygons and Spherical Curves in Metric Spaces with Curvature Bounded Below

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Abstract. This paper examines characterizations of closed curves in a geodesic metric space with curvature bounded below, including closed geodesic polygons and closed spherical curves bounding surfaces isometric to convex polygons and circles in the model space.

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1. Introduction and preliminaries

In this paper, we investigate the characterizations of closed curves in a geodesic metric space with curvature bound below in the sense of Alexandrov. We discuss the characterizations for closed geodesic polygons in the space that bound surfaces isometric to regions bounded by closed convex polygons in the model space R_K, and we also look at the characterizations for closed spherical curves in the space that bound surfaces isometric to regions bounded by circles in the model space R_K with the same perimeter.

Alexandrov [1–5, 13] introduced lower and upper curvature bounds on metric spaces without Riemannian structure, which extended concepts to arbitrary spaces. This led to the theorems of Riemannian geometry, which defined bounded curvature as bounded sectional curvature. Examples include the Remannian manifolds with sectional curvature are not less than K and its convex subset, and Hilbert spaces. Let (X, d) be a metric space and $\gamma : [a, b] \to X$ a curve in X. The length $\ell(\gamma)$ of γ is defined by

$$\ell(\gamma) = \sup \sum_{i=1}^{k} d(\gamma(t_{i-1}), \gamma(t_i)),$$

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where the supremum is taken over all partitions $a = t_0 < t_1 < \cdots < t_k = b$ of $[a, b]$. Hence,

$$d^*(x, y) := \inf\{\ell(\gamma)|\ \gamma \text{ is a curve from } x \text{ to } y\},$$

for all x and $y \in X$, defines a metric on X with distance values in $[0, \infty]$. If $d = d^*$, then (X, d) is called a length space.

A geodesic in X is an isometry from $R = (-\infty, \infty)$ into X. We may also refer to the image of this isometry as a geodesic. A geodesic path joining two points x and y is a map $c : [0, a] \subset R \to X$ such that $c(0) = x$ and $c(a) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, a]$. Usually, the image $c([0, a])$ is called a geodesic segment joining x and y, and if there is a unique geodesic segment joining two points x and y, then $[x, y]$ is denoted the geodesic segment joining them. The metric space (X, d) is called a geodesic space if each pair of two points of X is joined by a geodesic segment.

Definition 1. [5] *Let K be a real number. The R_K is one of the following spaces, depending on the sign of K: R^2, if $K = 0$, the Euclidean sphere of radius $1/\sqrt{K}$, if $K > 0$, and the hyperbolic plane with curvature K, if $K < 0$.*

We can learn more about the R_K spaces in [4, 7–11]. A geodesic triangle $\triangle(p, q, r)$ in X is a triangle with points p, q, r as its vertices and three chosen geodesics $[p, q]$, $[q, r]$, $[p, r]$ as its sides. A comparison triangle in R_K for the geodesic triangle $\triangle(p, q, r)$ in X is a triangle $\triangle(\tilde{p}, \tilde{q}, \tilde{r})$ in R_K such that $d(p, q) = d_K(\tilde{p}, \tilde{q})$, $d(q, r) = d(\tilde{q}, \tilde{r})$, and $d(p, r) = d(\tilde{p}, \tilde{r})$. Such a triangle $\triangle(\tilde{p}, \tilde{q}, \tilde{r})$ always exists if $d(p, q) + d(q, r) + d(p, r) < \frac{2\pi}{\sqrt{K}}$ and it is unique up to isometries.

Given a pair of a triangle $\triangle(p, q, r)$ in X and its comparison triangle $\triangle(\tilde{p}, \tilde{q}, \tilde{r})$ in R_K, the comparison point for a point $x \in [q, r]$ is the point denoted by $\tilde{x}$ in $[\tilde{q}, \tilde{r}]$ such that $d(q, x) = d(\tilde{q}, \tilde{x})$, and the comparison angle at q of the triangle $\triangle(p, q, r)$ is the angle at $\tilde{q}$ of triangle $\triangle(\tilde{p}, \tilde{q}, \tilde{r})$. $\angle_p(q, r)$ denotes the angle at p of $\triangle(p, q, r)$ in X. We let $\widetilde{\angle}_p(q, r)$ or $\angle_{\tilde{p}}(\tilde{q}, \tilde{r})$ denote the angle at $\tilde{p}$ of a triangle $\triangle(\tilde{q}, \tilde{p}, \tilde{r})$ in R_K. Sometimes, for convenience we let a triangle $\widetilde{\triangle}(p, q, r)$ in R_K be a comparison triangle of $\triangle(p, q, r)$ in X.

Definition 2. [6] *Let X be a length space. A locally complete space X is a space with curvature bounded below by a real number K if every point $x \in X$ has a neighborhood $U(x)$ the following condition is satisfied:*
(A) for any four distinct points $p, q, r, s \in U(x)$, $\widetilde{\angle}_s(q, p) + \widetilde{\angle}_s(q, r) + \widetilde{\angle}_s(p, r) \leq 2\pi$.

For spaces in which, locally, any two points are joined by a geodesic, in particular for locally compact spaces, the condition (A) in Definition 2 can be replaced by the condition: (B) for any triangle $\triangle(p, q, r)$ in $U(x)$ and any point s on the side $[q, r]$ the inequality $d(p, s) \geq d(\tilde{p}, \tilde{s})$ is satisfied, where $\tilde{s}$ is the corresponding point of s on the side $[\tilde{q}, \tilde{r}]$ of comparison triangle $\widetilde{\triangle}(p, q, r)$.

Let X be a space with curvature bounded below by K and α and β be two geodesics starting at a point p in X. The angle between α and β is defined by

$$\lim_{s \to 0} \cos^{-1}\left(\frac{d^2(p, \alpha(s)) + d^2(p, \beta(s)) - d^2((\alpha(s), \beta(s))}{2d(p, \alpha(s))d(p, \beta(s))}\right).$$

The angle at p of a triangle $\triangle(q,p,r)$ is the angle between $[p,q]$ and $[p,r]$.

