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Some Types of Tri-Lindelöfness Spaces

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Abstract. In this study, we investigate the Lindelöf property in the context of three topologies, introducing the concept of tri-Lindelöf spaces. Additionally, we analyze the characteristics of these spaces in relation to traditional Lindelöf spaces. Several theoretical results are presented and proven, extending various well-known theorems on Lindelöf spaces to the setting of three topologies. Furthermore, illustrative examples are provided to support and clarify the findings.

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1. Introduction

Topology plays a crucial role in the analysis of differential equations, particularly in function spaces, continuity, and stability. Studies on Volterra integro-differential equations and fractional differential equations rely on topological concepts such as compactness and convergence to ensure solution existence and uniqueness [1, 2]. Additionally, understanding perturbation effects, like white noise, involves topological stability, reinforcing the connection between topology and applied mathematics [3].

Topology is a branch of mathematics that studies the properties of spaces that remain unchanged under continuous deformations such as stretching or bending. It focuses on concepts like continuity, connectedness, and compactness, providing a fundamental

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framework for analyzing geometric and abstract structures [4–10]. One of the key areas in set-theoretic topology focuses on developing and exploring the relationships between different classes of topological spaces that lie between Lindelöf spaces. In this context, the class of Lindelöf spaces plays a crucial role, as it naturally occupies an intermediate position between these classes. A Lindelöf space is defined as a topological space (Q, ϱ) in which every open cover of Q has a countable subcover. Extending this concept, a tri-topological space $(Q, \varrho_1, \varrho_2, \varrho_3)$ is called a tri-Lindelöf space if every ϱ_i-open cover of Q has a tripartite countable subcover of Q, where $i = 1, 2, 3$.

A notable characteristic of these classes of spaces is that several important separation axioms, such as normality and the Hausdorff condition, coincide within these classes. This property makes them highly significant both theoretically and practically, whether addressing problems from a purely topological perspective or applications in other areas of mathematics. A set Q is called a tri-topological space if each point in Q has a fundamental system of almost open neighborhoods. It is worth noting that the concept of almost open sets in topological groups was first studied by Ghosh and Lahiri. The notion of a tri-topological group, which is the tri-topologized version of a topological group, has also been discussed in earlier works.

The concept of a Lindelöf space in a topological space (Q, ϱ) was introduced by [11]. Recent research [12–14] has further explored and expanded upon these ideas. This paper investigates the concept of tri-Lindelöf and tri-metalindelöf spaces, presenting associated conclusions.

In the next section, we introduce the fundamental concepts of topological spaces and tri-topological spaces, along with key notions in tri-topological spaces, such as open and closed sets, derived sets, closure sets, interior and exterior sets, and separation axioms. We then discuss the Lindelöf property in tri-topological spaces, analyze its characteristics, and examine its applicability to other spaces. Additionally, we review well-known definitions that will be utilized throughout this work. Finally, we introduce and explore tripartite metalindelöfness spaces, highlighting their structural properties and significance.

The terms ϱ_u, ϱ_{dis}, ϱ_{cof}, and ϱ_{coc} represent the usual topology, discrete topology, cofinite topology, and cocountable topology, respectively. The concept of bitopological spaces is represented as $Q = (Q, \varrho_1, \varrho_2)$, where ϱ_1 and ϱ_2 are two distinct topologies on Q. This aligns with prior research on bitopological spaces, where each topology satisfies a set of axioms. In [15], the notions of pairwise Hausdorff, pairwise regular, and pairwise normal spaces were discussed using well-established results such as Tietze extension theorems.

The primary objective of this paper is to introduce and investigate a new class of tripartite Lindelöf spaces, termed tripartite metalindelöf spaces. Tri-topological spaces are sets equipped with three distinct topologies, denoted as:

$$Q = (Q, \varrho_1, \varrho_2, \varrho_3),$$

where ϱ_1, ϱ_2, and ϱ_3 are topologies on Q. These spaces exhibit structural variations that correspond to well-established properties in classical topology.

2. Preliminaries

Before presenting the main results of this study, it is essential to establish the fundamental concepts related to tri-topological spaces. In this section, we review key definitions and properties that will serve as the foundation for our discussion. These include fundamental notions of topological spaces, tri-topological spaces, open and closed sets, derived sets, closure and interior sets, and separation axioms.

Additionally, we introduce the Lindelöf property in tri-topological spaces, which extends the classical Lindelöf concept to settings with multiple topologies. The definitions and results presented here will be instrumental in proving subsequent theorems and exploring the structure of tri-Lindelöf and tri-metalindelöf spaces.

Definition 1. *[16] Let $Q \neq \emptyset$ and let ϱ be a collection of subsets of Q, denoted by $\varrho \subseteq P(Q) = \{O \mid O \subseteq Q\}$. The collection ϱ is called a topological space on Q if it satisfies the following conditions:*

(i) $\emptyset, Q \in \varrho$ (the empty set and the whole space are in ϱ).

(ii) ϱ is closed under finite intersections.

(iii) ϱ is closed under arbitrary unions.

Definition 2. *[13] Let $Q \neq \emptyset$ and let $\varrho_i \subseteq P(Q)$ for $i = 1, 2, 3$. We say that $(Q, \varrho_1, \varrho_2, \varrho_3)$ is a tri-topological space if each ϱ_i is a topology on Q, for all $i = 1, 2, 3$.*

Example 1. *The space $(Q, \varrho_1, \varrho_2, \varrho_3)$ a tri-topological space. To see this, consider $Q = \{x, y, z\}$, with the following collections of subsets:*

$$\varrho_1 = \{\emptyset, Q, \{x\}\} \subseteq P(Q),$$

$$\varrho_2 = \{\emptyset, Q, \{x\}, \{y\}, \{x, y\}\} \subseteq P(Q),$$
$$\varrho_3 = \{\emptyset, Q, \{y\}, \{z\}, \{y, z\}\} \subseteq P(Q).$$

Since each ϱ_i satisfies the conditions of a topology for $i = 1, 2, 3$, it follows that $(Q, \varrho_1, \varrho_2, \varrho_3)$ is a tri-topological space. However, consider the collection $\varrho = \{\emptyset, Q, \{x\}, \{y\}\}$. This does not form a topology on Q because it does not satisfy all the conditions required for a topology.

Definition 3. [15] *Let $(Q, \varrho_1, \varrho_2, \varrho_3)$ be a tri-topological space and let $O \subseteq Q$. Then:*

(i) O is called a ϱ_i-open set if $O \in \varrho_i$ for some $i \in \{1, 2, 3\}$.

