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Some New Fractional Hermite-Hadamard type Inequalities For Generalized Class of Godunova-Levin Functions By Means of Interval Center-Radius Order Relation with Applications

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Abstract. The purpose of this article is to establish several new forms of Hermite-Hadamard inequalities by utilizing fractional integral operators via a totally interval midpoint-radius order relation for differentiable Godunova-Levin mappings. Moreover, in order to verify our main results, we construct some non-trivial examples and remarks that lead to other generalized convex mappings with different settings. Furthermore, we exploit special cases of Hölder's, Young's, and Minkowski-type inequalities in order to develop new bounds of Hermite-Hadamard inequality. Finally, we relate our key results with special means and demonstrate some of their applications.

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Key Words and Phrases: Hermite-Hadamard, cr-order, Fractional Operators, Interval mappings

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1. Introduction

Convex analysis provides a powerful mathematical framework for analyzing problems in various fields, especially due to the well-behaved properties of convex sets and functions. Its uses span a variety of fields, including control theory [56], economics [54], machine learning [38], and optimization [53]. In control theory [55], systems are often formulated as convex problems, where the system needs to minimize energy or error subject to dynamic constraints; in signal processing, it aids in the design of codes that minimize transmission errors, enhancing communication reliability [24]. Convex analysis is closely related to economic theory, particularly in the study of utility functions [14], which represent rational consumer preferences where utility increases with consumption, but at a diminishing rate. For more recent applications in diverse disciplines of applied sciences, we refer to [22, 26, 27, 59, 64] and the references therein.

Interval analysis is a mathematical methodology that allows numerical algorithms to address uncertainty more rigorously. It has applications in a variety of domains, including numerical computation, global optimization, control systems, engineering, and computer graphics. Borwein et al. [17] initially defined convex interval-valued functions (`IVFs`) in 1981, and since then, several researchers have extended and promoted different types of convexity by using `IVFs`. For example, include preinvex [48], harmonic convex [41], Godunova-Levin [5], (h_1, h_2)-convex [9], log-convex [49], coordinated convex [62], and various others [21, 31, 34, 39, 40, 43, 51, 52] and the references therein. It's important to remember that the partial order relation defines these convex `IVFs`, meaning that any two intervals may not be comparable. This indicates that the maximum-minimum problem cannot be solved since it is impossible to determine which of them is the greatest or smallest interval using these orderings. Hu and Wang [23] introduced the cr-order, which takes into account the midpoint and radius of two intervals to address this limitation. This order is total, meaning that any two interval numbers are comparable. In [61] authors provided the appropriate optimization conditions for the constrained optimization issue of interval-valued objective function and provided a novel definition of convex `IVF` using cr-order.

Among the several different types of inequalities, the Hermite-Hadamard type inequality is a fundamental component of convex analysis, offering crucial perspectives and instruments for theoretical investigation and real-world applications in numerous scientific domains. The inequality is defined as follows:

Consider $\Phi : \Omega \subseteq \mathscr{R} \to \mathscr{R}$ a convex mapping on the interval Ω with $\varepsilon_1, \varepsilon_2 \in \Omega$. Then, the inequality listed below is true:

$$\Phi\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right) \leq \frac{1}{\varepsilon_2-\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_2} \Phi(\theta)\, \mathrm{d}\theta \leq \frac{\Phi(\varepsilon_1)+\Phi(\varepsilon_2)}{2}. \tag{1}$$

The Hermite-Hadamard inequality is frequently employed in optimization problems involving convex functions. It helps establish bounds on integrals of convex functions, which can be crucial for finding optimal solutions in various mathematical models. Various authors study this inequality using different methodologies, including different kinds of interval-valued order relations, stochastic and fuzzy-valued mappings, and various kinds of fractional operators. For example, in [50], authors used generalized convex mappings, also known as preinvex functions, and developed various variations of Hermite and Hadamard inequalities for interval-valued functions; in [63], authors used h-convex mappings on coordinates in the sense of interval-valued functions and developed Hadamard and Bullen type inclusions; in [10], authors used (h_1, h_2)-Godunova and Levin functions and developed Hermite-Hadamard and Jensen type inequalities; in [35], authors used harmonical convex mappings and developed double inequalities using inclusion relation; in [18] Dragomir developed Hermite–Hadamard's type inequalities for operator convex functions; in [19] authors show some new generalization of Hermite–Hadamard and Mercer forms of inequalities for geometric–arithmetic convexity by using interval maps. Some other important results and inequalities connected to these employing different types of fractional operators, including Hadamard, Atangana–Baleanu, Caputo–Fabrizio and Riemann–Liouville fractional integrals (see refs. [1, 2, 6, 28, 30, 42, 60]).