The condition (B) is equivalent to the following condition:
$(\widetilde{B})$ for any triangle $\triangle(p,q,r)$ in $U(x)$, $\angle_p(q,r) \geq \widetilde{\angle}_p(q,r)$, $\angle_q(p,r) \geq \widetilde{\angle}_q(p,r)$, and $\angle_r(p,q) \geq \widetilde{\angle}_r(p,q)$, where $\widetilde{\triangle}(p,q,r)$ is a comparision triangle in R_K of the triangle $\triangle(p,q,r)$.

Spaces with curvature bounded below were defined above using local conditions. However, for complete spaces, the global conditions may be deduced from the corresponding local ones. The metric space X considered in this work is complete. We then call X a metric space with curvature bounded below in the large.

Theorem 1. [5] *If X is a metric space with curvature bounded below by K in the large, where $K > 0$, then* $\dim(X) \leq \pi/\sqrt{K}$ *and any triangle in X has perimeter no greater than $2\pi/\sqrt{K}$.*

Theorem 2. [12] *Let X be a metric space with curvature bounded below by K in the large, $\triangle(p,q,r)$ a triangle in X and $\triangle(\tilde{p},\tilde{q},\tilde{r})$ a triangle in R_K. If $d(p,q) = d(\tilde{p},\tilde{q})$, $d(p,r) = d(\tilde{p},\tilde{r})$, and $\angle_p(q,r) = \angle_{\tilde{p}}(\tilde{q},\tilde{r})$, then $d(q,r) \leq d(\tilde{q},\tilde{r})$.*

2. Closed geodesic polygons

A closed curve in a metric space X is a continuous map of an oriented circle in the 2-dimensional Euclidean space. A chain V on a closed curve γ is a set of points corresponding to finitely many parameter values in order. The points in V are called the vertices of the chain. If γ consists of geodesic segments joining adjacent pairs in V, then γ and V form a closed geodesic polygon with a vertex chain in V. A subset A of a metric space (X,d) is defined as convex if, for any two points $x, y \in A$, the segment joining x and y is also included in A. $C(A)$ represents the convex hull of a subset A, defined as the smallest convex set that contains A. An isometry between two metric spaces (X,d) and (Y,d^*) is a function $i : X \to Y$ such that $d(x,y) = d^*(i(x),i(y))$, for all points $x, y \in X$. We will start by describing a triangle in metric space with curvature bounded below whose convex hull is isometric to a comparison triangle's convex hull in model space R_K.

We note that if $p_1, p_2, p_3, p_4, p_5 = p_1$ is a closed polygon in X, a metric space with curvature bounded below by K, and $p_1', p_2', p_3', p_4', p_5' = p_1'$ is a closed polygon in R_K with $d(p_1,p_2) = d(p_1',p_2')$, $d(p_2,p_3) = d(p_2',p_3')$, $d(p_3,p_4) = d(p_3',p_4')$, $d(p_4,p_5) = d(p_4',p_5')$ and $d(p_1,p_4) = d(p_1',p_4'$ then

$$\angle_{p_1}(p_2,p_4) \leq \angle_{p_1}(p_2,p_3) + \angle_{p_1}(p_4,p_3) \geq \angle_{p_1'}(p_2',p_3') + \angle_{p_1'}(p_4',p_3') \geq \angle_{p_1'}(p_2',p_4').$$

Hence, it is not possible to compare the values of $\angle_{p_1}(p_2,p_4)$ and $\angle_{p_1'}(p_2',p_4')$. Therefore, we must suppose that the equation $\angle_{p_1}(p_2,p_4) = \angle_{p_1}(p_2,p_3) + \angle_{p_1}(p_4,p_3)$ is true in order to obtain $\angle_{p_1}(p_2,p_4) \geq \angle_{p_1'}(p_2',p_4')$.

Lemma 1. *Let X be a metric space with curvature bounded below by K in the large. Let $\triangle(p,q,r)$ be a triangle in X and $\triangle(p',q',r')$ be its comparison triangle in R_K, x be a point on $[p,q]$, and $x' \in [p',q']$ be a comparison point of x. If $\angle_p(x,r) = \angle_{p'}(x',r')$, then $d(x,r) = d(x',r')$.*

Proof. Since X is a metric space with curvature bounded below by K in the large, we have that $d(x,r) \geq d(x',r')$. Using Theorem 2, we have $d(x,r) \leq d(x',r')$. Hence, we have the result.

Theorem 3. *Let X be a metric space with curvature bounded below by K in the large. Let $\triangle(p,q,r)$ be a triangle in X and $\triangle(p',q',r')$ be its comparison triangle in R_K. Suppose that the following statements hold:*

(i) $\angle_r(p,q) = \angle_{r'}(p',q')$;

(ii) $\angle_p(x,r) = \angle_p(q,r)$, $\angle_q(x,r) = \angle_q(p,r)$ *and* $\angle_r(p,q) = \angle_r(p,x) + \angle_r(q,x)$ *for any* $x \in [p,q]$.

Then the convex hull of $\triangle(p,q,r)$ is isometric to the convex hull of $\triangle(p',q',r')$.

Proof. By Lemma 1, we have that $d(x,r) = d(x',r')$ if $x \in [p,q]$ and $x' \in [p',q']$ such that $d(x,p) = d(x',p')$. Let j be the map from the convex hull $C(\triangle(p',q',r'))$ in R_K to the convex hull $C(\triangle(p,q,r))$ in X which, for every $x' \in [p',q']$, sends the geodesic segment $[r',x']$ isometrically onto the geodesic segment $[r,x]$. We claim that j is an isometry onto its image; it then follows that the unique geodesic joining any two points of the image of j will be contained in the image, so j maps $C(\triangle(p',q',r'))$ onto $C(\triangle(p,q,r))$. Consider two points $a' \in [r',x']$ and $b' \in [r',y']$ in $C(\triangle(p',q',r'))$, where $x',y' \in [p',q']$ and x' is between q' and y'. Let $x = j(x')$, $y = j(y')$, $a = j(a')$ and $b = j(b')$. Since $d(r,x) = d(r',x')$, $d(r,y) = d(r',y')$ and $d(x,y) = d(x',y')$, we have that $\triangle(r',x',y')$ is a comparison triangle of $\triangle(r,x,y)$. Since X is a metric space with curvature bounded below by K in the large, we have $d(a,b) \geq d(a',b')$. As $\angle_r(a,b) \leq \angle_r(x,y) = \angle_{r'}(x',y') = \angle_{r'}(a',b')$, applying Theorem 2, we get that $d(a,b) \leq d(a',b')$. Therefore, $d(a,b) = d(a',b')$, as required.