(ii) O is called a ϱ_i-closed set if its complement, O^c, belongs to ϱ_i for some $i \in \{1, 2, 3\}$.

(iii) O is called a ϱ_i-clopen set if both O and O^c belong to ϱ_i for some $i \in \{1, 2, 3\}$.

Definition 4. [14] *Let $(Q, \varrho_1, \varrho_2, \varrho_3)$ be a tri-topological space, where $Q \neq \emptyset$ and $O \subseteq Q$. A point $q \in Q$ is called a tri-limit point of O if for every ϱ_i-open set u_q containing q, we have:*

$$u_q \cap (O \setminus \{q\}) \neq \emptyset, \quad \text{for all } i \in \{1, 2, 3\}.$$

The set of all tri-limit points in a tri-topological space is called the tri-derived set, denoted by:

$$O' = \{q \mid q \text{ is a tri-limit point of } O\}.$$

Lemma 1. *[13]**(Properties of the Tripartite Derived Set)** Let $(Q, \varrho_1, \varrho_2, \varrho_3)$ be a tri-topological space, and let $O, B \subseteq Q$. Then:*

(i) $\emptyset' = \emptyset$.

(ii) *If* $O \subseteq B$, *then* $O' \subseteq B'$.

(iii) $(O \cup B)' = O' \cup B'$.

(iv) $(O \cap B)' \subseteq O' \cap B'$.

Proof. **(i)** By contradiction: Assume that $\emptyset' \neq \emptyset$. Then, there exists $z \in \emptyset'$, meaning that for all ϱ_i-open sets u_z, we have:

$$u_z \cap (\emptyset - \{z\}) \neq \emptyset.$$

However, since $(\emptyset - \{z\}) = \emptyset$, we obtain:

$$u_z \cap \emptyset \neq \emptyset.$$

This contradicts the fact that the intersection of any set with the empty set is always empty. Therefore, $\emptyset' = \emptyset$.

Definition 5. [16] *Let $(Q, \varrho_1, \varrho_2, \varrho_3)$ be a tri-topological space, where $Q \neq \emptyset$ and $O \subseteq Q$. The tri-closure set of O is denoted by:*

$$\overline{O} = O \cup O'.$$

Lemma 2. [5] ***(Properties of the Tripartite Closure Set)*** *Let $(Q, \varrho_1, \varrho_2, \varrho_3)$ be a tri-topological space, and let $O, B \subseteq Q$. Then:*

(i) $\overline{\emptyset} = \emptyset$ *and* $\overline{Q} = Q$.

(ii) $\overline{O \cup B} = \overline{O} \cup \overline{B}$ *and* $\overline{O \cap B} \subseteq \overline{O} \cap \overline{B}$.

(iii) $\overline{O}$ *is a* ϱ_i*-closed set.*

(iv) $O = \overline{O}$ *if and only if* O *is a* ϱ_i*-closed set.*

(v) $q \in \overline{O}$ *if and only if for every* ϱ_i*-open set* u_q *such that* $q \in u_q$*, we have* $u_q \cap O \neq \emptyset$.

Proof. **(ii)** Let $O, B \subseteq Q$. Then:

$$\overline{O \cup B} = (O \cup B) \cup (O \cup B)' = (O \cup B) \cup (O' \cup B') = (O \cup O') \cup (B \cup B') = \overline{O} \cup \overline{B}.$$

Similarly, for the intersection property:

$$\overline{O \cap B} = (O \cap B) \cup (O \cap B)' \subseteq (O \cap B) \cup (O' \cap B') \subseteq (O \cup O') \cap (B \cup B') = \overline{O} \cap \overline{B}.$$

Definition 6. [8] *Let $(Q, \varrho_1, \varrho_2, \varrho_3)$ be a tri-topological space, where $Q \neq \emptyset$, and let $O \subseteq Q$. A point $q \in O$ is said to be a tri-interior point of O if there exists at least one neighborhood $N(q, \varepsilon)$ of q such that:*

$$N(q, \varepsilon) \subseteq O.$$

The set of all tri-interior points of O is called the tri-interior set, denoted by:

$$O^\circ \equiv \mathrm{INT}(O) = \overline{(O^c)}^c.$$

Lemma 3. [6] ***(Properties of the Tripartite Interior Set)*** *Let $(Q, \varrho_1, \varrho_2, \varrho_3)$ be a tri-topological space, and let $O, B \subseteq Q$. Then:*

(i) $\emptyset^\circ = \emptyset$ *and* $Q^\circ = Q$.

(ii) $(O \cap B)^\circ = O^\circ \cap B^\circ$ *and* $O^\circ \cup B^\circ \subseteq (O \cup B)^\circ$.

(iii) O° *is a* ϱ_i*-open set.*

(iv) $n \in O^\circ$ *if and only if there exists a* ϱ_i*-open set* u_n *such that* $n \in u_n \subseteq O$.

Definition 7. [7] *Let $(Q, \varrho_1, \varrho_2, \varrho_3)$ be a tri-topological space, where $Q \neq \emptyset$, and let $O \subseteq Q$. A point q is said to be a tri-exterior point of O if there exists at least one neighborhood $N(q, \varepsilon)$ of q such that:*

$$N(q, \varepsilon) \cap O = \emptyset.$$

The set of all tri-exterior points of O is called the tri-exterior set, denoted by:

$$\mathrm{EX}(O) = \mathrm{Int}(O^c) = \overline{O}^c.$$

Lemma 4. [14] ***(Properties of the Tripartite Exterior Set)*** *Let $(Q, \varrho_1, \varrho_2, \varrho_3)$ be a tri-topological space, and let $O, B \subseteq Q$. Then:*

(i) $\mathrm{EX}(\emptyset) = Q$ *and* $\mathrm{EX}(Q) = \emptyset$.

(ii) If $O \subseteq B$, *then* $\mathrm{EX}(B) \subseteq \mathrm{EX}(O)$.

(iii) $\mathrm{EX}(O)$ *is a* ϱ_i*-open set.*

(iv) $e \in \mathrm{EX}(O)$ *if and only if there exists a* ϱ_i*-open set* u_e *such that* $e \in u_e \subseteq O^c$.

Proof. **(iii)** Since $\mathrm{EX}(O) = \mathrm{Int}(O^c)$, we have:

$$\mathrm{EX}(O) = \overline{(O^c)}^c.$$

But since $\overline{O^c}^c = O$, it follows that:

$$\mathrm{EX}(O) = \overline{O}^c.$$

By the definition of the tri-closure set, we know that $\overline{O}$ is a ϱ_i-closed set. Since the complement of a closed set is an open set, it follows that $\mathrm{EX}(O)$ is a ϱ_i-open set. As a result, the interior set is also a ϱ_i-open set.

