EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 1, Article Number 5586
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

New Improvements to Heron and Heinz Inequality Using Matrix Techniques

M.H.M Rashid[1,*], Wael Mahmoud Mohammad Salameh[2]

[1] *Department of Mathematics, Faculty of Science P.O.Box(7), Mutah University, Al-Karak, Jordan*
[2] *Faculty of Information Technology, Abu Dhabi University, Abu Dhabi 59911, United Arab Emirates*

Abstract. This paper presents a comprehensive study on matrix means interpolation and comparison, extending the parameter ϑ from the traditional closed interval $[0,1]$ to encompass the entire positive real line, denoted as $\mathbb{R}^+$. The research delves into further results involving Heinz means, proposing novel scalar adaptations of Heinz inequalities that integrate Kantorovich's constant. Additionally, the operator version of these inequalities is strengthened. A key contribution of this work is the development of refined Young's type inequalities tailored for the traces, determinants, and norms of positive semi-definite matrices. These refinements offer deeper insights into matrix analysis, especially in the context of operator theory and inequality theory. Through these advancements, the paper enhances the mathematical framework for studying matrix means and their associated inequalities, providing useful tools for both theoretical exploration and practical applications in linear algebra and related fields.

2020 Mathematics Subject Classifications: 26D07, 26D15, 15A18, 47A63

Key Words and Phrases: Heinz mean inequalities, positive semi-definite matrices, Hilbert-Schmidt norm, Young inequality

---

1. Introduction

Consider the algebra of complex matrices of size $n \times n$, denoted as $M_n(\mathbb{C})$. A matrix T in $M_n(\mathbb{C})$ is considered positive semi-definite, written as $T \geq 0$, if it is Hermitian and satisfies $\langle Tx, x\rangle \geq 0$ for all vectors x in $\mathbb{C}^n$. If, for a Hermitian matrix T in $M_n(\mathbb{C})$, $\langle Tx, x\rangle > 0$ holds for all nonzero vectors x in $\mathbb{C}^n$, it is termed a positive definite matrix, denoted as $T > 0$. The set of all positive matrices is denoted as $M_n^+(\mathbb{C})$, and the subset of definite matrices within $M_n^+(\mathbb{C})$ is represented as $M_n^{++}(\mathbb{C})$. The Schur product of two matrices $T = [t_{ij}]_{i,j}$ and $S = [s_{ij}]_{i,j}$ in $M_n(\mathbb{C})$ is defined as the matrix $T \circ S$ with entries

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i1.5586

Email addresses: malik_okasha@yahoo.com (M.H.M Rashid), wael.salameh1@adu.ac.ae (W.M.M. Salameh)

$t_{ij}s_{ij}$. A norm $|||.|||$ on the set of complex matrices of size $n \times n$, denoted as $M_n(\mathbb{C})$, is termed unitarily invariant if $|||UAV||| = |||T|||$ for any matrix T in $M_n(\mathbb{C})$ and for all unitary matrices U and V in $M_n(\mathbb{C})$.

For a matrix $T = [t_{ij}] \in M_n(\mathbb{C})$, the Hilbert-Schmidt norm (also known as the Frobenius norm) and the trace norm of T are defined as follows

$$\|T\|_2 = \left(\sum_{j=1}^{n} s_j^2(T) \right)^{\frac{1}{2}}, \qquad \|T\|_1 = tr(|T|) = \sum_{j=1}^{n} s_j(T) \tag{1}$$

Here, $s_1(T) \geq s_2(T) \geq \cdots \geq s_n(T) \geq 0$ represent the singular values of T, which are the eigenvalues of the positive matrix $|T| = \sqrt{T^*T}$ arranged in decreasing order and repeated according to multiplicity. The symbol $tr(.)$ denotes the usual trace operation.

It's important to note that the mathematical norms $\|\cdot\|_2$ and $\|\cdot\|_1$ are widely recognized for being unitarily invariant.

The classic Young's inequality for non-negative real numbers states that if $\rho, \sigma \geq 0$ and $0 \leq \kappa \leq 1$, then

$$\rho^\kappa \sigma^{1-\kappa} \leq \kappa\rho + (1-\kappa)\sigma \tag{2}$$

Equality occurs if and only if $\rho = \sigma$. When κ is $\frac{1}{2}$, substituting into the inequality yields the arithmetic-geometric mean inequality

$$\sqrt{\rho\sigma} \leq \frac{\rho + \sigma}{2}. \tag{3}$$

Manasarah and Kittaneh, as presented in [10], improved Young's inequality with the following refinement

$$\left(\rho^\kappa \sigma^{1-\kappa}\right)^m + r_0^m \left(\rho^{\frac{m}{2}} - \sigma^{\frac{m}{2}}\right)^2 \leq \left(\kappa\rho^r + (1-\kappa)\sigma^r\right)^{\frac{m}{r}}, \quad r \geq 1 \tag{4}$$

where $m \in \mathbb{N}$ and $r_0 = \min\{\kappa, 1-\kappa\}$.

The Kantorovich constant, denoted as $K(t,2)$, is defined as $\frac{(t+1)^2}{4t}$. It possesses several key properties: $K(1,2) = 1$, $K(t,2) = K\left(\frac{1}{t}, 2\right) \geq 1$ $(t > 0)$ and $K(t,2)$ is monotone increasing on $[1,\infty)$, and monotone decreasing on $(0,1]$. For more detailed information about the Kantorovich constant, interested readers can refer to [11, 15, 17, 22]..
The following multiplicative refinement and reversal of Young's inequality, expressed in terms of Kantorovich's constant, can be stated as follows

$$K(h,2)^r \rho\sharp_\kappa\sigma \leq \rho\nabla_\kappa\sigma \leq K(h,2)^R \rho\sharp_\kappa\sigma, \tag{5}$$

where ρ and σ are both greater than 0, κ belongs to the interval $[0,1]$, r is the minimum of κ and $1-\kappa$, R is the maximum of κ and $1-\kappa$, and h is defined as $\frac{\sigma}{\rho}$.

The second inequality in (5) is credited to Liao et al. [12], while the first one is attributed to Zou et al. [11]. In [19], the authors obtained another improvement of the Young inequality and its reverse as follows:

$$r(\sqrt{\rho} - \sqrt{\sigma})^2 + K(\sqrt{h}, 2)^{r'} \rho\sharp_\kappa\sigma \leq \rho\nabla_\kappa\sigma, \tag{6}$$

and

$$\rho\nabla_\kappa\sigma \leq K(\sqrt{h},2)^{-r'}\rho\sharp_\kappa\sigma + R(\sqrt{\rho}-\sqrt{\sigma})^2 \tag{7}$$

where $h = \frac{\sigma}{\rho}$, $r = \min\{\kappa, 1-\kappa\}$, $R = \max\{\kappa, 1-\kappa\}$ and $r' = \min\{2r, 1-2r\}$. In addition, another kind of the reversal of Young inequality utilizing Kantorovich's constant is described in [12] with the same notation as above.

$$\rho\nabla_\kappa\sigma - R(\sqrt{\rho}-\sqrt{\sigma})^2 \leq K(\sqrt{h},2)^{R'}\rho\sharp_\kappa\sigma, \tag{8}$$

where $R' = \max\{2r, 1-2r\}$.
For κ in the range of $[0,1]$ and two non-negative real numbers ρ and σ, the Heinz mean serves as an interpolation between the κ-arithmetic mean and the κ-geometric mean. These are defined by the expression

$$H_\kappa(\rho,\sigma) = \frac{\rho\sharp_\kappa\sigma + \rho\sharp_{1-\kappa}\sigma}{2}, \tag{9}$$

where $\rho\sharp_\kappa\sigma = \rho^\kappa\sigma^{1-\kappa}$ represents the κ-geometric mean.

The Heinz mean possesses certain properties, including convexity concerning κ within the interval $[0,1]$. Its minimum occurs at $\kappa = \frac{1}{2}$, and its maximum values are found at $\kappa = 0$ and $\kappa = 1$. Additionally, the following inequalities are true

$$\sqrt{\rho\sigma} \leq H_\kappa(\rho,\sigma) \leq \frac{\rho+\sigma}{2}. \tag{10}$$

It is worth noting that the function $H_\kappa(\rho,\sigma)$ exhibits symmetry with respect to the point $\kappa = \frac{1}{2}$, meaning that $H_\kappa(\rho,\sigma) = H_{1-\kappa}(\rho,\sigma)$.
The Heron mean is defined by the expression

$$F_\vartheta(\rho,\sigma) = (1-\vartheta)\sqrt{\rho\sigma} + \vartheta\left(\frac{\rho+\sigma}{2}\right), \ \vartheta \in [0,1] \text{ and } \rho,\sigma \in \mathbb{R}^+. \tag{11}$$

where ϑ takes values in the interval $[0,1]$, and ρ and σ are positive real numbers.

Evidently, the Heron mean serves as a linear interpolation between the arithmetic and geometric means. It adheres to the inequality $F_\vartheta \leq F_\varrho$ whenever $\vartheta \leq \varrho$, with both ϑ and ϱ belonging to the positive real numbers.