We then describe that the convex hull of a closed geodesic polygon in a metric space with curvature bounded below is isometric to that of a polygon in the model space R_K.

Theorem 4. *Let X be a metric space with curvature bounded below by K in the large. Let σ be a closed geodesic polygon with ordered vertices p_1,p_2,p_3,p_4,p_1 with perimeter less than $\pi/\sqrt{K}$ in X and let σ' be a convex polygon with ordered vertices p_1',p_2',p_3',p_4',p_1' in the model space R_K. Suppose the following statements hold:*

(i) $C(\{p_1,p_2,p_3\})$ *and* $C(\{p_1,p_3,p_4\})$ *are isometric to* $C(\{p_1',p_2',p_3'\})$ *and* $C(\{p_1',p_3',p_4'\})$*, respectively;*

(ii) the geodesic $[p_1,p_3]$ *intersects the geodesic* $[p_2,p_4]$ *at a point* p*;*

(iii) $\angle_{p_2}(\tilde{p},p_3) = \angle_{p_2'}(p',p_3')$, $\angle_{p_2}(\tilde{p},p_1) = \angle_{p_2'}(p',p_1')$, $\angle_{p_4}(\tilde{p},p_1) = \angle_{p_4'}(p',p_1')$ *and* $\angle_{p_4}(\tilde{p},p_3) = \angle_{p_4'}(p',p_3')$ *for all* $\tilde{p} \in [p_2,p_4]$ *and* p' *is the intersection of* $[p_1',p_3']$ *and* $[p_2',p_4']$*;*

(iv) $\angle_{p_1}(p_2,p_4) = \angle_{p_1}(p_2,\tilde{p}) + \angle_{p_1}(\tilde{p},p_4)$ *and* $\angle_{p_3}(p_2,p_4) = \angle_{p_3}(p_2,\tilde{p}) + \angle_{p_3}(\tilde{p},p_4)$ *for all* $\tilde{p} \in [p_2,p_4]$.

Then the convex hull of σ is isometric the convex hull of σ'.

Proof. First, we shall prove that the point p is a corresponding point under both isometries to the point p'. Let p^* be a point on the geodesic segment $[p_1, p_3]$ such that $d(p^*, p_1) = d(p', p_1')$ and let $p'' \in [p_1', p_3']$ be a corresponding point of p under both isometries. Thus,

$$\begin{aligned} d(p_1, p_3) &= d(p_1, p) + d(p, p_3) \\ &\leq d(p_1, p^*) + d(p^*, p_3) \\ &= d(p_1', p') + d(p', p_3') \\ &= d(p_1', p_3'), \end{aligned}$$

and

$$\begin{aligned} d(p_1', p_3') &= d(p_1', p') + d(p', p_3') \\ &\leq d(p_1', p'') + d(p'', p_3') \\ &= d(p_1, p) + d(p, p_3) \\ &= d(p_1, p_3), \end{aligned}$$

and hence, $d(p_1, p_3) = d(p_1', p_3')$. So we have $p = p^*$ and $p' = p''$, as required.

By (iii) and (iv), we employ Theorem 3, $C(\{p_2, p_3, p_4\})$ is isometric to $C(\{p_2', p_3', p_4'\})$ and $C(\{p_1, p_2, p_4\})$ is isometric to $C(\{p_1', p_2', p_4'\})$. The next step is to confirm that $C(\sigma)$ and $C(\sigma')$ are isometric to each other. By the definition of convex hull, $C(\sigma)$ exists and is distinct, as we have noted. Let $i_1 : C(\{p_1, p_2, p_3\}) \to C(\{p_1', p_2', p_3'\})$ and $i_2 : C(\{p_1, p_3, p_4\}) \to C(\{p_1', p_3', p_4'\})$ be such that $i_1(p_j) = p_j'$, $j = 1, 2, 3$ and $i_2(p_k) = p_k'$, $k = 1, 3, 4$. Let i be a map from $C(\sigma)$ to $C(\sigma') = C(\{p_1', p_2', p_3'\}) \cup C(\{p_1', p_3', p_4'\})$ such that $i|_{C(\{p_1,p_2,p_3\})} = i_1$ and $i|_{C(\{p_1,p_3,p_4\})} = i_2$. We must demonstrate that i is an isometry from $C(\sigma)$ to $C(\sigma')$ by verifying that

($*$) i is an isometry onto its image, and

($**$) $C(\sigma) = C(\{p_1, p_2, p_3\}) \cup C(\{p_1, p_3, p_4\})$.

That i is surjective is obvious. Additionally, i is injective due to the circumstances of intersecting geodesic segments and isometric convex hulls. To prove ($*$), let $x_1, x_2 \in C(\sigma)$, $x_1' = i(x_1)$ and $x_2' = i(x_2)$. We shall verify that $d(x_1, x_2) = d(x_1', x_2')$. There is nothing to prove if $x_1, x_2 \in C(\{p_1, p_2, p_3\})$ or $x_1, x_2 \in C(\{p_1, p_2, p_4\})$ or $x_1, x_2 \in C(\{p_1, p_3, p_4\})$ or $x_1, x_2 \in C(\{p_2, p_3, p_4\})$. Without loss of generality, we assume that x_1 is in $C(p_1, p_2, p)$ and x_2 is in $C(p_3, p_4, p)$, where p is the point at which the geodesic segments $[p_1, p_3]$ and $[p_2, p_4]$ cross. Let x_1' and x_2' be corresponding points of x_1 and x_2, respectively. We suppose that the segment $[x_1', x_2']$ meets the segment $[p_1', p_3']$ at a point x_3' and meets the segment $[p_2', p_4']$ at a point x_4' such that $x_3' \in [x_1', x_4']$ (if $x_4' \in [x_1', x_3']$ we can prove in the same manner). Let x_3 and x_4 be two points such that $x_3' = i_1(x_3)$ and $x_4' = i_2(x_4)$. In R_K, we have $[x_1', x_4'] = [x_1', x_3'] \cup [x_3', x_4']$ and $[x_3', x_2'] = [x_3', x_4'] \cup [x_4', x_2']$. Due to the fact that $C(\{p_1, p_2, p_4\})$ is isometric to $C(\{p_1', p_2', p_4'\})$ and $[x_1', x_4']$ is in $C(\{p_1', p_2', p_4'\})$, we have that $[x_1, x_4] = [x_1, x_3] \cup [x_3, x_4]$ is in $C(\{p_1, p_3, p_4\})$ which such that $d(x_1, x_3) = d(x'_1, x'_3)$ and $d(x_3, x_4) = d(x'_3, x'_4)$, and then,

$$d(x_1, x_4) = d(x_1, x_3) + d(x_3, x_4) = d(x'_1, x'_3) + d(x'_3, x'_4) = d(x'_1, x'_4).$$