Definition 8. [12] *Let $(Q, \varrho_1, \varrho_2, \varrho_3)$ be a tri-topological space, where $Q \neq \emptyset$, and let $O \subseteq Q$. A point q is said to be a tri-boundary point of O if every neighborhood $N(q, \varepsilon)$ of q satisfies:*

$$N(q, \varepsilon) \cap O \neq \emptyset \quad \textit{and} \quad N(q, \varepsilon) \cap O^c \neq \emptyset.$$

The set of all tri-boundary points is called the tri-boundary set, denoted by:

$$\mathrm{Bd}(O) = \overline{O} - O^\circ = \overline{O} \cap \overline{O^c}.$$

Lemma 5. [1] ***(Properties of the Tripartite Boundary Set)*** *Let $(Q, \varrho_1, \varrho_2, \varrho_3)$ be a tri-topological space, and let $O, B \subseteq Q$. Then:*

(i) $\mathrm{Bd}(\emptyset) = \mathrm{Bd}(Q) = \emptyset$.

(ii) $\mathrm{Bd}(O)$ *is a ϱ_i-closed set.*

(iii) $b \in \mathrm{Bd}(O)$ *if and only if for every ϱ_i-open set u_b containing b, we have:*

$$u_b \cap O \neq \emptyset \quad \textit{and} \quad u_b \cap O^c \neq \emptyset.$$

Proof. **(iii)** Let $s \in \mathrm{Bd}(O)$, and let u_s be a ϱ_i-open set such that $s \in u_s$. Then, by definition:

$$s \in \overline{O} \cap \overline{O^c}.$$

This implies that:

$$s \in \overline{O} \quad \text{and} \quad s \in \overline{O^c}.$$

By the definition of closure, we have:

$$s \in O \cup O' \quad \text{and} \quad s \in O^c \cup (O^c)'.$$

This means that:

$$(s \in O \text{ or } s \in O') \quad \text{and} \quad (s \in O^c \text{ or } s \in (O^c)').$$

Thus, we conclude that:

$$u_s \cap (O \setminus \{s\}) \neq \emptyset \quad \text{and} \quad u_s \cap O^c \neq \emptyset.$$

Since $s \subseteq u_s$, we obtain:

$$u_s \cap O \neq \emptyset \quad \text{and} \quad u_s \cap O^c \neq \emptyset.$$

Hence, s satisfies the condition of a boundary point, completing the proof.

Definition 9. *[17] A topological space (Q, ϱ) is called a T_0-space if for all distinct points $q, s \in Q$ $(q \neq s)$, there exists an open set u_q such that:*

$$q \in u_q \quad \textit{and} \quad s \notin u_q,$$

or there exists an open set v_s such that:

$$s \in v_s \quad \textit{and} \quad q \notin v_s.$$

Definition 10. [8] *A tri-topological space $(Q, \varrho_1, \varrho_2, \varrho_3)$ is called a tri-T_0-space if for all distinct points $q, s \in Q$ $(q \neq s)$, there exists a ϱ_i-open set u_q such that:*

$$q \in u_q \quad \textit{and} \quad s \notin u_q,$$

or there exists a ϱ_j-open set v_s such that:

$$s \in v_s \quad \textit{and} \quad q \notin v_s,$$

where $i \neq j$ and $i, j \in \{1, 2, 3\}$.

Theorem 1. *Let $(Q, \varrho_1, \varrho_2, \varrho_3)$ be a tri-topological space. Then, the following statements are equivalent:*

(i) Q is a tri-T_0-space.

(ii) For all $q \neq s$, we have $q \notin \overline{\{s\}}$ or $s \notin \overline{\{q\}}$.

(iii) For all $q \neq s$, we have $\overline{\{q\}} \neq \overline{\{s\}}$.

Proof. **(i) ⇒ (ii):**

Let $q \neq s$. Since Q is a tri-T_0-space, there exists a ϱ_i-open set u_q such that $q \in u_q$ and $s \notin u_q$, or there exists a ϱ_j-open set v_s such that $s \in v_s$ and $q \notin v_s$, where $i, j \in \{1, 2, 3\}$. This implies that:

$$q \in u_q \quad \text{and} \quad u_q \cap \{s\} = \emptyset, \quad \text{or} \quad s \in v_s \quad \text{and} \quad v_s \cap \{q\} = \emptyset.$$

Therefore, we conclude that $q \notin \overline{\{s\}}$ or $s \notin \overline{\{q\}}$.

(ii) ⇒ (iii):

Suppose $q \neq s$. If $q \notin \overline{\{s\}}$ and $q \in \overline{\{q\}}$, then we must have $\overline{\{q\}} \neq \overline{\{s\}}$. Similarly, if $s \notin \overline{\{q\}}$ and $s \in \overline{\{s\}}$, then again $\overline{\{q\}} \neq \overline{\{s\}}$. Thus, condition (iii) holds.

(iii) ⇒ (i):

Suppose $q \neq s$ and we are given that $\overline{\{q\}} \neq \overline{\{s\}}$. Since $q \in \overline{\{q\}}$ and $s \in \overline{\{s\}}$, we must have:

$$q \notin Q - \overline{\{q\}} = v_s,$$

where v_s is a ϱ_i-open set in Q, since $\overline{\{q\}}$ is a ϱ_i-closed set. Furthermore, $s \in Q - \overline{\{q\}} = v_s$, where $i \in \{1, 2, 3\}$. Therefore, Q is a tri-T_0-space.

Definition 11. [16] *A tri-topological space $(Q, \varrho_1, \varrho_2, \varrho_3)$ is called a tri-T_1-space if for all distinct points $q, s \in Q$ $(q \neq s)$, there exists a ϱ_i-open set u_q such that:*

$$q \in u_q \quad \text{and} \quad s \notin u_q,$$

and there exists a ϱ_j-open set v_s such that:

$$s \in v_s \quad \text{and} \quad q \notin v_s,$$

where $i \neq j$ and $i, j \in \{1, 2, 3\}$.

Definition 12. [12] *A tri-topological space $(Q, \varrho_1, \varrho_2, \varrho_3)$ is called a tri-T_2-space if for all distinct points $q, s \in Q$ $(q \neq s)$, there exists a ϱ_i-open set u_q such that:*

$$q \in u_q,$$

and there exists a ϱ_j-open set v_s such that:

$$s \in v_s \quad \text{and} \quad u_q \cap v_s = \emptyset,$$

where $i \neq j$ and $i, j \in \{1, 2, 3\}$.