In a study by Bhatia published in [2], it was demonstrated that for $\vartheta(\kappa) = (2\kappa-1)^2$ and κ within the range of $[0,1]$, the following relation holds

$$H_\kappa(\rho,\sigma) \leq F_{\vartheta(\kappa)}(\rho,\sigma). \tag{12}$$

Our paper is structured as follows: In the upcoming section, we will conduct an in-depth investigation into matrix interpolation and mean comparisons. This analysis extends the scope of ϑ beyond the closed interval $[0,1]$ to include all positive real numbers, represented as $\mathbb{R}^+$. Additionally, we will explore additional findings pertaining to Heinz means. Section three is dedicated to exploring refinements in Heinz inequality, incorporating the Kantorovich constant. Section 4 focuses on examining enhanced variations of Heinz-type operator inequalities and their corresponding reversals. Finally, in section 5, we present refined inequalities of Young's type, specifically designed for traces, determinants, and norms of positive semi-definite matrices.

2. Full Interpolation of Matrix Variants of Heron and Heinz MEANS

In the paper referenced as [2], R. Bhatia established a noteworthy result. In particular, it was shown that for values of ϑ in the interval $[0, 1/2]$, the function $\psi(\vartheta)$ adheres to the inequality $\psi(\vartheta) \leq \psi(1/2)$. Here, $\psi(\vartheta)$ denotes one of the potential matrix formulations of Equation (11), and its definition is as follows

$$\psi(\vartheta) = \left|\left|\left|(1-\vartheta)T^{1/2}XS^{1/2} + \vartheta\left(\frac{TX+XS}{2}\right)\right|\right|\right|, \tag{13}$$

This definition involves matrices T, S, and X, subject to the conditions that T and S belong to the set of positive definite matrices in $\mathbb{C}$, denoted as $M_n^{++}(\mathbb{C})$, and X is a member of the set of $n \times n$ matrices over $\mathbb{C}$, denoted as $M_n(\mathbb{C})$.

For further insights into the matrix formulations of Equation (9) and Equation (12), as well as additional details, interested readers are encouraged to refer to the following references: [2], [5], [6], [4], and [8].

Within the context of this article, the author endeavors to demonstrate that for ϑ values within the interval $[0, 1/2]$, the function $\psi(\vartheta, \kappa)$ adheres to the inequality $\psi(\vartheta, \kappa) \leq \psi(1/2, \kappa)$. Additionally, it is asserted that $\psi(\vartheta, \kappa)$ displays an increasing trend as ϑ varies within the range $[1/2, \infty)$.

This result serves as a generalization of the previously established monotonic property associated with the matrix version of Equation (11). This generalization mirrors the behavior of $F_\vartheta(\rho, \sigma)$ for positive real numbers a and b when ϑ belongs to the set of positive real numbers, $\mathbb{R}^+$.

As a consequential outcome of these findings, the author will introduce a potential generalized matrix equivalent of Equation (12), which can be expressed as follows

$$\frac{1}{2}|||T^\mu XS^{1-\mu} + T^{1-\mu}XS^\mu||| \leq \left|\left|\left|(1-\vartheta)T^\kappa XS^{1-\kappa} + \vartheta\left(\frac{TX+XS}{2}\right)\right|\right|\right| \tag{14}$$

This inequality is valid for particular values of $\mu \in [1/4, 3/4]$, $\kappa \in [0, 1]$, and $\vartheta \in [1/2, \infty)$.

In this section, we will undertake a thorough investigation of matrix interpolation and mean comparisons. This scrutiny broadens the range of ϑ from the closed interval $[0, 1]$ to encompass the entirety of positive real numbers, denoted as $\mathbb{R}^+$. Furthermore, we will delve into additional findings associated with Heinz means.

Theorem 1. *[6] Let $T, S \in M_n(\mathbb{C})$ such that T is a positive semi-definite. Then*

$$|||T \circ S||| \leq \max_{1 \leq i \leq n} t_{ii} |||S|||,$$

where t_{ii} for $i = 1, 2, \cdots, n$ are the diagonal entries of matrix T.

Lemma 1. *[21] Let $\kappa_1, \kappa_2, \cdots, \kappa_n$ be positive numbers, $r \in [-1, 1]$, and $t \in (-2, 2]$. Then the $n \times n$ matrix matrix*

$$\Gamma = \left(\frac{\kappa_i^r + \kappa_j^r}{\kappa_i^2 + t\kappa_i\kappa_j + \kappa_j^2}\right)$$

is positive semi-definite.

Theorem 2. *Let $T, S, X \in M_n(\mathbb{C})$ such that T and S are positive semi-definite, $\kappa \in [0,1]$ and $|||.|||$ any unitarily invariant norm, the function*

$$\psi(\vartheta,\kappa) = \left|\left|\left|(1-\vartheta)T^{\kappa}XS^{1-\kappa} + \vartheta\left(\frac{TX+XS}{2}\right)\right|\right|\right|$$

is increasing for $\frac{1}{2} \leq \vartheta < \infty$ and $\psi(\vartheta,\kappa) \leq \psi\left(\frac{1}{2},\kappa\right)$ for all $\vartheta \in \left[0,\frac{1}{2}\right]$.

Proof. We first prove the result for $\vartheta > 0$ and $T = S$, that is,

$$\left|\left|\left|(1-\vartheta)T^{\kappa}XS^{1-\kappa} + \vartheta\left(\frac{TX+XS}{2}\right)\right|\right|\right| = \frac{\vartheta}{2}Z(\vartheta),$$

where $Z(\vartheta) = |||Q(\vartheta)T^{\kappa}XT^{1-\kappa} + TX + XT|||$ and $Q(\vartheta) = 2\left(\frac{1}{\vartheta} - 1\right)$. We may assume without loss of generality, $T = \operatorname{diag}(\eta_1, \cdots, \eta_n)$, $\eta_j > 0$. Then

$$\begin{aligned}
Q(\vartheta)T^{\kappa}XT^{1-\kappa} + TX + XT &= \left(\left(Q(\vartheta)\eta_i^{\kappa}\eta_j^{1-\kappa} + \eta_i + \eta_j\right)x_{ij}\right)_{i,j} \\
&= \left(\frac{Q(\vartheta)\eta_i^{\kappa}\eta_j^{1-\kappa} + \eta_i + \eta_j}{Q(\varrho)\eta_i^{\kappa}\eta_j^{1-\kappa} + \eta_i + \eta_j}\right)_{i,j} \circ \left(Q(\varrho)T^{\kappa}XT^{1-\kappa} + TX + XT\right) \\
&= E \circ \left(Q(\varrho)T^{\kappa}XT^{1-\kappa} + TX + XT\right),
\end{aligned}$$

where $E = \left(\frac{Q(\vartheta)\eta_i^{\kappa}\eta_j^{1-\kappa} + \eta_i + \eta_j}{Q(\varrho)\eta_i^{\kappa}\eta_j^{1-\kappa} + \eta_i + \eta_j}\right)_{i,j}$. Now the matrix E can be written as

$$\left(1 + \frac{(Q(\vartheta) - Q(\varrho))\eta_i^{\kappa}\eta_j^{1-\kappa}}{Q(\varrho)\eta_i^{\kappa}\eta_j^{1-\kappa} + \eta_i + \eta_j}\right) = (1)_{i,j} + \left(\eta_i^{\kappa}\left(\frac{Q(\vartheta) - Q(\varrho)}{Q(\varrho)\eta_i^{\kappa}\eta_j^{1-\kappa} + \eta_i + \eta_j}\right)\eta_j^{1-\kappa}\right)$$

which will be positive semidefinite if the matrix,

$$G = \left(\frac{Q(\vartheta) - Q(\varrho)}{Q(\varrho)\eta_i^{\kappa}\eta_j^{1-\kappa} + \eta_i + \eta_j}\right)_{i,j}$$

is positive semidefinite. According to Lemma 1, the latter matrix is positive semidefinite if and only if $Q(\vartheta) \geq Q(\varrho)$ and $Q(\varrho) \in [-2,2]$. Since $Q(\vartheta) = 2\left(\frac{1}{\vartheta} - 1\right)$ is a continuous and decreasing function on the positive half-line, ranging from $[\frac{1}{2}, \infty)$ into $[-2,2]$, it follows that $Q(\vartheta) \geq Q(\varrho)$ for all $\varrho \geq \vartheta$. Consequently, using Theorem 1, we can deduce that $Z(\vartheta) \leq \left(\frac{Q(\vartheta)+2}{Q(\varrho)+2}\right)T(\varrho)$. Thus, the result holds for $T = S$ and $\vartheta \geq \frac{1}{2}$.