Because $C(\{p_1, p_3, p_4\})$ is isometric to $C(\{p'_1, p'_3, p'_4\})$ and $[x'_3, x'_2]$ is in $C(\{p'_1, p'_3, p'_4\})$, we have $[x_3, x_2] = [x_3, x_4] \cup [x_4, x_2]$ is in $C(\{p_1, p_3, p_4\})$, which such that $d(x_3, x_4) = d(x'_3, x'_4)$ and $d(x_4, x_2) = d(x'_4, x'_2)$, and thus,

$$d(x_3, x_2) = d(x_3, x_4) + d(x_4, x_2) = d(x'_3, x'_4) + d(x'_4, x'_2) = d(x'_3, x'_2).$$

Hence, $[x_1, x_2] = [x_1, x_3] \cup [x_3, x_4] \cup [x_4, x_2]$ forms a geodesic seqment, and therefore,

$$\begin{aligned} d(x_1, x_2) &= d(x_1, x_3) + d(x_3, x_4) + d(x_4, x_2) \\ &= d(x'_1, x'_3) + d(x'_3, x'_4) + d(x'_4, x'_2) \\ &= d(x'_1, x'_2). \end{aligned}$$

We now demonstrate $(**)$. For convenience, we set $A = \{p_1, p_2, p_3\}$ and $B = \{p_1, p_3, p_4\}$. First, we must establish the convexity of $C(A) \cup C(B)$. Let x_1, x_2 be two points in $C(A) \cup C(B)$. The geodesic segment $[x_1, x_2]$ must be declared to be in $C(A) \cup C(B)$. It makes no difference if x_1, x_2 are both in $C(A)$ or $C(B)$. We assume, without loss of generality, that $x_1 \in C(A)$ and $x_2 \in C(B)$. Let x'_1 and x'_2 be corresponding points to x_1 and x_2, respectively, and t' the point of intersection of the two segments $[x'_1, x'_2]$ and $[p'_1, p'_3]$ and let $t' = i(t)$. Thus

$$d(x_1, x_2) = d(x'_1, x'_2) = d(x'_1, t') + d(t', x'_2) = d(x_1, t) + d(t, x_2).$$

This suggests that geodesic segments $[x_1, t]$ and $[t, x_2]$ form a geodesic segment connecting points x_1 and x_2. That is $[x_1, x_2] = [x_1, t] \cup [t, x_2] \subset C(A) \cup C(B)$. Accordingly, $C(A) \cup C(B)$ is convex. We obtain that $C(A \cup B) \subset C(A) \cup C(B)$, because $C(A \cup B)$ is the smallest convex set containing $A \cup B$. We also obtain $C(A) \cup C(B) \subset C(A \cup B)$ because both $C(A)$ and $C(B)$ are subsets of $C(A \cup B)$. Therefore, $C(A \cup B) = C(A) \cup C(B)$.

It is important to note that Theorem 4 depends on the intersection of two geodesics. We consider the space R^3 with usual metric as a metric space with curvature bounded below by 0 in the large. Let X be a triangle with points $(-1, 0, 0)$, $(1, 0, 0)$ and $(0, 1, 0)$ and Y be a triangle with points $(-1, 0, 0)$, $(1, 0, 0)$ and $(0, 0, 1)$. In R_0 space, we let X' be a triangle with points $(0, -1)$, $(0, 1)$ and $(1, 0)$ and B' be a triangle with points $(0, -1)$, $(0, 1)$ and $(-1, 0)$. Thus, we have that X' and Y' are corresponding triangles of X and Y, respectively. It is evident that $C(X')$ is isometric to $C(X)$ and $C(Y')$ is isometric to $C(Y)$ but that $C(X' \cup Y')$ is not isometric to $C(X \cup Y)$ since the segment connecting points $(-1, 0, 0)$ and $(1, 0, 0)$ do not meet the segment connecting points $(0, 1, 0)$ and $(0, 0, 1)$.

The following theorem can be proven using the concept of the proof of Theorem 4

Theorem 5. *Let X be a metric space with curvature bounded below by K in the large. Let σ be a closed geodesic polygon with ordered vertices $p_1, p_2, p_3, ..., p_n, p_1$ with perimeter less than $\pi/\sqrt{K}$ in X and σ' be a convex polygon with ordered vertices $p'_1, p'_2, p'_3, ..., p'_n, p'_1$ in the model space R_K, for $n > 4$. Suppose that the following statements hold:*

(i) *$C(\{p_1, p_2, ..., p_t\})$ is isometric to $C(\{p'_1, p'_2, ..., p'_t\})$ and $C(\{p_t, p_{t+1}, ..., p_n\})$ is isometric to $C(\{p'_t, p'_{t+1}, ..., p'_n\})$;*

(ii) *the geodesic segment $[p_1, p_t]$ intersects the geodesic segment $[p_i, p_j]$ at a point for some $i \in \{1, 2, ..., t-1\}$ and $j \in \{t+1, t+2, ..., n-1\}$.*

(iii) *$\angle_{p_i}(p^*, p_t) = \angle_{p'_i}(p', p'_t)$, $\angle_{p_i}(p^*, p_1) = \angle_{p'_i}(p', p'_1)$, $\angle_{p_j}(p^*, p_1) = \angle_{p'_j}(p', p'_1)$ and $\angle_{p_j}(p^*, p_t) = \angle_{p'_j}(p', p'_t)$ for all $p^* \in [p_i, p_j]$;*

(iv) *$\angle_{p_1}(p_i, p_j) = \angle_{p_1}(p_i, p^*) + \angle_{p_1}(p^*, p_j)$ and*
$\angle_{p_t}(p_i, p_j) = \angle_{p_t}(p_i, p^*) + \angle_{p_t}(p^*, p_j)$ for all $p^* \in [p_i, p_j]$

Then the convex hull of σ is isometric the convex hull of σ', that is the totally geodesic surface bounded by σ and the region bounded by σ' are isometric to each other.