Definition 13. [1] *A tri-topological space $(Q, \varrho_1, \varrho_2, \varrho_3)$ is called a tri-$T_{2\frac{1}{2}}$-space if for all distinct points $q, s \in Q$ $(q \neq s)$, there exist ϱ_i-closed sets O_q and B_s such that:*

$$q \in O_q, \quad s \in B_s, \quad \text{and} \quad O_q \cap B_s = \emptyset,$$

for some $i \in \{1, 2, 3\}$.

Definition 14. [3] *A tri-topological space $(Q, \varrho_1, \varrho_2, \varrho_3)$ is called a tri-regular space if for every point $q \notin O$, where O is a ϱ_i-closed set, there exist a ϱ_i-open set u_q and a ϱ_j-open set v_O such that:*

$$q \in u_q, \quad O \subseteq v_O, \quad \text{and} \quad u_q \cap v_O = \emptyset,$$

where $i \neq j$ and $i, j \in \{1, 2, 3\}$.

Theorem 2. *A space $(Q, \varrho_1, \varrho_2, \varrho_3)$ is a tri-regular space if and only if for all $q \in u_q$, where u_q is a ϱ_i-open set, there exists a ϱ_i-open set w_q such that:*

$$q \in w_q \subseteq \overline{w_q} \subseteq u_q.$$

Proof. $(\Rightarrow)$ Let $q \in u_q$. Since $q \notin u_q^c$, and u_q^c is a ϱ_i-closed set, we denote $u_q^c = O$. By the definition of a tri-regular space, there exist ϱ_i-open sets w_q and v_O such that:

$$q \in w_q, \quad O \subseteq v_O, \quad \text{and} \quad w_q \cap v_O = \emptyset.$$

Clearly, we have $w_q \subseteq \overline{w_q}$. To complete the proof, we need to show that $\overline{w_q} \subseteq u_q$. Since $w_q \cap v_O = \emptyset$, we can conclude that $w_q \subseteq v_O^c$. Taking closures, we obtain:

$$\overline{w_q} \subseteq \overline{v_O^c} = v_O^c.$$

But since $O^c = v_O^c$ and $O \subseteq v_O$, we have:

$$O^c \subseteq v_O^c.$$

Therefore, $v_O^c \subseteq u_q$, which implies:

$$\overline{w_q} \subseteq u_q.$$

Thus, the condition holds.

($\Leftarrow$) Suppose $q \notin O$ and O is a ϱ_i-closed set. Then, $q \in O^c$, which is a ϱ_i-open set. By the given condition, there exists a ϱ_i-open set w_q such that:

$$q \in w_q \subseteq \overline{w_q} \subseteq O^c.$$

Now, we have two conditions:

$$q \in w_q \quad \text{and} \quad O \subseteq \overline{w_q}^c.$$

Since both w_q and $\overline{w_q}^c$ are ϱ_i-open sets, it is sufficient to show:

$$w_q \cap \overline{w_q}^c = \emptyset.$$

Suppose not. Then there exists some $z \in w_q \cap \overline{w_q}^c$, which implies:

$$z \in w_q \quad \text{and} \quad z \in \overline{w_q}^c.$$

Since $z \in \overline{w_q}^c$, it follows that $z \notin w_q$, contradicting $z \in w_q$. Thus, our assumption is false, and we conclude:

$$w_q \cap \overline{w_q}^c = \emptyset.$$

By these conditions, we establish that $(Q, \varrho_1, \varrho_2, \varrho_3)$ is a tri-regular space.

Definition 15. *[13] A topological space (Q, ϱ) is called a T_3-space if it is both a T_1-space and a regular space.*

Definition 16. [13] *A tri-topological space $(Q, \varrho_1, \varrho_2, \varrho_3)$ is called a tri-T_3-space if it is both a tri-T_1-space and a tri-regular space.*

Definition 17. [15] *A tri-topological space $(Q, \varrho_1, \varrho_2, \varrho_3)$ is called a tri-normal space if for every two disjoint ϱ_i-closed sets O and B, there exist ϱ_i-open sets u_O and v_B such that:*

$$O \subseteq u_O, \quad B \subseteq v_B, \quad \text{and} \quad u_O \cap v_B = \emptyset.$$

Definition 18. [16] *A tri-topological space $(Q, \varrho_1, \varrho_2, \varrho_3)$ is called a tri-T_4-space if it is both a tri-T_1-space and a tri-normal space.*

Theorem 3. [8] *If $(Q, \varrho_1, \varrho_2, \varrho_3)$ is a tri-T_k-space, then it is also a tri-T_{k-1}-space.*

Theorem 4. *If a space $(Q, \varrho_1, \varrho_2, \varrho_3)$ is a tri-T_4-space, then it is also a tri-T_3-space.*

Proof. Let $(Q, \varrho_1, \varrho_2, \varrho_3)$ be a tri-T_4-space. By definition, a tri-T_4-space is both a tri-T_1-space and a tri-normal space. That is, for all two disjoint ϱ_i-closed sets O and B, there exist ϱ_i-open sets u_O and v_B such that:

$$O \subseteq u_O, \quad B \subseteq v_B, \quad \text{and} \quad u_O \cap v_B = \emptyset. \quad \text{(i)}$$

Now, let $b \in B$. Then, from (i), we have $b \in v_B$. Since $O \cap B = \emptyset$, it follows that $b \notin O$. (ii) From (i) and (ii), we conclude that $(Q, \varrho_1, \varrho_2, \varrho_3)$ is a tri-regular space. Since a tri-T_4-space is also a tri-T_1-space, and we have shown it is tri-regular, we conclude that $(Q, \varrho_1, \varrho_2, \varrho_3)$ is a tri-T_3-space.

Definition 19. *[13] Let (Q, ϱ) be a topological space, and let*

$$W = \{O_\alpha \mid \alpha \in \lambda, O_\alpha \subseteq Q\}$$

be a collection of subsets of Q. Then:

(i) W is called a cover of Q if and only if:

$$\bigcup_{\alpha \in \lambda} O_\alpha = Q.$$

(ii) W is called an open cover of Q if and only if W is a cover and each O_α is an open set, where $\alpha \in \lambda$.

(iii) W is called a closed cover of Q if and only if W is a cover and each O_α is a closed set, where $\alpha \in \lambda$.

(iv) A collection $C = \{B_\gamma \mid \gamma \in \Gamma\}$ is a subcover of W if and only if $C \subseteq W$ and $\bigcup_{\gamma \in \Gamma} B_\gamma = Q$.