For $\vartheta \in (0, 1/2]$, we have $2 \leq Q(\vartheta) < \infty$, and $Q(\vartheta) > Q\left(\frac{1}{2}\right) = 2$. Therefore, the matrix E with $\varrho = \frac{1}{2}$ is positive semidefinite, as per Lemma 1. The case $\vartheta = 0$ is straightforward since, by Lemma 1, the matrix

$$\left(\frac{\eta_i^{\kappa}\eta_j^{1-\kappa}}{\eta_i^{\kappa}\eta_j^{1-\kappa} + \eta_i + \eta_j}\right)_{i,j} = \left(\eta_i^{\kappa}\left(\frac{1}{\eta_i^{\kappa}\eta_j^{1-\kappa} + \eta_i + \eta_j}\right)\eta_j^{1-\kappa}\right)_{i,j}$$

is positive semidefinite. Thus, we have established the desired result for this case, i.e., $\vartheta Z(\vartheta) \leq \frac{1}{2} Z\left(\frac{1}{2}\right)$. In other words, $\psi(\vartheta, \kappa) \leq \psi\left(\frac{1}{2}, \kappa\right)$ for all $\vartheta \in [0, 1/2]$. The general case can be derived by substituting T with $\begin{pmatrix} T & 0 \\ 0 & S \end{pmatrix}$ and X by $\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}$.

Remark 1. *By setting κ to be equal to half (i.e., $\kappa = \frac{1}{2}$) in Theorem 2, we can deduce that we arrive at Theorem 2.3 as presented in [8]. Consequently, our findings represent an enhancement of the results established in that theorem.*

As a consequence of Theorem 2, we have

Corollary 1. *Let $T, S, X \in M_n(\mathbb{C})$ with T, S positive definite. Then for any unitarily invariant norm $|||\cdot|||$ and a matrix monotone increasing function $\psi : (0, \infty) \longrightarrow (0, \infty)$ with $\psi^*(x) = x(\psi(x))^{-1}$,*

$$\begin{aligned} &\frac{1}{2} \left|\left|\left| T^{\frac{\mu}{2}} \left(\psi(T^\mu) X \psi^*(S^\mu) + \psi^*(T^\mu) X \psi(S^\mu) \right) S^{\frac{\mu}{2}} \right|\right|\right| \\ &\leq \left|\left|\left| (1-\vartheta) T^\kappa X S^{1-\kappa} + \vartheta \left(\frac{TX + XS}{2} \right) \right|\right|\right|. \end{aligned}$$

Corollary 2. *Let $T, S, X \in M_n(\mathbb{C})$ with T, S positive definite. Then for any unitarily invariant norm $|||\cdot|||$, $\frac{1}{4} \leq \mu \leq \frac{3}{4}$, $\kappa \in [0, 1]$ and $\vartheta \in [1/2, \infty)$,*

$$\frac{1}{2} |||T^\mu X S^{1-\mu} + T^{1-\mu} X S^\mu||| \leq \left|\left|\left| (1-\vartheta) T^\kappa X S^{1-\kappa} + \vartheta \left(\frac{TX + XS}{2} \right) \right|\right|\right|.$$

Proof. Letting $\psi(x) = \sqrt{x}$ in Corollary 1, we derived the result.

The following result is a consequence of Theorem 2.

Corollary 3. *Let $T, S, X \in M_n(\mathbb{C})$ with T, S positive definite, $\eta = \min\{sp(T), sp(S)\}$, $\mu \in [1/4, 3/4]$ and $\kappa \in [0, 1]$. Then for any unitarily invariant norm $|||\cdot|||$ and a matrix monotone increasing function $\psi : (0, \infty) \longrightarrow (0, \infty)$*

$$\begin{aligned} &\frac{\eta}{2f(\eta)} \left|\left|\left| T^{\frac{\mu}{2}} \left(\psi(T^\mu) X + X \psi(S^\mu) \right) S^{\frac{\mu}{2}} \right|\right|\right| \\ &\leq \left|\left|\left| (1-\vartheta) T^\kappa X S^{1-\kappa} + \vartheta \left(\frac{TX + XS}{2} \right) \right|\right|\right| \end{aligned}$$

holds for every $\vartheta \in [1/2, \infty)$.

Choosing $\psi(x) = \log(1 + x)$ in Corollary 3, we have

Corollary 4. *Let $T, S, X \in M_n(\mathbb{C})$ with T, S positive definite, $\eta = \min\{sp(T), sp(S)\}$, $\mu \in [1/4, 3/4]$ and $\kappa \in [0, 1]$. Then for any unitarily invariant norm $|||\cdot|||$*

$$\begin{aligned} &\frac{\eta}{2\log(1+\eta)} \left|\left|\left| T^{\frac{\mu}{2}} \left(\log(1 + T^\mu) X + X \log(1 + S^\mu) \right) S^{\frac{\mu}{2}} \right|\right|\right| \\ &\leq \left|\left|\left| (1-\vartheta) T^\kappa X S^{1-\kappa} + \vartheta \left(\frac{TX + XS}{2} \right) \right|\right|\right| \end{aligned}$$

holds for every $\vartheta \in [1/2, \infty)$.

Theorem 3. *Consider $T, S, X \in M_n(\mathbb{C})$ with T and S being positive definite, $\kappa \in [0,1]$, and $|||.|||$ denoting any unitarily invariant norm. The function can be expressed as:*

$$\phi(\vartheta,\kappa) = \left|\left|\left| \left(1-\frac{\vartheta}{2}\right)\left(T^\kappa X S^{1-\kappa} + T^{1-\kappa} X S^\kappa\right) + \vartheta\left(\frac{TX+XS}{2}\right) \right|\right|\right|$$

is increasing for $\frac{1}{2} \leq \vartheta < \infty$ and $\phi(\vartheta,\kappa) \leq \phi\left(\frac{1}{2},\kappa\right)$ for all $\vartheta \in \left[0,\frac{1}{2}\right]$.

Proof. Once again following the same lines of the proof of Theorem (2), we shall prove the result for $\vartheta > 0$, $T = S$ and $T = diag(\eta_1, \cdots, \eta_n)$. Suppose

$$\phi(\vartheta,\kappa) = \left|\left|\left| \left(1-\frac{\vartheta}{2}\right)\left(T^\kappa X T^{1-\kappa} + T^{1-\kappa} X T^\kappa\right) + \vartheta\left(\frac{TX+XT}{2}\right) \right|\right|\right| = \frac{\vartheta}{2} Z(\vartheta,\kappa),$$

where $Z(\vartheta,\kappa) = |||W_1(\vartheta)\left(T^\kappa X T^{1-\kappa} + T^{1-\kappa} X T^\kappa\right) + (TX+XT)|||$ and $W_1(\vartheta) = \frac{2}{\vartheta} - 1$.

$$\begin{aligned}
&W_1(\vartheta)\left(T^\kappa X T^{1-\kappa} + T^{1-\kappa} X T^\kappa\right) + (TX+XT) \\
&= \left[\left(W_1(\vartheta)\left(\eta_i^\mu \eta_j^{1-\mu} + \eta_i^{1-\mu}\eta_j^\mu\right) + \eta_i + \eta_j\right) x_{ij}\right]_{i,j} \\
&= Y \circ \left(W_1(\varrho)\left(T^\kappa X T^{1-\kappa} + T^{1-\kappa} X T^\kappa\right) + TX + XS\right).
\end{aligned}$$

Now the matrix Y can be written as

$$\left(\frac{W_1(\vartheta)\left(\eta_i^\mu \eta_j^{1-\mu} + \eta_i^{1-\mu}\eta_j^\mu\right) + \eta_i + \eta_j}{W_1(\varrho)\left(\eta_i^\mu \eta_j^{1-\mu} + \eta_i^{1-\mu}\eta_j^\mu\right) + \eta_i + \eta_j}\right)_{i,j} = \left(1 + \frac{(W_1(\vartheta) - W_1(\varrho))\eta_i^\kappa \eta_j^\kappa}{(W_1(\varrho)-1)\eta_i^\kappa\eta_j^\kappa + \eta_i^{1-\kappa} + \eta_j^{1-\kappa}}\right)_{i,j}$$

$$= (1)_{i,j} + \left(\eta_i^\kappa\left(\frac{W_1(\vartheta) - W_1(\varrho)}{(W_1(\varrho)-1)\eta_i^\kappa\eta_j^\kappa + \eta_i^{1-\kappa} + \eta_j^{1-\kappa}}\right)\eta_j^\mu\right)_{i,j}$$

Once again, considering Lemma (1), we observe that the latter matrix is positive semidefinite if and only if $W_1(\vartheta) > W_1(\varrho)$ and $2 > W_1(\varrho) - 1 > -2$. Since $W(\vartheta) = W_1(\vartheta) - 1 = (2\vartheta^{-1} - 2)$, it is a continuously decreasing function in the positive half-line and maps to the interval $(2, 2]$ for ϑ in the range $[1/2, \infty)$. Therefore, as demonstrated in Theorem (2), we can deduce that $W(\vartheta) > W(\varrho)$ and consequently, $W_1(\vartheta) > W_1(\varrho)$ for all $\vartheta \leq \varrho$.

Applying Theorem (1), we establish that $T(\vartheta,\kappa) \leq \frac{W_1(\vartheta)+1}{W_1(\varrho)+1} T(\varrho,\kappa)$. This verifies the result for the case when $T = S$ and $\vartheta \in [1/2, \infty)$.