3. Spherical curves

If there is a point p in the metric space X and a positive real number r such that $d(x, p) = r$ for all x in γ, the curve γ is said to be spherical. The radius of γ is the actual value r. For example, a circle of radius $r > 0$ in the model space R_K is a closed spherical curve at a distance r from its center. In the subsequent discussion, we define γ_{ab} as a spherical curve in a metric space with curvature bound below with endpoints a and b, and $\gamma'_{a'b'}$ as a subarc of a circle in the model space R_K with endpoints a' and b'.

We describe a closed spherical curve bounding a surface isometric to a region bounded by a circle in the model space R_K.

Lemma 2. *Let X be a metric space with curvature bounded below by K in the large. Let γ be a spherical curve at a distance $r < \frac{\pi}{2\sqrt{K}}$ from a point p with endpoints a, b in X and γ' be a subarc of a circle of radius r centered at a point p' in R_K with endpoints a', b' such that $d(a, b) = d(a', b')$. If that $c \in \gamma$ is between a, b and $c' \in \gamma$ is between a', b' with conditions $d(a, c) = d(a', c')$ and $d(b, c) = d(b', c')$, then the geodesic segment $[p, c]$ intersects the geodesic segment $[a, b]$ at a point q which corresponds to the point q' of intersection of $[p', c']$ and $[a', b']$.*

Proof. We shall prove that the segment $[a, b]$ intersects the segment $[p, c]$ at a point. Suppose that q' is the point of intersection of the segments $[a', b']$ and $[p', c']$. Let q be a point on the segment $[p, c]$ such that $d(p, q) = d(p', q')$. Thus, q' is a corresponding point of q in two triangles $\triangle(p, a, c)$ and $\triangle(p, b, c)$, and hence, $d(a, q) = d(a', q')$ and $d(b, q) = d(b', q')$. Consequently,

$$d(a, b) \leq d(a, q) + d(q, b) = d(a', q') + d(q', b') = d(a', b') = d(a, b),$$

that is, the point q is the intersection of $[a, b]$ and $[p, c]$.

Lemma 3. *Let X be a metric space with curvature bounded below by K in the large. Let γ be a spherical curve at a distance $r < \frac{\pi}{2\sqrt{K}}$ from a point p with endpoints a, b in X and γ' be a subarc of a circle of radius r centered at a point p' in R_K with endpoints a', b'. Suppose that $c \in \gamma$ is between a, b and $c' \in \gamma$ is between a', b'. Assume that the following statements hold:*

(i) $\ell(\gamma) = \ell(\gamma') \leq \frac{\pi}{\sqrt{K}}$;

(ii) $d(a, b) = d(a', b')$;

(iii) $d(a, c) = d(a', c')$ *and* $d(b, c) = d(b', c')$;

(iv) $\angle_p(a, b) = \angle_{p'}(a', b')$ *and* $\angle_c(a, b) = \angle_{c'}(a', b')$;

(v) for any triangle $\triangle(u, v, w)$ *in* X, $\angle_u(v, x) = \angle_u(v, w)$ *and* $\angle_u(w, x) = \angle_u(w, v)$ *for all* $x \in [v, w]$;

(vi) for any triangle $\triangle(u, v, w)$ *in* X, $\angle_u(v, w) = \angle_u(v, x) + \angle_u(x, w)$ *for all* $x \in [v, w]$;

Then $C(\{p, a, c, b\})$ *is isometric to* $C(\{p', a', c', b'\})$.

Proof. By (ii) and (iii), we have that triangles $\triangle(a', b', c')$ and $\triangle(a', b', p')$ are corresponding triangles of $\triangle(a, b, c)$ and $\triangle(a, b, p)$, respectively. By (iv) and (v), and applying Theorem 4, we get that the convex hulls of triangles $\triangle(a', b', c')$ and $\triangle(a', b', p')$ are isometric to the convex hulls of triangles $\triangle(a, b, c)$ and $\triangle(a, b, p)$, respectively. By (vi), we obtain

$$\angle_a(p, c) = \angle_a(p, q) + \angle_a(q, c) = \angle_{a'}(p', q') + \angle_{a'}(q', c') = \angle_{a'}(p', c')$$

and

$$\angle_b(p, c) = \angle_b(p, q) + \angle_b(q, c) = \angle_{b'}(p', q') + \angle_{b'}(q', c') = \angle_{b'}(p', c'),$$

and using (iv) and Theorem 3, we have that the convex hulls of triangles $\triangle(a', p', c')$ and $\triangle(b', p', c')$ are isometric to the convex hulls of triangles $\triangle(a, p, c)$ and $\triangle(b, p, c)$, respectively. We prove, as the same proof of Theorem 4, that $C(\{p, a, c, b\})$ is isometric to $C(\{p', a', c', b'\})$.

Theorem 6. *Let X be a metric space with curvature bounded below by K in the large. Let γ be a spherical curve at a distance $r < \frac{\pi}{2\sqrt{K}}$ from a point p with endpoints a, b in X and γ' be a subarc of a circle of radius r centered at a point p' in R_K with endpoints a', b'. Suppose that $c \in \gamma$ is between a, b and $c' \in \gamma$ is between a', b'. Assume that the following statements hold:*

(i) $\ell(\gamma) = \ell(\gamma') \leq \frac{\pi}{\sqrt{K}}$;

(ii) $d(a, b) = d(a', b')$;

(iii) $C(\{p, a, c\})$ *is isometric to* $C(\{p', a', c'\})$ *and* $C(\{p, b, c\})$ *is isometric to* $C(\{p', b', c'\})$;

(iv) $\angle_c(a,b) = \angle_{c'}(a',b')$ *and* $\angle_p(a,b) = \angle_{p'}(a',b')$*;*

(v) $\angle_a(x,c) = \angle_a(b,c)$*,* $\angle_b(x,c) = \angle_b(a,c)$ *and* $\angle_c(a,b) = \angle_c(a,x) + \angle_c(x,b)$ *for any* $x \in [a,b]$*;*

(vi) $\angle_a(x,p) = \angle_a(b,p)$*,* $\angle_b(x,p) = \angle_b(a,p)$ *and* $\angle_p(a,b) = \angle_p(a,x) + \angle_p(x,b)$ *for any* $x \in [a,b]$*.*

Then $C(\{p,a,c,b\})$ *is isometric to* $C(\{p',a',c',b'\})$*.*

Proof. By Lemma 2, the segment $[a,b]$ intersects the segment $[p,c]$ at a point q and using Theorem 4, we get that $C(\{p,a,c,b\})$ is isometric to $C(\{p',a',c',b'\})$.