As the primary focus of this paper is on cr-interval order relations, recent advances in center-radius order relations should be recalled using a different type of convex mapping. In [58], the authors initially presented the idea of cr-order in covex sense. In comparison to other order relations, this relation is more compatible and possesses a various additional characteristics that other interval order relations lack. In [25] authors defined a new class of convex mapping for convex optimizing problems in the context of cr-order based on their work. In response to these discoveries, Liu et al. [36, 37] derived discrete versions of Jensen and Hermite-Hadamard inequalities based on two different types of generalized convex mappings by using cr-order. As a result of using superquadratic functions in a fractional frame of reference via cr-order relations, Khan and Saad [33] developed several novel bounds for various kinds of double inequalities. To explore entropy and mean characteristics, Fahad et al. [20] used geometric and arithmetic-cr-convex functions.

Afzal et al. [4, 8] created different types of discrete Jensen type and Hermite-Hadamard inequality utilizing the conventional Riemann integral operator by using the cr-h-Godunova-Levin function in convex and harmonic convex sense.

Theorem 1 (see [4]). *Let* $\mathsf{h} : (0,1) \to \mathscr{R}^+$ *such that* $\mathsf{h}\left(\frac{1}{2}\right) \neq 0$, *and* $\Phi : [\varepsilon_1, \varepsilon_2] \to \mathscr{R}^+_{\mathcal{I}}$ *be an cr-*h*-Godunova-Levin mapping,* $\varepsilon_1, \varepsilon_2 \in \mathscr{R}^+$, *then the inequality stated below holds true:*

$$\frac{\mathsf{h}\left(\frac{1}{2}\right)}{2}\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right) \preceq_{cr} \frac{1}{\varepsilon_2-\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_2}\Phi(v)\mathsf{d}v \preceq_{cr} [\Phi(\varepsilon_1)+\Phi(\varepsilon_2)]\int_0^1 \frac{\mathsf{d}\flat}{\mathsf{h}(\flat)}.$$

Sahoo et al. [45] employed fractional integral operators to construct the following double relation for cr-convex functions:

Theorem 2 (see [45]). *Let* $\Phi : [\varepsilon_1, \varepsilon_2] \to \mathscr{R}^+_{\mathcal{I}}$ *be an cr-convex function on* $[\varepsilon_1, \varepsilon_2]$, *then the inequality stated below holds true:*

$$\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right) \preceq_{cr} \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varepsilon_2-\varepsilon_1)^{\alpha}}\left(\mathrm{J}_{\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)^+}\Phi(\varepsilon_2)+\mathrm{J}^{\alpha}_{\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)^-}\Phi(\varepsilon_1)\right)$$
$$\preceq_{cr} \frac{\Phi(\varepsilon_1)+\Phi(\varepsilon_2)}{2}$$

Shah et al. [47] employed cr-convex stochastic processes to establish different variations of Hermite-Hadamard type relations in the Mercer sense.

Theorem 3 (see [47])**.** *Let $\Phi : [\varepsilon_1, \varepsilon_2]\times\Omega \to \mathscr{R}_{\mathcal{I}}^+$ be a γ-convex cr-interval-valued stochastic processes then the inequality stated below holds true:*

$$\frac{\left(1-e^{-\zeta}\right)}{\gamma\left(\frac{1}{2}\right)}\Phi\left(\eta_1+\eta_2-\frac{\varepsilon_1+\varepsilon_2}{2},.\right)$$
$$\preceq_{cr} \frac{1-\beta}{2}\left[\mathrm{J}^{\beta}_{\eta_1+\eta_2-\varepsilon_1^-}[\Phi]\left(\eta_1+\eta_2-\varepsilon_2\right)+\mathrm{J}^{\beta}_{\eta_1+\eta_2-\varepsilon_2^+}[\Phi]\left(\eta_1+\eta_2-\varepsilon_1\right)\right]$$
$$\preceq_{cr} \left[\Phi\left(\eta_1,.\right)+\Phi\left(\eta_2,.\right)-\frac{\Phi(\varepsilon_1,.)+\Phi(\varepsilon_2,.)}{2}\right]\Delta.$$

For some additional results and inequalities obtained using other types of generalized convex mappings under center radius order relations connected to developed results, please see the following publications [7, 12, 44, 46] and their references.