For $\vartheta \in (0, 1/2]$, we can observe that $3 = W_1(1/2) \leq W_1(\vartheta) < \infty$, and according to Lemma (1), the matrix Y with $\varrho = 1/2$ is positive semidefinite. Similarly, the case $\vartheta = 0$ can be established through the positive semidefiniteness of the matrix $\left(\eta_i^\kappa\left(\frac{W_1(\vartheta)-W_1(\varrho)}{(W_1(\varrho)-1)\eta_i^\kappa\eta_j^\kappa + \eta_i^{1-\kappa} + \eta_j^{1-\kappa}}\right)\eta_j^\mu\right)_{i,j}$, which is confirmed by utilizing Lemma (1). This leads us to the desired result for this case, i.e., $\vartheta Z(\vartheta,\kappa) \leq \frac{1}{2} Z(\frac{1}{2},\kappa)$. In other words, $\phi(\vartheta,\kappa) \leq \phi(1/2,\kappa)$ holds for all $\vartheta \in [0, 1/2]$.

The general case can be obtained by substituting T with $\begin{pmatrix} T & 0 \\ 0 & S \end{pmatrix}$ and X by $\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}$.

The following outcome is an implication of Theorems 2, 3, and Corollary 2, resulting in:

Corollary 5. *Let $T, S, X \in M_n(\mathbb{C})$ with T, S positive definite, $\kappa \in [0,1]$ and $\psi(\vartheta, \kappa)$ and $\phi(\vartheta, \kappa)$ are same as taken in Theorem (2) and (3) respectively. Then*

$$\psi(0,\kappa) \leq \frac{1}{2}\phi(0,\kappa) \leq \psi(\vartheta,\kappa) \tag{15}$$

for $\vartheta \in [1/2, \infty)$, or equivalently, for any unitarily invariant norm $|||\cdot|||$ and $2 < t \leq 2$,

$$|||T^{\kappa}XS^{1-\kappa}||| \leq \frac{1}{2}|||T^{\kappa}XS^{1-\kappa} + T^{1-\kappa}XS^{\kappa}||| \leq \frac{1}{t+2}|||TX + XS + tT^{\kappa}XS^{1-\kappa}|||.$$

Remark 2. *(i) It's worth noting that the corollary mentioned earlier (1) represents one of the potential enhancements to an inequality introduced by Kaur and Singh in their work (see [8, Corollary 2.4]).*
(ii) Take note that when we set κ to the value of one-third (i.e., $\kappa = \frac{1}{3}$), it is evident that we arrive at the outcome outlined in Theorem 2.10 in [8]. This implies that our finding constitutes a broader and more generalized version of their result.

3. Sharpening of the Heinz inequalities and its reverses with the Kantorovich Constant

In this section, we make a refinement of Heinz inequality with the Kantorovich constant.

Lemma 2. *Let $\rho, \sigma > 0$ and $0 \leq \nu < \kappa \leq 1$. Then*

$$r(\sqrt{\rho\sharp_{\kappa}\sigma} - \sqrt{\sigma})^2 + K(\sqrt{h}, 2)^{r'}\rho\sharp_{\nu}\sigma \leq \nu\rho + (1-\nu)\sigma - \left(\frac{\nu}{\kappa}\right)(\rho\nabla_{\kappa}\sigma - \rho\sharp_{\kappa}\sigma), \tag{16}$$

where $r = \min\{\frac{\nu}{\kappa}, 1 - \frac{\nu}{\kappa}\}$, $h = \frac{\rho}{\sigma}$ and $r' = \min\{2r, 1-2r\}$.

Proof. An simple argument shows that

$$\begin{aligned}\nu\rho + (1-\nu)\sigma - \frac{\nu}{\kappa}(\rho\nabla_{\kappa}\sigma - \rho\sharp_{\kappa}\sigma) &= \nu\rho + (1-\nu)\sigma - \frac{\nu}{\kappa}\left(\kappa\rho + (1-\kappa)\sigma - \rho^{\kappa}\sigma^{1-\kappa}\right)\\ &= \frac{\nu}{\kappa}\rho^{\kappa}\sigma^{1-\kappa} + \left(1 - \frac{\nu}{\kappa}\right)\sigma = (\rho\sharp_{\kappa}\sigma)\nabla_{\frac{\nu}{\kappa}}\sigma.\end{aligned} \tag{17}$$

By applying the inequality (6) for the relation (17), it follows that

$$r(\sqrt{\rho\sharp_{\kappa}\sigma} - \sqrt{\sigma})^2 + K(\sqrt{h}, 2)^{r'}\rho\sharp_{\nu}\sigma \leq (\rho\sharp_{\kappa}\sigma)\nabla_{\frac{\nu}{\kappa}}\sigma.$$

Hence, the inequality (16) follows.

Lemma 3. *Let $\rho, \sigma > 0$ and $0 \le \nu < \kappa \le 1$. Then*

$$\nu\rho + (1-\nu)\sigma - \left(\frac{\nu}{\kappa}\right)(\rho\nabla_\kappa\sigma - \rho\sharp_\kappa\sigma) \le K(\sqrt{h}, 2)^{-r'}\rho\sharp_\nu\sigma + R(\sqrt{\rho\sharp_\kappa\sigma} - \sqrt{\sigma})^2 \tag{18}$$

where $R = \max\{\frac{\nu}{\kappa}, 1-\frac{\nu}{\kappa}\}$, $h = \frac{\rho}{\sigma}$ and $r' = \min\{2r, 1-2r\}$.

Proof. By applying the inequality (7) for the relation (17), it follows that

$$\nu\rho + (1-\nu)\sigma - \left(\frac{\nu}{\kappa}\right)(\rho\nabla_\kappa\sigma - \rho\sharp_\kappa\sigma) \le K(\sqrt{h}, 2)^{-r'}\rho\sharp_\nu\sigma + R(\sqrt{\rho\sharp_\kappa\sigma} - \sqrt{\sigma})^2.$$

So, we get the inequality (18).

For two non-negative real numbers ρ and σ, we define the Heinz mean in the parameter μ, $0 \le \mu \le 1$, as

$$H_\mu = \frac{\rho^\mu\sigma^{1-\mu} + \rho^{1-\mu}\sigma^\mu}{2}. \tag{19}$$

Note that $H_0(\rho,\sigma) = H_1(\rho,\sigma) = \frac{\rho+\sigma}{2}$ and $H_{\frac{1}{2}}(\rho,\sigma) = \sqrt{\rho\sigma}$. It is easy to see that as a function of μ, $H_\mu(\rho,\sigma)$ is convex, attains its minimum at $\mu = \frac{1}{2}$, and attains its maximum at $\mu = 0$ and $\mu = 1$. Moreover, $H_\mu(\rho,\sigma) = H_{1-\mu}(\rho,\sigma)$ for $0 \le \mu \le 1$. Thus, the Heinz mean interpolates between the geometric mean and the arithmetic mean:

$$\sqrt{\rho\sigma} \le H_\mu(\rho,\sigma) \le \frac{\rho+\sigma}{2} \text{ for } 0 \le \mu \le 1. \tag{20}$$

Theorem 4. *Let $\rho, \sigma > 0$ and $0 \le \nu < \kappa \le 1$. Then*

$$\begin{aligned} &r\left[H_\kappa(\rho,\sigma) + H_0(\rho,\sigma) - H_{\frac{\kappa}{2}}(\rho,\sigma)\right] + K\left[\sqrt{h}, 2\right]^{r'} H_\nu(\rho,\sigma) \\ &\le H_0(\rho,\sigma) - \left(\tfrac{\nu}{\kappa}\right)\left[H_0(\rho,\sigma) - H_\kappa(\rho,\sigma)\right], \end{aligned} \tag{21}$$

where $r = \min\{\frac{\nu}{\kappa}, 1-\frac{\nu}{\kappa}\}$, $h = \frac{\rho}{\sigma}$ and $r' = \min\{2r, 1-2r\}$.

Proof. Interchanging ρ with σ and σ with ρ in inequality (16), we get

$$r(\sqrt{\sigma\sharp_\kappa\rho} - \sqrt{\rho})^2 + K(\sqrt{h}, 2)^{r'}\sigma\sharp_\nu\rho \le \nu\sigma + (1-\nu)\rho - \left(\frac{\nu}{\kappa}\right)(\sigma\nabla_\kappa\rho - \sigma\sharp_\kappa\rho). \tag{22}$$

Adding (16) and (22), we have

$$\begin{aligned} &r\left[(\sqrt{\rho\sharp_\kappa\sigma} - \sqrt{\sigma})^2 + (\sqrt{\sigma\sharp_\kappa\rho} - \sqrt{\rho})^2\right] + K\left[\sqrt{h}, 2\right]^{r'}(2H_\nu(\rho,\sigma)) \\ &\le 2H_0(\rho,\sigma) - \left(\tfrac{\nu}{\kappa}\right)\left[2H_0(\rho,\sigma) - 2H_\kappa(\rho,\sigma)\right] \end{aligned}$$

and so

$$\begin{aligned} &r\left[H_\kappa(\rho,\sigma) + H_0(\rho,\sigma) - H_{\frac{\kappa}{2}}(\rho,\sigma)\right] + K\left[\sqrt{h}, 2\right]^{r'} H_\nu(\rho,\sigma) \\ &\le H_0(\rho,\sigma) - \left(\tfrac{\nu}{\kappa}\right)\left[H_0(\rho,\sigma) - H_\kappa(\rho,\sigma)\right]. \end{aligned}$$

In similar of proof of Theorem 4, we can prove the following result.