By Theorem 6, we have the following corollary.

Corollary 1. *Let* X *be a metric space with curvature bounded below by* K *in the large. Let* γ *be a spherical curve at a distance* $r < \frac{\pi}{2\sqrt{K}}$ *from a point* p *with endpoints* a,b *in* X *and* γ' *be a subarc of a circle of radius* r *centered at a point* p' *in* R_K *with endpoints* a',b'*. Suppose that* $a = c_1, c_2, ..., c_n = b \in \gamma$ *are consecutive points on* γ *and* $a' = c'_1, c'_2, ..., c'_n = b' \in \gamma$ *are consecutive points on* γ'*. Assume that the following statements hold:*

(i) $\ell(\gamma) = \ell(\gamma') \leq \frac{\pi}{\sqrt{K}}$*;*

(ii) $d(a,b) = d(a',b')$*;*

(iii) $C(\{p,c_1,c_2,...,c_t\})$ *and* $C(\{p,c_t,c_{t+1},...,c_n\})$ *are isometric to* $C(\{p',c'_1,c'_2,...,c'_t\})$ *and* $C(\{p',c'_t,c'_{t+1},,...,c_n\})$*, respectively, for some* $t \in \{2,3,...,n-1\}$*;*

(iv) $\angle_{c_t}(a,b) = \angle_{c'_t}(a',b')$ *and* $\angle_p(a,b) = \angle_{p'}(a',b')$*;*

(v) $\angle_a(x,c_t) = \angle_a(b,c_t)$*,* $\angle_b(x,c_t) = \angle_b(a,c_t)$ *and* $\angle_{c_t}(a,b) = \angle_{c_t}(a,x) + \angle_{c_t}(x,b)$ *for any* $x \in [a,b]$*;*

(vi) $\angle_a(x,p) = \angle_a(b,p)$*,* $\angle_b(x,p) = \angle_b(a,p)$ *and* $\angle_p(a,b) = \angle_p(a,x) + \angle_p(x,b)$ *for any* $x \in [a,b]$*.*

Then $C(\{p,c_1,c_2,...,c_n\})$ *is isometric to* $C(\{p',c'_1,c'_2,...,c'_n\})$*.*

Proof. First, we will demonstrate that the segments $[a,b]$ and $[p,c]$ cross at a specific location. Assume that the intersection of the segments $[a',b']$ and $[p',c']$ is at q'. A point on the segment $[p,c]$ with the property $d(p,q) = d(p',q')$ is called q. Because q' is a point that corresponds to q, $d(a,q) = d(a',q')$ and $d(b,q) = d(b',q')$. Consequently,

$$d(a,b) \leq d(a,q) + d(q,b) = d(a',q') + d(q',b') = d(a',b') = d(a,b),$$

that is the point q is the intersection of $[a,b]$ and $[p,c]$.

We can see from (iv), (v), and (vi) that $\triangle(a,b,c)$ and $\triangle(a,b,p)$ are isometric to $\triangle(a',b',c')$ and $\triangle(a',b',p')$, respectively. We proceed in the same way as Theorem 4, having established that $C(\{p,a,c,b\})$ is isometric to $C(\{p',a',c',b'\})$.

Theorem 7. *Let X be a metric space with curvature bounded below by K in the large, and γ be a spherical curve at a distance $r < \frac{\pi}{2\sqrt{K}}$ from a point p with endpoints a, b in X. Let γ' be a subarc of a circle of radius r centered at a point p' in R_K with endpoints a', b'. Suppose that the following statements hold:*

(i) $\ell(\gamma) = \ell(\gamma') \leq \frac{\pi}{\sqrt{K}}$*;*

(ii) $d(a, b) = d(a', b')$*;*

(iii) $\ell(\gamma_{xy}) = \ell(\gamma'_{x'y'})$ *if and only if* $d(x, y) = d(x', y')$ *for all* $x, y \in \gamma$ *and* $x', y' \in \gamma'$*;*

(iv) $\angle_p(a, b) = \angle_{p'}(a', b')$ *and* $\angle_c(a, b) = \angle_{c'}(a', b')$*;*

(v) for any triangle $\triangle(u, v, w)$ *in* X*,* $\angle_u(v, x) = \angle_u(v, w)$ *and* $\angle_u(w, x) = \angle_u(w, v)$ *for all* $x \in [v, w]$*;*

(vi) for any triangle $\triangle(u, v, w)$ *in* X*,* $\angle_u(v, w) = \angle_u(v, x) + \angle_u(x, w)$ *for all* $x \in [v, w]$*;*

Then $\cup_{e\in\gamma}[p, e] = C(\{p\} \cup \gamma)$ *and* $C(\{p\} \cup \gamma)$ *is isometric to* $C(\{p'\} \cup \gamma')$*, that is the totally geodesic surface bounded by* γ *and the region bounded by* γ' *are isometric to each other.*