This study is regarded fresh and significant since it presents new and original conclusions using center-radius interval order relations. Furthermore, this is the first time that Hermite-Hadamard and its numerous versions, including product form, weighted form, and employing symmetric mappings, are produced by Atangana-Baleanu fractional integral operators under cr-order relation. Additionally, we utilize a number of additional known results, such as Minkowski, Holder, and Young, in the development of these results. Furthermore, we provide bounds of these inequalities in terms of special functions and many applications in terms of special means.

We are especially inspired by the works of these authors [4, 11, 13, 20, 33] to introduced a new and improved form of several inequalities, which undergo multiple improvements and reverses in various circumstances. This note is structured into five parts, starting with an introduction and foundational discussion of the topic related to preliminary. In Section 3, we develop numerous innovative versions of the double inequality, including its product and weighted forms, using various other well-known inequalities. In Section 4, we tie our conclusions to special means and demonstrate their applications. Finally, in Section 5, we provide a precise conclusion and possible future work.

2. Preliminaries

In this section, we present some well-known definitions and outcomes that can be utilized to support the paper's core findings. Furthermore, we will go over some fundamental ideas relating to fractional and interval calculus. Some fundamental topics are not fully covered here; thus, we refer to [20]. Prior to proceeding, we correct a few notations that are utilized in the article.

- $\mathscr{R}_{\mathcal{I}}$: space of intervals in $\mathscr{R}$;
- $\underline{\Phi} = \overline{\Phi}$: interval maps become dysfunctional;
- $\subseteq$: inclusion interval order relation;
- $\preceq_{\texttt{cr}}$: cr-interval order relation;
- $\leq$: standard order relation;
- IVF: interval-valued function;

2.1. Set-valued Analysis

The space containing all subsets of $\mathscr{R}$ in n-dimensional interval space $\mathscr{R}_{\mathcal{I}}$.

$$\mathscr{R}_{\mathcal{I}} = \{[\varepsilon_1, \varepsilon_2] : \varepsilon_1, \varepsilon_2 \in \mathscr{R} \textit{ and } \varepsilon_1 \leq \varepsilon_2\},$$

To define the Hausdorff metric in $\mathscr{R}_{\mathcal{I}}$, use this formula:

$$\mathtt{H}(\varepsilon_1, \varepsilon_2) = \max\{\mathtt{d}(\varepsilon_1, \varepsilon_2), \mathtt{d}(\varepsilon_2, \varepsilon_1)\}, \tag{2}$$

where $\mathtt{d}(\varepsilon_1, \varepsilon_2) = \sup_{\nu\in\varepsilon_1} \mathtt{d}(\nu, \varepsilon_2)$, and $\mathtt{d}(\nu, \varepsilon_2) = \min_{\mu\in\varepsilon_2} \mathtt{d}(\nu, \mu) = \min_{\mu\in\varepsilon_2} |\nu - \mu|$.

Remark 1. *The Hausdorff metric (2) can also be expressed as follows:*

$$\mathtt{H}([\underline{\varepsilon_1}, \overline{\varepsilon_1}], [\underline{\varepsilon_2}, \overline{\varepsilon_2}]) = \max\{|\underline{\varepsilon_1} - \underline{\varepsilon_2}|, |\overline{\varepsilon_1} - \overline{\varepsilon_2}|\}.$$

In interval space, we call this the Moore metric.

For instance, if $\zeta_1 = [\underline{\varepsilon_1}, \overline{\varepsilon_1}]$ and $\zeta_2 = [\underline{\varepsilon_2}, \overline{\varepsilon_2}]$ are two closed intervals, then the Minkowski sum, scalar multiplication, and difference are defined as follows:

$$\zeta_1 + \zeta_2 = \{\varepsilon_1 + \varepsilon_2 \mid \varepsilon_1 \in \zeta_1, \varepsilon_2 \in \zeta_2\} \text{ and } \Gamma\zeta_1 = \{\Gamma\varepsilon_1 \mid \varepsilon_1 \in \zeta_1\}.$$

and

$$\zeta_1 - \zeta_2 = [\underline{\varepsilon_1} - \overline{\varepsilon_2}, \overline{\varepsilon_1} - \underline{\varepsilon_2}],$$