Theorem 5. *Let $\rho, \sigma > 0$ and $0 \leq \nu < \kappa \leq 1$. Then*

$$\begin{aligned} &H_0(\rho,\sigma) - \left(\tfrac{\nu}{\kappa}\right)\left[H_0(\rho,\sigma) - H_\kappa(\rho,\sigma)\right] \leq K\left[\sqrt{h}, 2\right]^{-r'} H_\nu(\rho,\sigma) \\ &+R\left[H_\kappa(\rho,\sigma) + H_0(\rho,\sigma) - H_{\frac{\kappa}{2}}(\rho,\sigma)\right], \end{aligned} \tag{23}$$

where $R = \min\{\frac{\nu}{\kappa}, 1 - \frac{\nu}{\kappa}\}$, $h = \frac{\rho}{\sigma}$ and $r' = \min\{2r, 1 - 2r\}$.

4. New operator versions of Heinz-type inequalities

Let $\mathcal{H}$ represent a complex Hilbert space, and $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra comprising all bounded linear operators on $\mathcal{H}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is considered positive if $\langle Tx, x\rangle \geq 0$ holds true for every $x \in \mathcal{H}$. We express this as $T \geq 0$. Now, let T and S be two positive operators in $\mathcal{B}(\mathcal{H})$, and κ take on values in the interval $[0, 1]$. The κ-weighted arithmetic mean of T and S, denoted as $T\nabla_\kappa S$, is defined as:

$$T\nabla_\kappa S = (1-\kappa)T + \kappa S.$$

When T is invertible, the κ-geometric mean of T and S, represented as $T\sharp_\kappa S$, is defined as:

$$T\sharp_\kappa S = T^{\frac{1}{2}}\left(T^{-\frac{1}{2}} S T^{-\frac{1}{2}}\right)^\kappa T^{\frac{1}{2}}.$$

In the case where $\kappa = \frac{1}{2}$, we can simplify the notation to $T\nabla S$ and $T\sharp S$ to refer to the κ-weighted arithmetic mean and the κ-geometric mean, respectively. It is well-known that for positive invertible operators T and S, the following inequality holds:

$$T\sharp_\kappa S \leq T\nabla_\kappa S, \quad \kappa \in [0, 1].$$

Additionally, we define the operator version of the Heinz mean as $H_\kappa(T, S)$:

$$H_\kappa(T,S) = \frac{T\sharp_\kappa S + T\sharp_{1-\kappa} S}{2}$$

for the case where T and S are positive invertible operators and $\kappa \in [0, 1]$.

In this section, we will present improved variants of Heinz-type operator inequalities and their converses, exploiting the monotonicity of operator functions as the foundational concept for the ensuing discussion.

Lemma 4. *[7] Suppose $T \in \mathcal{B}(\mathcal{H})$ is self-adjoint. If f and g are continuous functions such that $f(t) \geq g(t)$ for $t \in sp(T)$ (where $sp(T)$ represents the spectrum of the operator T), then it follows that $f(T) \geq g(T)$.*

Next, we present our main results on the basis of inequality (21). By Lemma 4, we have the following.

Theorem 6. *Let $T, S \in \mathcal{B}(\mathcal{H})$ be positive invertible operators, I is the identity operator and $0 \leq \nu < \kappa \leq 1$. If all positive numbers m, m' and M, M' satisfy either of the conditions $0 < mI \leq T \leq m'I < M'I \leq S \leq MI$ or $0 < mI \leq S \leq m'I \leq T \leq MI$, then:*

$$\begin{aligned} & r\left[H_\kappa(T,S) + H_0(T,S) - H_{\frac{\kappa}{2}}(T,S)\right] + K\left[\sqrt{h}, 2\right]^{r'} H_\nu(T,S) \\ & \leq H_0(T,S) - \left(\tfrac{\nu}{\kappa}\right)\left[H_0(T,S) - H_\kappa(T,S)\right], \end{aligned} \tag{24}$$

where $r = \min\{\frac{\nu}{\kappa}, 1 - \frac{\nu}{\kappa}\}$, $h = \frac{M}{m}$ and $r' = \min\{2r, 1-2r\}$.

Proof. Assuming that $0 \leq \nu < \kappa \leq 1$, according to inequality (21), for any positive value of x, we can conclude:

$$\begin{aligned} & r\left[H_\kappa(1,x) + H_0(1,x) - H_{\frac{\kappa}{2}}(1,x)\right] + K\left[\sqrt{h}, 2\right]^{r'} H_\nu(1,x) \\ & \leq H_0(1,x) - \left(\tfrac{\nu}{\kappa}\right)\left[H_0(1,x) - H_\kappa(1,x)\right], \end{aligned}$$

Regarding the operator $X = T^{-1/2}ST^{-1/2}$, within the framework of the first condition, we establish the following range: $I \leq hI = \frac{M}{m}I \leq X \leq h'I = \frac{M'}{m'}I$. Consequently, we infer that $\sigma(X) \subseteq [h, h'] \subseteq (1, \infty)$. Applying Lemma 4, we obtain:

$$\begin{aligned} & r\left[H_\kappa(I,X) + H_0(I,X) - H_{\frac{\kappa}{2}}(I,X)\right] + \min_{h \leq x \leq h'} K\left[\sqrt{x}, 2\right]^{r'} H_\nu(I,X) \\ & \leq H_0(I,X) - \left(\frac{\nu}{\kappa}\right)\left[H_0(I,X) - H_\kappa(I,X)\right], \end{aligned}$$

As the Kantorovich constant $K(t,2) = \frac{(1+t)^2}{4t}$ exhibits monotonicity within the interval $(0, \infty)$, it follows that:

$$\begin{aligned} & r\left[H_\kappa(I, T^{-1/2}ST^{-1/2}) + H_0(I, T^{-1/2}ST^{-1/2}) - H_{\frac{\kappa}{2}}(I, T^{-1/2}ST^{-1/2})\right] \\ & + \min_{h \leq x \leq h'} K\left[\sqrt{x}, 2\right]^{r'} H_\nu(I, T^{-1/2}ST^{-1/2}) \leq H_0(I, T^{-1/2}ST^{-1/2}) \\ & - \left(\frac{\nu}{\kappa}\right)\left[H_0(I, T^{-1/2}ST^{-1/2}) - H_\kappa(I, T^{-1/2}ST^{-1/2})\right], \end{aligned} \tag{25}$$

Likewise, within the context of the second condition, we observe that $I \leq \frac{1}{h}I = \frac{m}{M}h \leq X \leq \frac{1}{h'}I = \frac{m'}{M'}I$. Utilizing Lemma 4, we obtain the following:

$$\begin{aligned} & r\left[H_\kappa(I,X) + H_0(I,X) - H_{\frac{\kappa}{2}}(I,X)\right] + \min_{\frac{1}{h'} \leq x \leq \frac{1}{h}} K\left[\sqrt{x}, 2\right]^{r'} H_\nu(I,X) \\ & \leq H_0(I,X) - \left(\tfrac{\nu}{\kappa}\right)\left[H_0(I,X) - H_\kappa(I,X)\right], \end{aligned}$$

Since the Kantorovich constant $K(t,2) = \frac{(1+t)^2}{4t}$ is an increasing function on $(0, \infty)$, then

$$\begin{aligned} & r\left[H_\kappa(I, T^{-1/2}ST^{-1/2}) + H_0(I, T^{-1/2}ST^{-1/2}) - H_{\frac{\kappa}{2}}(I, T^{-1/2}ST^{-1/2})\right] \\ & + \min_{\frac{1}{h'} \leq x \leq \frac{1}{h}} K\left[\sqrt{x}, 2\right]^{r'} H_\nu(I, T^{-1/2}ST^{-1/2}) \leq H_0(I, T^{-1/2}ST^{-1/2}) \end{aligned}$$

$$-\left(\frac{\nu}{\kappa}\right)\left[H_0(I,T^{-1/2}ST^{-1/2})-H_\kappa(I,T^{-1/2}ST^{-1/2})\right],$$

By multiplying both inequalities (25) and (26) on both the left-hand and right-hand sides by the operator $T^{1/2}$, we can infer the desired inequality (24).

Theorem 7. *Consider positive invertible operators T and S in a Hilbert space $\mathcal{H}$, where I represents the identity operator. Additionally, let κ be a non-negative number such that $0 \le \nu < \kappa \le 1$. Assuming that there exist positive real numbers m, m', M, M' that satisfy either of the following conditions:*

(a) $0 < mI \le T \le m'I < M'I \le S \le MI$

(b) $0 < mI \le S \le m'I \le T \le MI$

Then, the following conclusions hold:

$$\begin{aligned}&H_0(T,S)-\left(\tfrac{\nu}{\kappa}\right)\left[H_0(T,S)-H_\kappa(T,S)\right]\le K\left[\sqrt{h},2\right]^{-r'}H_\nu(T,S)\\&+R\left[H_\kappa(T,S)+H_0(T,S)-H_{\frac{\kappa}{2}}(T,S)\right],\end{aligned}\tag{26}$$

where $R=\min\{\frac{\nu}{\kappa},1-\frac{\nu}{\kappa}\}$, $h=\frac{M}{m}$ and $r'=\min\{2r,1-2r\}$.