Proof. We will first demonstrate that $\cup_{e\in\gamma}[p, e] = C(\{p\} \cup \gamma)$. By the definition of $C(\{p\} \cup \gamma)$, we have that $\cup_{e\in\gamma}[p, e] \subset C(\{p\} \cup \gamma)$. The next step is to confirm that $\cup_{e\in\gamma}[p, e]$ is convex . Let $m_1, m_2 \in \cup_{e\in\gamma}[p, e]$. Therefore, $m_1 \in [p, e_1]$ and $m_2 \in [p, e_2]$ for some $e_1, e_2 \in \gamma$. If $e_1 = e_2$, there is nothing to prove; hence, we can assume that $e_1 \neq e_2$ without losing generality. Let e'_1, e'_2 be two points on γ such that $\ell(\gamma'_{a'e'_1}) = \ell(\gamma_{ae_1})$ and $\ell(\gamma'_{a'e'_2}) = \ell(\gamma_{ae_2})$. Assuming without loss of generality, that e'_1 lies between a' and e'_2. Since $\ell(\gamma_{a,e_1}) = \ell(\gamma_{a',e'_1})$ and $\ell(\gamma_{a,e_2}) = \ell(\gamma_{a',e'_2})$, we have that $\ell(\gamma_{e_1e_2}) = \ell(\gamma_{e'_1e'_2})$. By (iii), $d(a, e_1) = d(a', e'_1)$, $d(e_1, e_1) = d(e'_1, e'_2)$ and $d(e_2, b) = d(e'_2, b)$, and hence we have those triangles $\triangle(p', a', e'_1)$, $\triangle(p', e'_1, e'_2)$ and $\triangle(p', e'_2, b')$ are comparison triangles of $\triangle(p, a, e_1)$, $\triangle(p, e_1, e_2)$ and $\triangle(p, e_2, b)$, respectively. As X is metric space with curvature bounded below, $\angle_p(a, e_1) \geq \angle_{p'}(a', e'_1)$, $\angle_p(e_1, e_2) \geq \angle_{p'}(e'_1, e'_2)$ and $\angle_p(e_2, b) \geq \angle_{p'}(e'_2, b')$. Since

$$
\begin{aligned}
\angle_{p'}(a', b') &= \angle_{p'}(a', e'_1) + \angle_{p'}(e'_1, e'_2) + \angle_{p'}(e'_2, b') \\
&\leq \angle_p(a, e_1) + \angle_p(e_1, e_2) + \angle_p(e_2, b) \\
&= \angle_p(a, b) \\
&= \angle_{p'}(a', b'),
\end{aligned}
$$

we get $\angle_p(e_1, e_2) = \angle_{p'}(e'_1, e'_2)$. Because $d(p, m_1) = d(p', m'_1)$, $d(p, m_2) = d(p', m'_2)$ and $\angle_p(e_1, e_2) = \angle_p(m_1, m_2)$, by using Theorem 2, $d(m_1, m_2) \leq d(m'_1, m'_2)$. As $\ell(\gamma_{e_1e_2}) = \ell(\gamma_{e'_1e'_2})$, by (iii), we have $d(e_1, e_2) = d(e'_1, e'_2)$. Now we have that a triangle $\triangle(p', e'_1, e'_2)$ is a comparison triangle of a triangle $\triangle(p, e_1, e_2)$. As a result, X is a metric space with curvature bounded below, $d(m_1, m_2) \geq d(m'_1, m'_2)$. Consequently, $d(m_1, m_2) = d(m'_1, m'_2)$. That means that for any point on $[m_1, m_2]$, lies on a segment $[p, t]$, for some $t \in \gamma$. We may infer that, the geodesic segment $[m_1, m_2]$ is contained in $\cup_{e\in\gamma}[p, e]$.

We will then prove that $C(\{p'\} \cup \gamma')$ is isometric to $C(\{p\} \cup \gamma)$. Define a map i from $C(\{p'\} \cup \gamma')$ to $C(\{p\} \cup \gamma)$ in such a way that every geodesic segment $[p', w']$ from p' to a point w' on γ' is sent isometrically onto the segment $[p, w]$ where w is a point on γ with $\ell(\gamma_{aw}) = \ell(\gamma_{a'w'})$. The fact that i is a bijection is evident. Simply confirming that i maintains distances between points will demonstrate that i is an isometry from $C(\{p'\}\cup\gamma')$ onto $C(\{p\}\cup\gamma)$. Let x_1' and x_2' be two points on segments $[p', y_1']$ and $[p', y_2']$, respectively, for some $y_1', y_2' \in \gamma'$. On corresponding geodesic segments $[p, y_1]$ of $[p', y_1']$ and $[p, y_2]$ of $[p', y_2']$, we let x_1 and x_2 be the points corresponding to x_1' and x_2', respectively. We can verify $d(x_1, x_2) = d(x_1', x_2')$ similarly as above, the result is completely proven.

We describe characterizations of a closed spherical curve in a metric space with curvature bounded below by K in the large and having the same length as a circle in the model space R_K in the last theorem.

Theorem 8. *Let X be a metric space with curvature bounded below by K in the large, and γ be a closed spherical curve at a distance $r < \frac{\pi}{2\sqrt{K}}$ from a point p. Let γ' be a circle of radius r centered at a point p' in R_K. Suppose that the following statements hold:*

(i) $\ell(\gamma) = \ell(\gamma')$;

(ii) $\ell(\gamma_{ab}) = \ell(\gamma'_{a'b'})$ *iff* $d(a, b) = d(a', b')$ *iff* $\angle_p(a, b) = \angle_{p'}(a', b')$, *for all* $a, b \in \gamma$ *and* $a', b' \in \gamma'$;

(iii) for any triangle $\triangle(u, v, w)$ *in* X, $\angle_u(v, x) = \angle_u(v, w)$ *and* $\angle_u(w, x) = \angle_u(w, v)$ *for all* $x \in [v, w]$;

(iv) for any triangle $\triangle(u, v, w)$ *in* X, $\angle_u(v, w) = \angle_u(v, x) + \angle_u(x, w)$ *for all* $x \in [v, w]$.

Then $C(\gamma)$ is isometric to $C(\gamma')$, that is, the totally geodesic surface bounded by γ and the disk bounded by γ' are isometric to each other.