with the product

$$\zeta_1 \cdot \zeta_2 = [\min\{\underline{\varepsilon_1\varepsilon_2}, \underline{\varepsilon_1}\overline{\varepsilon_2}, \overline{\varepsilon_1}\underline{\varepsilon_2}, \overline{\varepsilon_1\varepsilon_2}\}, \sup\{\underline{\varepsilon_1\varepsilon_2}, \underline{\varepsilon_1}\overline{\varepsilon_2}, \overline{\varepsilon_1}\underline{\varepsilon_2}, \overline{\varepsilon_1\varepsilon_2}\}],$$

and the division

$$\frac{\zeta_1}{\zeta_2} = \left[\min\left\{\frac{\underline{\varepsilon_1}}{\underline{\varepsilon_2}}, \frac{\underline{\varepsilon_1}}{\overline{\varepsilon_2}}, \frac{\overline{\varepsilon_1}}{\underline{\varepsilon_2}}, \frac{\overline{\varepsilon_1}}{\overline{\varepsilon_2}}\right\}, \max\left\{\frac{\underline{\varepsilon_1}}{\overline{\varepsilon_2}}, \frac{\underline{\varepsilon_1}}{\overline{\varepsilon_2}}, \frac{\overline{\varepsilon_1}}{\underline{\varepsilon_2}}, \frac{\overline{\varepsilon_1}}{\overline{\varepsilon_2}}\right\}\right],$$

where $0 \notin \zeta_2$.

Definition 1 (see [4]). *For any two intervals the center-radius order relation is defined as* $\zeta_1 = [\underline{\varepsilon_1}, \overline{\varepsilon_2}] = \langle \omega_c, \omega_r \rangle = \left\langle \frac{\underline{\varepsilon_1}+\overline{\varepsilon_2}}{2}, \frac{\overline{\varepsilon_2}-\underline{\varepsilon_1}}{2} \right\rangle$, $\zeta_2 = [\underline{\varepsilon_1}, \overline{\varepsilon_2}] = \langle \Omega_c, \Omega_r \rangle = \left\langle \frac{\underline{\varepsilon_1}+\overline{\varepsilon_2}}{2}, \frac{\overline{\varepsilon_2}-\underline{\varepsilon_1}}{2} \right\rangle$, *where*

$$\zeta_1 \preceq_{\mathtt{cr}} \zeta_2 \iff \begin{cases} \omega_c < \Omega_c, & \text{if } \omega_c \neq \Omega_c; \\ \omega_r \leq \Omega_r, & \text{if } \omega_r = \Omega_r. \end{cases}$$

The relation $\preceq_{\mathtt{cr}}$ *satisfies the following relational properties for any three intervals* $\zeta_1 = [\underline{\varepsilon_1}, \overline{\varepsilon_2}] = \langle \omega_c, \omega_r \rangle$, $\zeta_2 = [\underline{\varepsilon_1}, \overline{\varepsilon_2}] = \langle \Omega_c, \Omega_r \rangle$ *and* $\zeta_3 = [\underline{\eta_1}, \overline{\eta_2}] = \langle \eta_c, \eta_r \rangle$: *Reflexivity:* $\zeta_1 \preceq_{\mathtt{cr}} \zeta_1$. *Anti-symmetry:* $\zeta_1 \preceq_{\mathtt{cr}} \zeta_2$ *and* $\zeta_2 \preceq_{\mathtt{cr}} \zeta_1$. *Transitivity:* $\zeta_1 \preceq_{\mathtt{cr}} \zeta_2$ *and* $\zeta_2 \preceq_{\mathtt{cr}} \zeta_3$ *then* $\zeta_1 \preceq_{\mathtt{cr}} \zeta_3$. *Comparability:* $\zeta_2 \preceq_{\mathtt{cr}} \zeta_3$ *or* $\zeta_3 \preceq_{\mathtt{cr}} \zeta_2$.