A space (Q, ϱ) is called compact if every open cover of Q has a finite subcover.

Definition 20. [6] *Let $(Q, \varrho_1, \varrho_2, \varrho_3)$ be a tri-topological space, and let*

$$W = \{O_\alpha \mid \alpha \in \lambda, O_\alpha \subseteq Q\}$$

be a collection of subsets of Q. Then:

(i) W is called a tri-cover of Q if and only if:

$$\bigcup_{\alpha \in \lambda} O_\alpha = Q.$$

(ii) W is called a tri-open cover of Q if and only if W is a tri-cover and each O_α is a ϱ_i-open set, where $\alpha \in \lambda$ and $i \in \{1, 2, 3\}$.

(iii) *W is called a tri-closed cover of Q if and only if W is a tri-cover and each O_α is a ϱ_i-closed set, where $\alpha \in \lambda$ and $i \in \{1, 2, 3\}$.*

(iv) *A collection $C = \{B_\gamma \mid \gamma \in \Gamma\}$ is called a tripartite subcover of W if and only if:*

- *(a)* $C \subseteq W$,
- *(b)* $\bigcup_{\gamma\in\Gamma} B_\gamma = Q$.

A space $(Q, \varrho_1, \varrho_2, \varrho_3)$ is called a tripartite compact space if every tri-open cover of Q has a finite tripartite subcover.

Definition 21. [7] *Let $(Q, \varrho_1, \varrho_2, \varrho_3)$ be a tri-topological space. The space Q is said to be tripartite locally compact if each point of Q has a tri-open neighborhood whose tri-closure is tri-compact.*

Note: Every tri-compact space is tri-locally compact.

3. Tri-Lindelöf and Nearly Tri-Lindelöf Spaces

In this section, we discuss the concept of Lindelöf spaces in classical topological spaces, as well as their extension to tri-topological spaces. Additionally, we introduce the notion of nearly Lindelöf spaces in tri-topological spaces and explore their fundamental properties and related theorems.

Definition 22. *[14] Let (Q, ϱ) be a topological space. It is called a Lindelöf space if every open cover of Q has a countable subcover.*

Definition 23. *Let $(Q, \varrho_1, \varrho_2, \varrho_3)$ be a tri-topological space. It is called a tri-Lindelöf space if every ϱ_i-open cover of Q has a tripartite countable subcover.*

Note: A set A is said to be countable if either it is finite or if there exists an injective function from A into the set of natural numbers $\mathbb{N}$.

Definition 24. *Let $(Q, \varrho_1, \varrho_2, \varrho_3)$ be a tri-topological space, and let $\tilde{C}$ be a cover of Q. We say that $\tilde{C}$ is a tri-open cover if:*

$$\tilde{C} \subseteq \bigcup_{i=1}^{3} \upsilon_i.$$

Remark 1. *A tri-topological space $(Q, \varrho_1, \varrho_2, \varrho_3)$ is a second-countable base with respect to ϱ_1, ϱ_2, and ϱ_3.*

Definition 25. *A tri-topological space $(Q, \varrho_1, \varrho_2, \varrho_3)$ is called a tri-S-Lindelöf space if and only if it is a Lindelöf space, a pairwise Lindelöf space, and a tripartite Lindelöf space.*

Theorem 5. *If a tri-topological space Q is a second-countable space, then it is a tripartite Lindelöf space.*

Proof. Since Q is second-countable, each topology $\varrho_1, \varrho_2, \varrho_3$ has a countable basis. Given any open cover in each topology, a countable subcover exists by definition of second-countability. Hence, Q is Lindelöf in each topology, making it a tripartite Lindelöf space.

Remark 2. *Every tripartite compact space is a tripartite Lindelöf space, but the converse need not be true.*

Theorem 6. *Let $(\mathbb{R}, \varrho_u, \varrho_u, \varrho_u)$ be a tri-topological space. Then, $(\mathbb{R}, \varrho_u, \varrho_u, \varrho_u)$ is a tri-Lindelöf space but not a tri-compact space.*

Proof. Since the set

$$O = \{(q, s) \mid q, s \in \mathbb{Q}\}$$

is a countable base with respect to ϱ_i of $\mathbb{R}$, for $i = 1, 2, 3$, we conclude that $\mathbb{R}$ is a second-countable space. By the previous theorem, this implies that $(\mathbb{R}, \varrho_u, \varrho_u, \varrho_u)$ is a tri-Lindelöf space. However, the set (q, s) for all $q < s$ in $\mathbb{Q}$ is not closed, meaning that $(\mathbb{R}, \varrho_u, \varrho_u, \varrho_u)$ is not compact. Therefore, $(\mathbb{R}, \varrho_u, \varrho_u, \varrho_u)$ is a tri-Lindelöf space but not a tri-compact space.

Corollary 1. *Every second-countable topological space is a tripartite topological space $(X, \vartheta_1, \vartheta_2, \vartheta_3)$. Thus, X is a tri-S-Lindelöf space.*

Proof. Clearly by above theorem

Remark 3. *A compact subset of a tri-T_2-space is tri-closed, but a tri-Lindelöf subset of a tri-T_2-space need not be tri-closed.*

Definition 26. *A space $(Q, \varrho_1, \varrho_2, \varrho_3)$ is called a tripartite space if and only if the countable intersection of tri-open sets is open.*

Theorem 7. *If L is a Lindelöf subset of a tri-T_2-space Q, then for each $m \notin L$, we can separate m and L into two disjoint tri-open sets in Q.*

Proof. Since Q is a tri-T_2-space, for each $x \in L$, there exist disjoint tri-open sets U_x and V_x such that $x \in U_x$ and $m \in V_x$. The collection $\{U_x \mid x \in L\}$ forms an open cover of L, which has a countable subcover $\{U_{x_i} \mid i \in \mathbb{N}\}$ because L is Lindelöf. Let $U = \bigcup_{i \in \mathbb{N}} U_{x_i}$ and $V = \bigcap_{i \in \mathbb{N}} V_{x_i}$. Then U and V are disjoint tri-open sets separating L and m, proving the theorem.

Definition 27. *[17] Suppose (Q, ϱ) is a topological space. A subset $Y \subseteq Q$ is called nearly open if:*

$$Y = (\overline{Y})^\circ \quad \textit{in } \varrho.$$

Definition 28. *Suppose $(Q, \varrho_1, \varrho_2, \varrho_3)$ is a tri-topological space. A subset $Y \subseteq Q$ is called a tripartite nearly open set if:*

$$(\overline{Y})^\circ \textit{ is open in } \varrho_1, \quad (\overline{Y})^\circ \textit{ is open in } \varrho_2, \quad \textit{and} \quad (\overline{Y})^\circ \textit{ is open in } \varrho_3.$$

Definition 29. *A collection $\tilde{U} = \{U_\alpha \mid \alpha \in \Delta\}$ is called a nearly open cover of the tri-topological space $(Q, \varrho_1, \varrho_2, \varrho_3)$ if:*

(i) Each U_α is a tri-nearly open set for all $\alpha \in \Delta$.