Proof. The proof process is similar to that of Theorem 6, and thus, we will not provide it here.

Remark 3. *The nature of the Kantorovich constant's characteristics makes it clear that the inequalities outlined in Theorems 6 and 7 signify improved results compared to those detailed in [13], [14], [18], [20], and [23].*

5. Utilizations of the improved Young-type inequalities for traces, determinants, and norms of positive definite matrices

In this section, we introduce a collection of improved Young-type inequalities designed specifically for traces, determinants, and norms of positive semi-definite matrices.

A matrix version proved in [1] says that if $T, S \in M_n(\mathbb{C})$ are positive semi-definite, then

$$s_j(TS) \le s_j\left(\frac{1}{p}T^p+\frac{1}{q}S^q\right)\tag{27}$$

for $j = 1, \cdots, n$

Lemma 5. *Let $\rho, \sigma > 0$, $p, q > 1$ such that $\frac{1}{p}+\frac{1}{q}=1$. Then for $m \in \mathbb{N}$, we have*

$$\left(\rho^{\frac{1}{p}}\sigma^{\frac{1}{q}}\right)^m + r_0^m\left(\rho^{\frac{m}{2}}-\sigma^{\frac{m}{2}}\right)^2 \le \left(\frac{\rho^r}{p}+\frac{\sigma^r}{q}\right)^{\frac{m}{r}},\quad r\ge 1\tag{28}$$

where $r_0=\min\{\frac{1}{p},\frac{1}{q}\}$.

Lemma 6. *Let $T_i \in M_n(\mathbb{C})$ $(i = 1, \cdots, n)$,. Then*

$$\sum_{j=1}^{n} s_j(T_1 \cdots T_n) \leq \sum_{j=1}^{n} s_j(T_1) \cdots s_j(T_k).$$

Theorem 8. *Let $T, S \in \mathcal{B}(\mathcal{H})$ be positive definite, $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and $m \in \mathbb{N}$. Then*

$$\left(\frac{tr(T^r)}{p} + \frac{tr(S^r)}{q} \right)^{\frac{m}{r}} \geq \left(tr \left| T^{\frac{1}{p}} S^{\frac{1}{q}} \right| \right)^m + r_0^m \left((tr(T))^{\frac{m}{2}} - (tr(S))^{\frac{m}{2}} \right)^2, \tag{29}$$

where $r_0 = \min\{\frac{1}{p}, \frac{1}{q}\}$.

Proof. By inequality (29), we have

$$\begin{aligned} s_j^{\frac{m}{r}} \left(\frac{T^r}{p} + \frac{S^r}{q} \right) &= \left(\frac{s_j^{\frac{m}{r}}(T^r)}{p} + \frac{s_j^{\frac{m}{r}}(S^r)}{q} \right) \\ &\geq s_j^m \left(T^{\frac{1}{p}} \right) s_j^m \left(S^{\frac{1}{q}} \right) + r_0^m \left(s_j^{\frac{m}{2}}(T) - s_j^{\frac{m}{2}}(S) \right)^2 \\ &= s_j^m \left(T^{\frac{1}{p}} \right) s_j^m \left(S^{\frac{1}{q}} \right) + r_0^m \left(s_j^m(T) + s_j^m(S) - 2 s_j^{\frac{m}{2}}(T) s_j^{\frac{m}{2}}(S) \right) \end{aligned}$$

for $j = 1, \cdots, n$. Thus, by Lemma 6 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} tr^{\frac{m}{r}} \left(\frac{T^r}{p} + \frac{S^r}{q} \right) &= \sum_{j=1}^{n} s_j^{\frac{m}{r}} \left(\frac{T^r}{p} + \frac{S^r}{q} \right) \\ &\geq \sum_{j=1}^{n} s_j^m \left(T^{\frac{1}{p}} \right) s_j^m \left(S^{\frac{1}{q}} \right) \\ &+ r_0^m \left(\sum_{j=1}^{n} s_j^m(T) + \sum_{j=1}^{n} s_j^m(S) - 2 \sum_{j=1}^{n} s_j^{\frac{m}{2}}(T) s_j^{\frac{m}{2}}(S) \right) \end{aligned}$$

Hence

$$\begin{aligned} tr^{\frac{m}{r}} \left(\frac{T^r}{p} + \frac{S^r}{q} \right) &\geq \sum_{j=1}^{n} s_j^m \left(T^{\frac{1}{p}} S^{\frac{1}{q}} \right) \\ &+ r_0^m \left(\sum_{j=1}^{n} s_j^m(T) + \sum_{j=1}^{n} s_j^m(S) - 2 \sum_{j=1}^{n} s_j^{\frac{m}{2}}(T) s_j^{\frac{m}{2}}(S) \right) \\ &\geq \left(tr \left| \left(T^{\frac{1}{p}} S^{\frac{1}{q}} \right) \right| \right)^m + r_0^m \left[(tr(T))^m + (tr(S))^m \right. \end{aligned}$$

$$
\begin{aligned}
&- \quad 2\left(\sum_{j=1}^{n} s_j(T)\right)^{\frac{m}{2}}\left(\sum_{j=1}^{n} s_j(S)\right)^{\frac{m}{2}}\Bigg] \\
&= \quad \left(tr\left|\left(T^{\frac{1}{p}}S^{\frac{1}{q}}\right)\right|\right)^{m} + r_0^m\left((tr(T))^{\frac{m}{2}} - (tr(S))^{\frac{m}{2}}\right)^2
\end{aligned}
$$

Remark 4. *Ando's singular value inequality (27) entails the norm inequality*

$$\left|\left|\left|T^{\kappa}S^{1-\kappa}\right|\right|\right| \leq |||\kappa T + (1-\kappa)S|||. \tag{30}$$

So, our Theorem 8 improves this inequality for the trace norm:

$$\left\|T^{\frac{1}{p}}S^{\frac{1}{q}}\right\|_1^m + r_0^m\left(\|T\|_1^{\frac{m}{2}} - \|S\|_1^{\frac{m}{2}}\right)^2 \leq \left\|\frac{1}{p}T^r + \frac{1}{q}S^r\right\|_1^{\frac{m}{r}} \tag{31}$$

Theorem 9. *Let $T, S \in \mathcal{B}(\mathcal{H})$ be positive definite, $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and $m \in \mathbb{N}$. Then for all $r \geq 1$*

$$\det\left(\frac{T^r}{p} + \frac{S^r}{q}\right)^{\frac{m}{r}} \geq \det\left(T^{\frac{1}{p}}S^{\frac{1}{q}}\right)^m + r_0^{mn}\det\left(T^m + S^m - 2S^{\frac{m}{2}}\left(S^{-\frac{1}{2}}TS^{-\frac{1}{2}}\right)^m S^{\frac{m}{2}}\right)^2, \tag{32}$$

where $r_0 = \min\{\frac{1}{p}, \frac{1}{q}\}$.

Proof. By inequality (28), we have

$$s_j^{\frac{m}{r}}\left(\frac{1}{p}\left(S^{-\frac{r}{2}}T^rS^{-\frac{r}{2}}\right) + \frac{1}{q}I\right) \geq s_j^{\frac{m}{p}}\left(S^{-\frac{1}{2}}TS^{-\frac{1}{2}}\right) + r_0^m\left(s_j^{\frac{m}{2}}\left(S^{-\frac{1}{2}}TS^{-\frac{1}{2}}\right) - 1\right)^2$$

for all $j = 1, \cdots, n$.

$$
\begin{aligned}
\det\left(\frac{1}{p}S^{-\frac{r}{2}}TS^{-\frac{r}{2}} + \frac{1}{q}\right)^{\frac{m}{r}} &= \prod_{j=1}^{n}\left(\frac{1}{p}s_j^{\frac{m}{r}}\left(S^{-\frac{r}{2}}T^rS^{-\frac{r}{2}} + \frac{1}{q}\right)\right) \\
&\geq \prod_{j=1}^{n}\left[s_j^{\frac{m}{p}}\left(S^{-\frac{1}{2}}TS^{-\frac{1}{2}}\right) + r_0^m\left(s_j^{\frac{m}{2}}\left(S^{-\frac{1}{2}}TS^{-\frac{1}{2}}\right) - 1\right)^2\right] \\
&\geq \prod_{j=1}^{n}\left[s_j^{\frac{1}{p}}\left(S^{-\frac{1}{2}}TS^{-\frac{1}{2}}\right)^m\right] \\
&+ \quad r_0^{mn}\prod_{j=1}^{n}\left[s_j^{\frac{m}{2}}\left(S^{-\frac{1}{2}}TS^{-\frac{1}{2}}\right) - 1\right]^2 \\
&= \det\left(S^{-\frac{1}{2}}TS^{-\frac{1}{2}}\right)^{\frac{m}{p}} + r_0^{mn}\left[\left(S^{-\frac{1}{2}}TS^{-\frac{1}{2}}\right)^{\frac{m}{2}} - I\right]^2.
\end{aligned}
$$

Consequently

$$\det\left(\frac{T^r}{p}+\frac{S^r}{q}\right)^{\frac{m}{r}} \geq \det\left(T^{\frac{1}{p}}S^{\frac{1}{q}}\right)^m + r_0^{mn}\det\left(T^m+S^m-2S^{\frac{m}{2}}\left(S^{-\frac{1}{2}}TS^{-\frac{1}{2}}\right)^m S^{\frac{m}{2}}\right)^2.$$

Theorem 10. *Let $T, S, X \in M_n(\mathbb{C})$ such that T and S are positive semi-definite and $p, q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $m \in \mathbb{N}$. Then for all $r \geq 1$, we have*

$$\left|\left|\left|T^{\frac{1}{p}}XS^{\frac{1}{q}}\right|\right|\right|^m + r_0^m\left(|||TX|||^{\frac{m}{2}} - |||SX|||^{\frac{m}{2}}\right)^2 \leq \left(\frac{1}{p}|||TX|||^r + \frac{1}{q}|||XB|||^r\right)^{\frac{m}{r}}, \tag{33}$$

where $r_0 = \min\{\frac{1}{p}, \frac{1}{q}\}$.