Theorem 4 (see [4]). *Let* $\Phi : [\varepsilon_1, \varepsilon_2] \to \mathscr{R}^+_{\mathcal{I}}$ *be an interval set-valued map given by* $\Phi = [\underline{\Phi}, \bar{\Phi}]$. *Then the* Φ *is Riemann integrable on* $[\varepsilon_1, \varepsilon_2]$ *iff* $\underline{\Phi}$ *and* $\bar{\Phi}$ *are Riemann integrable on* $[\varepsilon_1, \varepsilon_2]$ *and*

$$\int_{\varepsilon_1}^{\varepsilon_2} \Phi(\mathbf{e})\mathtt{d}\mathbf{e} = \left[\int_{\varepsilon_1}^{\varepsilon_2} \underline{\Phi}(\mathbf{e})\mathtt{d}\mathbf{e}, \int_{\varepsilon_1}^{\varepsilon_2} \bar{\Phi}(\mathbf{e})\mathtt{d}\mathbf{e} \right]$$

We shall refer to the set of all Riemann integrable interval-valued maps on $[\varepsilon_1, \varepsilon_2]$ *as* $\mathcal{IR}_{([\varepsilon_1, \varepsilon_2])}$.

Theorem 5 (see [4]). *Let* $\Phi, \mathscr{H} : [\varepsilon_1, \varepsilon_2] \to \mathscr{R}^+_{\mathcal{I}}$ *given by* $\Phi = [\underline{\Phi}, \bar{\Phi}]$, *and* $\mathscr{H} = [\underline{\mathscr{H}}, \bar{\mathscr{H}}]$. *If* $\Phi, \mathscr{H} \in \mathcal{IR}_{([\varepsilon_1, \varepsilon_2])}$, *and* $\Phi(\mathbf{e}) \preceq_{cr} \mathscr{H}(\mathbf{e})$, $\forall\, \mathbf{e} \in [\varepsilon_1, \varepsilon_2]$, *then*

$$\int_{\varepsilon_1}^{\varepsilon_2} \Phi(\mathbf{e})\mathtt{d}\mathbf{e} \preceq_{cr} \int_{\varepsilon_1}^{\varepsilon_2} \mathscr{H}(\mathbf{e})\mathtt{d}\mathbf{e}.$$

Example 1. *Consider* $\Phi = [v+1, 2v+2]$ *and* $\mathscr{H} = [v^2+2, 3v+2]$, $\forall\ v \in [0,1]$.

$$\Phi_c = \frac{3v+3}{2}, \Phi_r = \frac{v+1}{2}, \mathscr{H}_c = \frac{v^2+3v+4}{2} \quad \text{and} \quad \mathscr{H}_r = \frac{3v-v^2}{2}.$$

From Definition 1, we have $\Phi(v) \preceq_{cr} \mathscr{H}(v)$, $\forall\ v \in [0,1]$.

Since,

$$\int_0^1 [v+1, 2v+2]\mathtt{d}v = \left[\frac{3}{2}, 3\right].$$

and

$$\int_0^1 [v^2+2, 2v+2]\mathtt{d}v = \left[\frac{7}{3}, \frac{7}{2}\right]$$

From Theorem 5, *we have*

$$\int_0^1 \Phi(v)\mathtt{d}v \preceq_{cr} \int_0^1 \mathscr{H}(v)\mathtt{d}v.$$

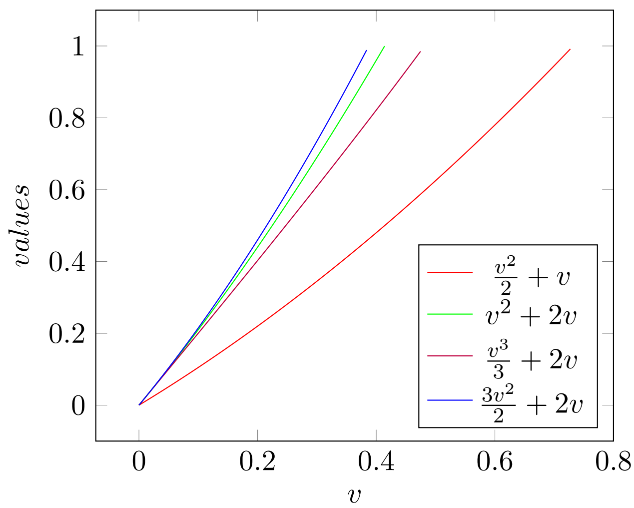

Figure 1: Graphical validation of Theorem 5 .

Definition 2 (see [4]). *Let $\Phi : [\varepsilon_1, \varepsilon_2] \to \mathscr{R}^+_{\mathcal{I}}$ be an cr set-valued map given by $\Phi = [\underline{\Phi}, \bar{\Phi}]$; then, Φ is said to be cr-convex if*

$$\Phi(\flat\varepsilon_1 + (1-\flat)\varepsilon_2) \preceq_{cr} \flat\Phi(\varepsilon_1) + (1-\flat)\Phi(\varepsilon_2),$$

holds for all $\varepsilon_1, \varepsilon_2 \in \mathfrak{B} \subset \mathscr{R}$ and $\flat \in [0,1]$.