(ii) $\bigcup_{\alpha \in \Delta} U_\alpha = Q$.

Definition 30. *Suppose $(Q, \varrho_1, \varrho_2, \varrho_3)$ is a tri-topological space, and let $\tilde{U} = \{U_\alpha \mid \alpha \in \Delta\}$ be a tri-nearly open cover of Q. A collection $S_\gamma = \{U_{\alpha\gamma} \mid \gamma \in \Gamma\}$ is called a tri-nearly subcover of Q if:*

$$\bigcup_{\gamma \in \Gamma} U_{\alpha\gamma} = Q.$$

Definition 31. *[11] Suppose (Q, ϱ) is a topological space. We say that Q is a nearly compact space if every nearly open cover of Q has a finite nearly subcover.*

Definition 32. *Suppose $(Q, \varrho_1, \varrho_2, \varrho_3)$ is a tri-topological space. We say that Q is a tri-nearly compact space if every tri-nearly open cover of Q has a finite tri-nearly subcover.*

Definition 33. *[12] Suppose (Q, ϱ) is a topological space. The space Q is called a nearly Lindelöf space if every nearly open cover of Q has a countable nearly subcover.*

Definition 34. *Suppose $(Q, \varrho_1, \varrho_2, \varrho_3)$ is a tri-topological space. The space Q is called a tri-nearly Lindelöf space if every tri-nearly open cover of Q has a tripartite countable nearly subcover.*

Corollary 2. *Every tri-nearly compact space is a tri-nearly Lindelöf space, but the converse need not be true.*

Proof. Let Q be a tri-nearly compact space. By definition, every tri-nearly open cover of Q has a finite ϱ_j-subcover of Q. Thus, for each tri-nearly open cover on Q, there exists a countable ϱ_j-subcover of Q, for all $i \neq j$, where $i, j \in \{1, 2, 3\}$. Therefore, Q is a tri-nearly Lindelöf space.

However, the converse need not be true. For example, the space $(\mathbb{R}, \varrho_{u1}, \varrho_{u2}, \varrho_{u3})$ is a tri-nearly Lindelöf space but is not a tri-nearly compact space.

Theorem 8. *A tri-nearly Lindelöf space is preserved under an onto tri-continuous function.*

Proof. Let $i \neq j$ where $i, j \in \{1, 2, 3\}$. Let $F : (Q, \varrho_1, \varrho_2, \varrho_3) \to (Y, \sigma_1, \sigma_2, \sigma_3)$ be a surjective continuous function, and suppose that Q is a tri-nearly Lindelöf space. We aim to show that Y is also a tri-nearly Lindelöf space. Assume $\tilde{U} = \{U_\alpha \mid \alpha \in \vartheta\}$ is a nearly ϑ_i-open cover of Y, meaning that each U_α is a tri-open set for all $\alpha \in \vartheta$. Since F is continuous, it follows that $F^{-1}(U_\alpha)$ is a tri-open set in Q for each $\alpha \in \vartheta$. Additionally, since F is surjective, we obtain:

$$F^{-1}(\tilde{U}) = \{F^{-1}(U_\alpha) \mid \alpha \in \vartheta\}$$

which forms a tri-open cover of Q. Since Q is tri-nearly Lindelöf, there exists a countable ϑ_i-subcover

$$\{F^{-1}(U_\alpha) \mid \alpha \in \sigma\}$$

where $\sigma \subseteq \vartheta$ and $|\sigma| \leq \aleph_0$. Thus, we have:

$$Q \subseteq \bigcup_{\alpha \in \sigma} F^{-1}(U_\alpha).$$

Since F is onto, it follows that:

$$Y = F(Q) \subseteq F\left(\bigcup_{\alpha \in \sigma} F^{-1}(U_\alpha)\right) \subseteq \bigcup_{\alpha \in \sigma} U_\alpha.$$

Hence, $\tilde{U}$ has a countable ϑ_i-subcover of Y, proving that Y is a tri-nearly Lindelöf space.

Remark 4. *A compact subset of a tri-nearly T_2-space is closed, but a tri-nearly Lindelöf subset of a tri-nearly T_2-space need not be closed.*

Definition 35. *A space $(Q, \varrho_1, \varrho_2, \varrho_3)$ is called a tripartite nearly tri-space if the countable intersection of nearly open sets is open.*

4. Metalindelöfness spaces in Tri-topological Spaces

In this section, we explore the concept of locally tri-metalindelöfness in tri-topological spaces and examine various fundamental properties of these spaces.

Definition 36. *A tri-topological space $(Q, \varrho_1, \varrho_2, \varrho_3)$ is called a tri-metalindelöf space if every tri-open cover of $(Q, \varrho_1, \varrho_2, \varrho_3)$ has a point-countable parallel refinement.*

Theorem 9. *A countable tri-metalindelöf space is a tri-compact space.*

Proof. Let Q be a countable tri-metalindelöf space, and let $\mathcal{U}$ be an arbitrary tri-open cover of Q. Since Q is tri-metalindelöf, there exists a point-countable refinement $\mathcal{V}$ such that each point in Q is contained in at most countably many sets of $\mathcal{V}$. Since Q is countable, $\mathcal{V}$ itself must be countable. By the Lindelöf property of $\mathcal{V}$, there exists a countable subcover of $\mathcal{U}$, proving that Q is tri-compact.

Theorem 10. *A separable tri-metalindelöf space $(Q, \varrho_1, \varrho_2, \varrho_3)$ is tri-Lindelöf.*

Proof. Since Q is separable, it has a countable dense subset D. Given any tri-open cover $\mathcal{U}$ of Q, each point of D is contained in at most countably many sets of a point-countable refinement $\mathcal{V}$ due to the tri-metalindelöf property. Since D is countable, we can extract a countable subcollection $\mathcal{V}'$ covering D. The closure of D, which is Q, is covered by $\mathcal{V}'$, ensuring a countable subcover of $\mathcal{U}$. Thus, Q is tri-Lindelöf.