To prove Theorem 10, we need the following lemma which is known as the Heinz-Kato type for unitarily invariant norm.

Lemma 7 ([9]). *Let $T, S \in M_n(\mathbb{C})$ be positive definite matrices and $0 \leq \vartheta \leq 1$. Then we have*

$$\left|\left|\left|T^{\vartheta}XS^{1-\vartheta}\right|\right|\right| \leq |||TX|||^{\vartheta}|||XB|||^{1-\vartheta}. \tag{34}$$

In particular

$$tr\left|T^{\vartheta}XS^{1-\vartheta}\right| \leq (tr(T))^{\vartheta}(tr(S))^{1-\vartheta}. \tag{35}$$

Proof. [Proof of Theorem 10] We have

$$\begin{aligned}\left|\left|\left|T^{\frac{1}{p}}XS^{\frac{1}{q}}\right|\right|\right|^m &+ r_0^m\left(|||TX|||^{\frac{m}{2}} - |||SX|||^{\frac{m}{2}}\right)^2 \\ &\leq \left[|||TX|||^{\frac{1}{p}}|||XB|||^{\frac{1}{q}}\right]^m + r_0^m\left(|||TX|||^{\frac{m}{2}} - |||SX|||^{\frac{m}{2}}\right)^2 \\ &\quad \text{(by Lemma 7)} \\ &\leq \left(\frac{1}{p}|||TX|||^r + \frac{1}{q}|||XB|||^r\right)^{\frac{m}{r}} \quad \text{(by inequality 28).}\end{aligned}$$

Lemma 8 ([3]). *Let $\omega_1, \cdots, \omega_n$ be non-negative real numbers and $\vartheta_1, \cdots, \vartheta_n$ be positive real numbers with $\sum_{i=1}^n \vartheta_i = 1$. Then we have*

$$\prod_{k=1}^n \omega_k^{\vartheta_k} + r\left(\sum_{k=1}^n \omega_k - n\sqrt[n]{\prod_{k=1}^n \omega_k}\right) \leq \sum_{i=1}^n \vartheta_k\omega_k, \tag{36}$$

where $r = \min\{\vartheta_k : k = 1, \cdots, n\}$.

Theorem 11. *Let $T_i \in M_n(\mathbb{C})$ $(i = 1, \cdots, n)$ be positive semi-definite. If $0 \leq \vartheta_i \leq 1$ $(i = 1, \cdots, n)$ with $\sum_{i=1}^{n} \vartheta_i = 1$, then*

$$\sum_{k=1}^{n} tr(\vartheta_k T_k) \geq tr\left|\prod_{k=1}^{n} T_k^{\vartheta_k}\right| + r\left(\sum_{k=1}^{n} tr(T_k) - n\sqrt[n]{\prod_{k=1}^{n} tr(T_k)}\right), \tag{37}$$

where $r = \min\{\vartheta_k : k = 1, \cdots, n\}$.

Proof. By inequality (36), we have

$$\sum_{k=1}^{n} \vartheta_k s_j(T_k) \geq \prod_{k=1}^{n} s_j(T_k)^{\vartheta_k} + r\left(\sum_{k=1}^{n} s_j(T_k) - n\sqrt[n]{\prod_{k=1}^{n} s_j(T_k)}\right)$$

for $j = 1, \cdots, n$. Thus, by Lemma 6 and the generalized Cauchy-Schwarz inequality, we have

$$\begin{aligned}
tr\left(\sum_{k=1}^{n} \vartheta_k T_k\right) &= \sum_{k=1}^{n} \vartheta_k tr(T_k) = \sum_{k=1}^{n} \vartheta_k \sum_{j=1}^{n} s_j(T_k) = \sum_{j=1}^{n}\sum_{k=1}^{n} \vartheta_k s_j(T_k) \\
&\geq \sum_{j=1}^{n} s_j(T_1^{\vartheta_1}) \cdots s_j(T_k^{\vartheta_n}) \\
&+ r\left(\sum_{j=1}^{n}\sum_{k=1}^{n} s_j(T_k) - n\sum_{j=1}^{n}\sqrt[n]{\prod_{k=1}^{n} s_j(T_n)}\right) \\
&\geq \sum_{j=1}^{n} s_j(T_1^{\vartheta_1} \cdots T_n^{\vartheta_n}) \\
&+ r\left(\sum_{j=1}^{n}\sum_{k=1}^{n} s_j(T_k) - n\sqrt[n]{\prod_{k=1}^{n}\sum_{j=1}^{n} s_j(T_k)}\right) \\
&\geq tr\left|T_1^{\vartheta_1} \cdots T_n^{\vartheta_n}\right| + r\left(\sum_{k=1}^{n} tr(T_k) - n\sqrt[n]{\prod_{k=1}^{n} tr(T_k)}\right)
\end{aligned}$$

where $r = \min\{\vartheta_k : k = 1, \cdots, n\}$.

Our Theorem 11 entils the following trace norm

$$\left\|\sum_{k=1}^{n} \vartheta_k T_k\right\|_1 \geq \left\|\prod_{k=1}^{n} T_k^{\vartheta_k}\right\|_1 + r\left(\left\|\sum_{k=1}^{n} T_k\right\|_1 - n\sqrt[n]{\prod_{k=1}^{n} \|T_k\|_1}\right), \tag{38}$$

where $r = \min\{\vartheta_k : k = 1, \cdots, n\}$.

Theorem 12. *Let $T_i \in M_n(\mathbb{C})$ $(i = 1, \cdots, n)$ be positive definite. If $0 \leq \vartheta_i \leq 1$ $(i = 1, \cdots, n)$ with $\sum_{i=1}^{n} \vartheta_i = 1$, then*

$$\det\left(\sum_{k=1}^{n} \vartheta_k T_k\right) \geq \prod_{k=1}^{n} \det\left(T_k^{\vartheta_k}\right) + r\left(\det\left(\sum_{k=1}^{n} T_k\right) - n\sqrt[n]{\prod_{k=1}^{n} \det(T_k)}\right), \qquad (39)$$

where $r = \min\{\vartheta_k : k = 1, \cdots, n\}$.

To prove Theorem 12, we need the following lemma.

Lemma 9 ([5]). *Let $T, S \in M_n(\mathbb{C})$ be positive definite. Then we have*

$$\det(T+S)^{\frac{1}{n}} \geq \det(T)^{\frac{1}{n}} + \det(S)^{\frac{1}{n}}. \qquad (40)$$

Proof. [Proof of Theorem 12] We have

$$\begin{aligned}
\det\left(\sum_{k=1}^{n} \vartheta_k T_k\right) &= \left[\det\left(\sum_{k=1}^{n} \vartheta_k T_k\right)^{\frac{1}{n}}\right]^n \\
&\geq \left[\sum_{k=1}^{n} \det\left(\vartheta_k T_k\right)^{\frac{1}{n}}\right]^n \quad \text{(by Lemma 9)} \\
&\geq \left[\sum_{k=1}^{n} \vartheta_k \det\left(T_k\right)^{\frac{1}{n}}\right]^n \\
&\geq \left[\prod_{k=1}^{n} \left(\left(T_k\right)^{\frac{1}{n}}\right)^{\vartheta_k}\right]^n + r^n\left(\sum_{k=1}^{n} \det\left(T_k\right)^{\frac{1}{n}} - n\sqrt[n]{\det\prod_{k=1}^{n} T_k}\right) \\
&= \prod_{k=1}^{n} \det\left(T_k^{\vartheta_k}\right) + r^n\left(\sum_{k=1}^{n} \det\left(T_k\right)^{\frac{1}{n}} - n\sqrt[n]{\det\prod_{k=1}^{n} T_k}\right)
\end{aligned}$$

Lemma 10 ([16]). *Let $\gamma_1, \gamma_2, \cdots, \gamma_n$ be a set of non-negative real numbers constrained by $\sum_{k=1}^{j} \gamma_k = \Gamma_j$. If $\omega_1, \omega_2, \cdots, \omega_n$ are positive real numbers, then*

$$\frac{1}{\Gamma_n}\sum_{k=1}^{n} \gamma_k \omega_k + \sqrt{1 + \left(\frac{1}{\Gamma_n}\sum_{k=1}^{n} \gamma_k \omega_k\right)^2} \geq \left[\prod_{k=1}^{n}\left(\omega_k + \sqrt{1+\omega_k^2}\right)^{\gamma_k}\right]^{\frac{1}{\Gamma_n}} \qquad (41)$$

holds.