Proof. Let $x, y \in \gamma$ and $x', y' \in \gamma'$ be such that $\ell(\gamma_{xy}) = \ell(\gamma'_{x'y'}) \leq \frac{\pi}{\sqrt{K}}$. We can conclude from (ii) that $d(x, y) = d(x', y')$. We establish a map j_1 from $C(\{p'\} \cup \gamma'_{x'y'})$ to $C(\{p\} \cup \gamma_{xy})$ such that each segment $[p', z']$ from p' to z' on $\gamma'_{x'y'}$ is transferred on to the geodesic segment $[p, z]$ from p to a point z on γ_{xy}, where z is the point such that $\ell(\gamma_{xz}) = \ell(\gamma'_{x'z'})$ and a map j_2 is defined from $C(\{p'\} \cup \gamma'_{y'x'})$ to $C(\{p\} \cup \gamma_{yx})$ similar to j_1. Lemma 3 indicates that j_1 and j_2 are isometries. We will now demonstrate that $C(\gamma')$ and $C(\gamma)$ are isometric to each other. By the definition of convex hull, we observe that $C(\gamma)$ exists and is unique. Let i be a map from $C(\gamma') = C(\gamma'_{x'y'}) \cup C(\gamma'_{y'x'})$ to $C(\gamma)$ in such a way that the function i on $C(\gamma'_{x'y'})$ is j_1 and on $C(\gamma'_{y'x'})$ is j_2. We must demonstrate that i is an isometry from $C(\gamma')$ to $C(\gamma)$, we must show that i is an isometry onto its image and $C(\gamma) = C(\gamma_{xy}) \cup C(\gamma_{yx}) = C(\gamma_{xy} \cup \gamma_{yx})$. It is clear that i is surjective. Additionally, as we shown in Lemma 3, i is injective as a result of the requirements of intersecting geodesics and isometric convex hulls. Let u_1' and u_2' be in $C(\gamma')$ and $u_1 = i(u_1')$ and $u_2 = i(u_2')$. We will demonstrate that $d(u_1, u_2) = d(u_1', u_2')$. If $u_1', u_2' \in C(\gamma'_{x'y'})$ or $u_1', u_2' \in C(\gamma'_{y'x'})$, neither case can be proven. We assume that

$u'_1 \in C(\gamma'_{x'y'})$ and $u'_2 \in C(\gamma'_{y'x'})$. Let $u'_1 \in [p', v'_1]$ and $u'_2 \in [p', v'_2]$ for some $v'_1 \in C(\gamma'_{x'y'})$ and $v'_2 \in C(\gamma'_{y'x'})$. On X, we let $[q, v_1]$ be the geodesic segment containing u_1 and let $[q, v_2]$ be the geodesic segment containing u_2 where $v_1 \in C(\gamma_{xy})$ and $v_2 \in C(\gamma_{yx})$. If $\ell(\gamma'_{v'_1v'_2}) \leq \ell(\gamma')/2$, we then have $\gamma'_{v'_1v'_2} = \gamma'_{v'_1y'} \cup \gamma'_{y'v'_2}$. By (ii), $C(\gamma'_{v'_1y'})$ is isometric to $C(\gamma_{v_1y})$ by j_1 and $C(\gamma'_{v'_2y'})$ is isometric to $C(\gamma_{v_2y})$ by j_2, we thus get that $C(\gamma'_{v'_1v'_2})$ is isometric to $C(\gamma_{v_1v_2})$ by i. Consequently, we get $d(u_1, u_2) = d(u'_1, u'_2)$. Additionally, we also have $d(u_1, u_2) = d(u'_1, u'_2)$ if $\ell(\gamma'_{v'_2v'_1}) \leq \ell(\gamma')/2$. We will now demonstrate that $C(\gamma) = C(\gamma_{xy}) \cup C(\gamma_{yx}) = C(\gamma_{xy} \cup \gamma_{yx})$. It is necessary to demonstrate that the set $C(\gamma_{xy} \cup \gamma_{yx})$ is convex. Without losing generality, we suppose that x_1 is in $C(\gamma_{xy})$ and x_2 is in $C(\gamma_{yx})$. Let $[q, w_1]$ and $[q, w_2]$ be the segments containing x_1 and x_2, respectively, where $[q, w_1]$ is the segment containing x_1 and $[q, w_2]$ is the segment containing x_2. Since j_1 is the isometry from $C(\gamma'_{x'y'})$ to $C(\gamma_{xy})$ and j_2 is the isometry from $C(\gamma'_{y'x'})$ to $C(\gamma_{yx})$, we let two points w'_1 and w'_2 in R_K be the points corresponding to w_1 and w_2, respectively, and let two points x'_1 and x'_2 in R_K be the points corresponding to x_1 and x_2, respectively.

If $\ell(\gamma'_{w'_1w'_2}) \leq \ell(\gamma')/2$, then $\gamma'_{w'_1w'_2} = \gamma'_{w'_1y'} \cup \gamma'_{y'w'_2}$ is the result. As $C(\gamma'_{w'_1y'})$ is isometric to $C(\gamma_{w_1y})$ by j_1 and $C(\gamma'_{w'_2y'})$ is isometric to $C(\gamma_{w_2y})$ by j_2, we thus obtain that $C(\gamma'_{w'_1w'_2})$ is isometric to $C(\gamma_{w_1w_2})$ by i. Consequently, $d(x_1, x_2) = d(x'_1, x'_2)$ is what we have. Let x'' be the point where $[x'_1, x'_2]$ and $[x', y']$ intersect, and let $\hat{x} = j_1(x'') = j_2(x'') = i(x'')$. Hence,

$$d(x_1, x_2) = d(x'_1, x'_2) = d(x'_1, x'') + d(x'', x'_2) = d(x'_1, \hat{x}) + d(\hat{x}, x'_2),$$

so $[x_1, x_2] = [x_1, \hat{x}] \cup [\hat{x}, x_2] \subset C(\gamma_{xy}) \cup C(\gamma_{yx})$. Therefore, $C(\gamma_{xy}) \cup C(\gamma_{yx})$ is a convex set. If $\ell(\gamma'_{w'_2w'_1}) \leq \ell(\gamma')/2$, we proceed in the same proof to have that $C(\gamma_{xy}) \cup C(\gamma_{yx})$ is a convex set as in the case $\ell(\gamma'_{w'_1w'_2}) \leq \ell(\gamma')/2$. Accordingly, we can conclude that $C(\gamma')$ is isometric to $C(\gamma)$. The theorem's proof is now complete.

4. Conclusion

The totally geodesic surface enclosed by a closed spherical curve at a distance $r < \frac{\pi}{2\sqrt{K}}$ from a point in a metric space with curvature bounded below by K is isometric to the region bounded by a circle of radius r in R_K, provided that The closed spherical curve and the circle possess identical lengths, and the angle properties in this metric space have similarities to those in R_K.

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