Definition 37. *A tri-topological space* $(Q, \varrho_1, \varrho_2, \varrho_3)$ *is called tripartite countably metalindelöf if every countable tri-open cover of* $(Q, \varrho_1, \varrho_2, \varrho_3)$ *has a point-countable parallel refinement.*

Theorem 11. *The tri-topological space* $(Q, \varrho_1, \varrho_2, \varrho_3)$ *is a tri-metalindelöf space. It is also tripartite countably metacompact.*

Proof. Since $(Q, \varrho_1, \varrho_2, \varrho_3)$ is tri-metalindelöf, every open cover of Q has a point-countable open refinement. To show that it is also tripartite countably metacompact, let $\mathcal{U}$ be a countable open cover of Q. By the tri-metalindelöf property, there exists a point-countable refinement $\mathcal{V}$ of $\mathcal{U}$, meaning each point of Q is contained in at most countably many sets of $\mathcal{V}$. Since $\mathcal{V}$ is countable, we can extract a countable subcover from $\mathcal{U}$, proving that Q is tripartite countably metacompact.

Theorem 12. *Every tri-Lindelöf tripartite countably metacompact space* $(Q, \varrho_1, \varrho_2, \varrho_3)$ *is a tri-metalindelöf space.*

Proof. Let $\tilde{U} = \{U_\alpha \mid \alpha \in \Delta\}$ be a tri-open cover of Q. Since Q is tri-Lindelöf, there exists a tripartite countable subcover, say

$$\tilde{A} = \{A_{\alpha i}\}_{i=1}^{\infty}.$$

Furthermore, since Q is tripartite countably metacompact, the subcover $\tilde{A}$ has a point-countable parallel refinement $\check{G}$ of $\tilde{U}$. Hence, $(Q, \varrho_1, \varrho_2, \varrho_3)$ is a tri-metalindelöf space.

Theorem 13. *Every tri-metalindelöf countably metacompact space* $(Q, \varrho_1, \varrho_2, \varrho_3)$ *is a tri-metacompact space.*

Proof. Let $(Q, \varrho_1, \varrho_2, \varrho_3)$ be a tri-metalindelöf countably metacompact space. By the tri-metalindelöf property, for each $i \in \{1, 2, 3\}$, for any open cover $\mathcal{U}_i$ of Q, there exists a countable subcover $\mathcal{U}_i^c \subseteq \mathcal{U}_i$. By countable metacompactness, for each $i \in \{1, 2, 3\}$, for the subcover $\mathcal{U}_i^c$, there exists a countable subcover $\mathcal{V}_i \subseteq \mathcal{U}_i^c$ such that for each point $x \in Q$, there exists a neighborhood N_x of x that intersects only finitely many sets of $\mathcal{V}_i$. Therefore, $(Q, \varrho_1, \varrho_2, \varrho_3)$ is tri-metacompact.

Example 2. *The tri-topological space* $(\mathbb{R}, \varrho_{dis}, \varrho_{dis}, \varrho_{dis})$ *is tri-metalindelöf since* $\varrho_{dis(i)}$ *forms a tri-open cover*

$$V = \{\{x\} \mid x \in \mathbb{R}\}$$

of $\mathbb{R}$*. It is also tripartite countably metacompact. Clearly,* $(\mathbb{R}, \varrho_{dis}, \varrho_{dis}, \varrho_{dis})$ *is a tri-metalindelöf space.*

Theorem 14. *Every tripartite countably metacompact topological space* $(Q, \varrho_1, \varrho_2, \varrho_3)$ *is tri-compact.*

Proof. Let $(Q, \varrho_1, \varrho_2, \varrho_3)$ be a tripartite countably metacompact topological space. By the definition of countable metacompactness, every countable open cover of Q has a finite subcover.

We are required to show that Q is tri-compact, which means that every open cover of Q has a finite subcover with respect to the topology $\varrho_1, \varrho_2, \varrho_3$.

Since Q is countably metacompact, for any countable collection of open sets $\{U_i\}_{i\in\mathbb{N}}$ that cover Q, there exists a finite subcollection $\{U_{i_1}, U_{i_2}, \ldots, U_{i_k}\}$ such that their union covers Q.

For each of the topologies $\varrho_1, \varrho_2, \varrho_3$, we have the same result since the topologies are compatible with the metacompactness condition. Therefore, we can find finite subcovers for each of the topologies, implying that Q is tri-compact.

Hence, the space $(Q, \varrho_1, \varrho_2, \varrho_3)$ is tri-compact.

Theorem 15. *The product of a tri-compact space Q and a tri-metalindelöf space Y is tri-metalindelöf, where $(Q, \varrho_1, \varrho_2, \varrho_3)$ and $(Y, \sigma_1, \sigma_2, \sigma_3)$ are tri-topological spaces.*

Proof. Let $f : Q \times Y \to Y$ be the tri-projection function, defined by

$$f(q, s) = s,$$

such that $(q, s) \in Q \times Y$. Then, $f : Q \times Y \to Y$ is a tri-perfect function. Since Y is tri-metalindelöf, it follows that $Q \times Y$ is also tri-metalindelöf.

Theorem 16. *Let $f : (Q, \varrho_1, \varrho_2, \varrho_3) \to (Y, \sigma_1, \sigma_2, \sigma_3)$ be a continuous, tri-closed, and onto function. Then, if Q is tri-metalindelöf, then Y is also tri-metalindelöf.*

Proof. Let $\tilde{A} = \{U_\alpha \mid \alpha \in \Delta\} \cup \{V_\beta \mid \beta \in \Gamma\}$ be any tri-open cover of Y, where $\{U_\alpha \mid \alpha \in \Delta\}$ consists of tripartite σ-open members of $\tilde{A}$. Since f is a continuous and onto function, the set

$$\tilde{U} = \{f^{-1}(U_\alpha) \mid \alpha \in \Delta\} \cup \{f^{-1}(V_\beta) \mid \beta \in \Gamma\}$$

is a tri-open cover of Q. Given that Q is tri-metalindelöf, there exists a point-countable tri-open parallel refinement of $\tilde{U}$, denoted by

$$\tilde{U}^* = \{f^{-1}(U^*_\alpha) \mid \alpha \in \Delta\} \cup \{f^{-1}(V^*_\beta) \mid \beta \in \Gamma\}.$$

Thus,

$$\tilde{A}^* = \{U^*_\alpha \mid \alpha \in \Delta\} \cup \{V^*_\beta \mid \beta \in \Gamma\}$$

is a point-countable tri-open parallel refinement of $\tilde{A}$. Therefore, Y is tri-metalindelöf.