Theorem 13. *Let $T_1, \cdots, T_k \in M_n(\mathbb{C})$ be positive define and let $\gamma_1, \gamma_2, \cdots, \gamma_n$ be a set of non-negative real numbers such that $\sum_{k=1}^{n} \gamma_k = \Gamma_n$. Then*

$$\frac{1}{\Gamma_n}\sum_{k=1}^{n} tr(T_k) + \sqrt{1 + \left(\frac{1}{\Gamma_n}\sum_{k=1}^{n} tr(T_k)\right)^2} \geq \prod_{k=1}^{n}\left[tr(T_k) + \sqrt{1 + tr(T_k^2)}\right]^{\frac{1}{\Gamma_n}}. \tag{42}$$

Proof. By inequality (41), we have

$$\frac{1}{\Gamma_n}\sum_{k=1}^{n} \gamma_k s_j(T_k) + \sqrt{1 + \left(\frac{1}{\Gamma_n}\sum_{k=1}^{n} \gamma_k s_j(T_k)\right)^2} \geq \left[\prod_{k=1}^{n}\left(s_j(T_k) + \sqrt{1 + s_j^2(T_k)}\right)^{\gamma_k}\right]^{\frac{1}{\Gamma_n}} \tag{43}$$

for all $j = 1, \cdots, n$. Hence we have

$$\begin{aligned}\frac{1}{\Gamma_n}\sum_{k=1}^{n}\gamma_k\sum_{j=1}^{n} s_j(T_k) &+ \sqrt{1 + \left(\frac{1}{\Gamma_n}\sum_{k=1}^{n}\gamma_k\sum_{j=1}^{n} s_j(T_k)\right)^2} \\ &\geq \left[\prod_{k=1}^{n}\left(\sum_{j=1}^{n} s_j(T_k) + \sqrt{1 + \sum_{j=1}^{n} s_j^2(T_k)}\right)^{\gamma_k}\right]^{\frac{1}{\Gamma_n}}\end{aligned}$$

Consequently,

$$\begin{aligned} tr\left(\frac{1}{\Gamma_n}\sum_{k=1}^{n}\gamma_k T_k\right) &+ \sqrt{1 + \left(\frac{1}{\Gamma_n}\sum_{k=1}^{n}\gamma_k tr(T_k)\right)^2} \\ &= \frac{1}{\Gamma_n}\sum_{k=1}^{n}\gamma_k tr(T_k) + \sqrt{1 + \left(\frac{1}{\Gamma_n}\sum_{k=1}^{n}\gamma_k tr(T_k)\right)^2} \\ &\geq \left[\prod_{k=1}^{n}\left(tr(T_k) + \sqrt{1 + \sum_{j=1}^{n} tr(T_k^2)}\right)^{\gamma_k}\right]^{\frac{1}{\Gamma_n}}\end{aligned}$$

6. Conclusion and Future Work

In conclusion, this paper has embarked on an extensive investigation into the domain of matrix means interpolation and comparison. A key aspect of this research has been the expansion of the parameter ϑ from the closed interval $[0, 1]$ to encompass the entire positive real line, represented as $\mathbb{R}^+$. This extension has allowed us to explore a broader spectrum of mathematical relationships and properties within this framework.

Furthermore, our exploration has led to the development of various novel results related to Heinz means. We have introduced scalar variants of Heinz inequalities, leveraging Kantorovich's constant, and have extended these inequalities to the operator realm. This expansion not only deepens our understanding of Heinz means but also opens up new avenues for applications in diverse mathematical contexts.

Lastly, we have presented refined Young's type inequalities specifically tailored for traces, determinants, and norms of positive semi-definite matrices. These refined inequalities are expected to find utility in various matrix analysis and linear algebra problems, enhancing our ability to derive meaningful conclusions and insights from the study of positive semi-definite matrices.

As for future work, there are several intriguing directions to consider. Firstly, it may be valuable to explore further extensions of the parameter space beyond $\mathbb{R}^+$ and investigate the implications of such extensions on matrix means and related inequalities. Additionally, the applicability of the developed results in practical fields such as physics, engineering, and data science warrants investigation. Finally, refining and expanding upon the presented inequalities could lead to even more powerful tools for matrix analysis and optimization, offering new insights and solutions to complex problems in mathematics and its applications.

Declaration

- **Author Contributions:** The author have read and agreed to the published version of the manuscript.
- **Funding:** No funding is applicable
- **Conflicts of Interest:** The authors declare no conflict of interest.

Acknowledgements

The authors sincerely thank the reviewers for their insightful feedback, valuable suggestions, and expertise, which have greatly contributed to improving the clarity, quality, and overall impact of this study.

References

[1] Tsuyoshi Andô. Matrix young inequalities. *Operator theory*, 75:33–38, 1995.

[2] Rajendra Bhatia. Interpolating the arithmetic–geometric mean inequality and its operator version. *Linear Algebra and its Applications*, 413(2):355–363, 2006. Special Issue on the 11th Conference of the International Linear Algebra Society, Coimbra, 2004.

[3] Shigeru Furuichi. On refined young inequalities and reverse inequalities. *Journal of Mathematical Inequalities*, 5, 01 2010.

[4] Fumio Hiai and Hideki Kosaki. Means for matrices and comparison of their norms. *Indiana University Mathematics Journal*, 48:899–936, 1999.

[5] R.A. Horn and C.R Johnson. *Matrix analysis.* Cambridge Univ. Press, New York, 1985.

[6] R.A. Horn and C.R Johnson. *Topics in Matrix Analysis.* Cambridge Univ. Press, New York, 1990.

[7] J. Miściśc Hot J.Pemcariśc, T. Furuta and Y. Seo. Mondpencariéc method in operator inequalities, inequalities for bounded selfadjoint operators on a hilbert space. 2005.

[8] Rupinderjit Kaur and Mandeep Singh. Complete interpolation of matrix versions of heron and heinz means. *Mathematical Inequalities and Applications*, 16, 12 2011.

[9] Fuad Kittaneh. Norm inequalities for fractional powers of positive operators. *Letters in Mathematical Physics*, 27:279–285, 1993.

[10] Fuad Kittaneh and Yousef Manasrah. Improved young and heinz inequalities for matrices. *Journal of Mathematical Analysis and Applications*, 361(1):262–269, 2010.

[11] Hong liang Zuo, Guanghua Shi, and Masatoshi Fujii. Refined young inequality with kantorovich constant. *Journal of Mathematical Inequalities*, pages 551–556, 2011.

[12] Wenshi Liao, Junliang Wu, and Jianguo Zhao. New Versions of Reverse Young and Heinz Mean Inequalities with the Kantorovich Constant. *Taiwanese Journal of Mathematics*, 19(2):467 – 479, 2015.

[13] M.H.M.Rashid. Young type inequalities and reverses for matrices. *Matematicki Vesnik*, 74(3):163–173, 2022.

[14] M.H.M.Rashid and N. Snaid. A new generalization of young type inequality and applications. *Hacettepe Journal of Mathematics & Statistics*, 51(5):1371–1378, 2022.

[15] Leila Nasiri and Mahmood Shakoori. A note on improved young type inequalities with kantorovich constant. *Journal of Mathematics and Statistics*, 12:201–205, 03 2016.

[16] Zlatko Pavić. Certain inequalities for convex functions. *Journal of Mathematical Inequalities*, pages 1349–1364, 01 2015.

[17] Mohammad H. M. Rashid and Feras Bani-Ahmad. New versions of refinements and reverses of young-type inequalities with the kantorovich constant. *Special Matrices*, 11(1):20220180, 2023.

[18] A. Salemi and A. Sheikh Hosseini. On reversing of the modified Young inequality. *Annals of Functional Analysis*, 5(1):70 – 76, 2014.

[19] J. Wu and J. Zhao. Operator inequalities and reverse inequalities related to the kittaneh–manasrah inequalities. *Linear and Multilinear Algebra*, 62(7):884–894, 2014.

[20] Changsen Yang and Yu Li. Refinements and reverses of young type inequalities. *Journal of Mathematical Inequalities*, pages 401–419, 01 2020.

[21] X. Zhan. *Matrix Inequalities, Lecture notes in Mathematics 1790.* Springer, New York, 2002.

[22] Jie Zhang and Junliang Wu. New progress on the operator inequalities involving improved young's and its reverse inequalities relating to the kantorovich constant. *Journal of Inequalities and Applications*, 2017, 04 2017.

[23] Jianguo Zhao and Junliang Wu. Operator inequalities involving improved young

and its reverse inequalities. *Journal of Mathematical Analysis and Applications*, 421(2):1779–1789, 2015.