Lemma 6. *Let $f : (Q, \varrho_1, \varrho_2, \varrho_3) \to (Y, \sigma_1, \sigma_2, \sigma_3)$ be a continuous and onto function. If $\tilde{A} = \{A_\alpha \mid \alpha \in \Delta\}$ is a point-countable family of subsets of Q, then $\{f(A_\alpha) \mid \alpha \in \Delta\}$ is a point-countable family of subsets of Y.*

Proof. Let $f : (Q, \varrho_1, \varrho_2, \varrho_3) \to (Y, \sigma_1, \sigma_2, \sigma_3)$ be a continuous and onto function. Suppose $\tilde{A} = \{A_\alpha \mid \alpha \in \Delta\}$ is a point-countable family of subsets of Q, i.e., for each point $x \in Q$, the set $\{\alpha \in \Delta \mid x \in A_\alpha\}$ is countable. We aim to prove that $\{f(A_\alpha) \mid \alpha \in \Delta\}$ is a point-countable family of subsets of Y.

Consider any point $y \in Y$. Since f is onto, there exists some $x \in Q$ such that $f(x) = y$. Since $\tilde{A} = \{A_\alpha \mid \alpha \in \Delta\}$ is a point-countable family of subsets of Q, the set $\{\alpha \in \Delta \mid x \in A_\alpha\}$ is countable. Thus, the set $\{\alpha \in \Delta \mid f(x) \in f(A_\alpha)\}$ is also countable, since for any $\alpha \in \Delta$, we have $f(x) \in f(A_\alpha)$ if and only if $x \in A_\alpha$. Therefore, $\{f(A_\alpha) \mid \alpha \in \Delta\}$ is a point-countable family of subsets of Y.

Q.E.D.

Definition 38. *A space $(Q, \varrho_1, \varrho_2, \varrho_3)$ is called a tri-metalindelöf space if every tri-open cover of Q has a tri-open locally countable refinement.*

Definition 39. *A subset ψ of a space $(Q, \varrho_1, \varrho_2, \varrho_3)$ is said to be tri-paralindelöf relative to Q if every tri-open cover of ψ by members of ϱ has a tripartite locally countable parallel refinement in Q by members of ϱ.*

Corollary 3. *Every tri-paralindelöf space is tri-metalindelöf.*

Proof. Clearly by above lemma

Theorem 17. *Every tri-closed subspace of a tri-metalindelöf space $(Q, \varrho_1, \varrho_2, \varrho_3)$ is tri-metalindelöf.*

Proof. Let X be a tri-closed subspace of the tri-metalindelöf space $(Q, \varrho_1, \varrho_2, \varrho_3)$. To show that X is tri-metalindelöf, we need to prove that for every open cover $\mathcal{U} = \{U_\alpha\}$ of X, there exists a countable subcover $\mathcal{V} = \{V_\beta\}$ such that $X \subseteq \bigcup_\beta V_\beta$, and each V_β is a tri-open set.

Since X is a subspace of Q, the open cover $\mathcal{U}$ of X is also an open cover of Q. Since $(Q, \varrho_1, \varrho_2, \varrho_3)$ is tri-metalindelöf, there exists a countable subcover $\mathcal{W} = \{W_\gamma\}$ of Q, where each W_γ is a tri-open set, such that $Q = \bigcup_\gamma W_\gamma$.

Now, because X is tri-closed in Q, each set $W_\gamma \cap X$ is tri-open in X. Therefore, the collection $\mathcal{V} = \{W_\gamma \cap X\}$ forms a countable subcover of X and each $W_\gamma \cap X$ is tri-open in X.

Hence, X is tri-metalindelöf.

Theorem 18. *Every tri-metalindelöf subset of a tri-T_2 locally indiscrete space $(Q, \varrho_1, \varrho_2, \varrho_3)$ is tri-closed.*

Proof. Let $f : (Q, \varrho_1, \varrho_2, \varrho_3) \to (Y, \sigma_1, \sigma_2, \sigma_3)$ be a bijective and continuous map. Suppose that $(Y, \sigma_1, \sigma_2, \sigma_3)$ is a tri-T_2 and tripartite locally indiscrete space, and that $(Q, \varrho_1, \varrho_2, \varrho_3)$ is tri-metalindelöf. Then, f is a tri-homomorphism. It suffices to show that f is tri-closed. Let A be a proper tri-closed subset of Q. Since A is a tri-metalindelöf subset of Q, and $f : (Q, \varrho_1, \varrho_2, \varrho_3) \to (Y, \sigma_1, \sigma_2, \sigma_3)$ is continuous, it follows that $f(A)$

is a tri-metalindelöf subset of Y. Thus, $f(A)$ is a tri-closed subset of Y. Therefore, $f : (Q, \varrho_1, \varrho_2, \varrho_3) \to (Y, \sigma_1, \sigma_2, \sigma_3)$ is a tri-closed function. Hence, the result follows.

Corollary 4. *Let $f : (Q, \varrho_1, \varrho_2, \varrho_3) \to (Y, \sigma_1, \sigma_2, \sigma_3)$ be a tripartite bijective continuous map. If $(Y, \sigma_1, \sigma_2, \sigma_3)$ is a tri-T_2 and tripartite locally indiscrete space, and $(Q, \varrho_1, \varrho_2, \varrho_3)$ is tri-paralindelöf, then f is a tri-homomorphism.*

Proof. Clearly by above theorem

5. Conclusion and Future Directions

In this study, we have introduced and explored the concepts of Lindelöfness, metalindelöfness, and nearly Lindelöfness in the setting of tri-topological spaces. By extending classical topological properties to this generalized framework, we established several theoretical results and analyzed their behavior. Various separation axioms were examined, revealing intricate relationships between tri-compactness, tri-Lindelöfness, and tri-metalindelöfness. The findings contribute to the broader understanding of general topology, offering new insights into the structure and properties of tri-topological spaces.

Future research can extend these findings in several directions. One promising avenue is the investigation of finer separation axioms such as tri-regularity and tri-normality, as well as their implications in different tri-topological settings. Another important direction is the extension of these concepts to fuzzy tri-topological spaces, which could provide useful applications in both theoretical and applied mathematics. Additionally, further exploration of the behavior of tri-topological spaces under various types of mappings, including continuous, perfect, and quotient functions, would enhance our understanding of their structural properties.

Moreover, a deeper study of the interplay between tri-nearly compactness and tri-nearly Lindelöfness in different classes of spaces could lead to new theoretical developments. Exploring the potential applications of these topological structures in mathematical analysis, functional analysis, and related disciplines may also yield significant contributions. Overall, the results obtained in this paper lay a strong foundation for further research into tri-topological spaces, paving the way for new discoveries and applications in topology and beyond.

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