Definition 3 (see [4]). *Let $\Phi : [\varepsilon_1, \varepsilon_2] \to \mathscr{R}^+_{\mathcal{I}}$ be an cr set-valued map given by $\Phi = [\underline{\Phi}, \bar{\Phi}]$ and $\mathsf{h} : (0,1) \to \mathscr{R}$ be non-negative function; then, Φ is said to be cr-h-convex if*

$$\Phi(\flat\varepsilon_1 + (1-\flat)\varepsilon_2) \preceq_{cr} \mathsf{h}(\flat)\Phi(\varepsilon_1) + \mathsf{h}(1-\flat)\Phi(\varepsilon_2),$$

holds for all $\varepsilon_1, \varepsilon_2 \in \mathfrak{B} \subset \mathscr{R}$ and $\flat \in (0,1)$.

Definition 4 (see [4]). *Let $\Phi : [\varepsilon_1, \varepsilon_2] \to \mathscr{R}^+_{\mathcal{I}}$ be an cr set-valued map given by $\Phi = [\underline{\Phi}, \bar{\Phi}]$ and $\mathsf{h} : (0,1) \to \mathscr{R}$ be non-negative function; then, Φ is said to be cr-h-Godunova-Levin if*

$$\Phi(\flat\varepsilon_1 + (1-\flat)\varepsilon_2) \preceq_{cr} \frac{\Phi(\varepsilon_1)}{\mathsf{h}(\flat)} + \frac{\Phi(\varepsilon_2)}{\mathsf{h}(1-\flat)},$$

holds for all $\varepsilon_1, \varepsilon_2 \in \mathfrak{B} \subset \mathscr{R}$ and $\flat \in (0,1)$. The class of all cr-h-Godunova-Levin convex mappings are denoted by $\mathtt{SGX}(\mathsf{h}, [\varepsilon_1, \varepsilon_2], \mathscr{R}^+_{\mathcal{I}})$.

Remark 2.

- *If $\mathsf{h}(\flat) = \frac{1}{\flat^s}$, then Definition 4 recovers cr-s-convex functions in [4].*
- *If $\mathsf{h}(\flat) = 1$, then Definition 4 recovers cr-p-functions in [4].*
- *If $\mathsf{h}(\flat) = \frac{1}{\flat}$, then Definition 4 recovers cr-convex functions in [45]*

Definition 5 (see [15]). *Let $\Phi : [\varepsilon_1, \varepsilon_2] \to \mathscr{R}^+_{\mathcal{I}}$ be an set-valued map given by $\Phi = [\underline{\Phi}, \bar{\Phi}]$. The interval-valued left-sided and right-sided Atangana-Baleanu fractional integral of function Φ and order $\varsigma > 0$ is defined by*

$$
\begin{aligned}
{}^{\mathtt{AB}}_{\varepsilon_1}\mathtt{I}^{\varsigma}_{\mathtt{t}}\{\Phi(\mathtt{t})\} &= \frac{1-\varsigma}{\mathbb{B}(\varsigma)}\Phi(\mathtt{t}) + \frac{\varsigma}{\mathbb{B}(\varsigma)\Gamma(\varsigma)}\int_{\varepsilon_1}^{\mathtt{t}} \Phi(\flat)(\mathtt{t}-\flat)^{\varsigma-1}\,\mathrm{d}\flat,\\
{}^{\mathtt{AB}}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{\Phi(\mathtt{t})\} &= \frac{1-\varsigma}{\mathbb{B}(\varsigma)}\Phi(\mathtt{t}) + \frac{\varsigma}{\mathbb{B}(\varsigma)\Gamma(\varsigma)}\int_{\mathtt{t}}^{\varepsilon_2} \Phi(\flat)(\flat-\mathtt{t})^{\varsigma-1}\,\mathrm{d}\flat,
\end{aligned}
$$

where $\varepsilon_1 < \varepsilon_2, \varsigma \in (0,1], \Gamma(\flat) = \int_0^\infty \mathtt{t}^{\flat-1}\mathtt{e}^{-\mathtt{t}}\,\mathrm{d}\mathtt{t}$ is the special function, $\mathbb{B}(\varsigma) > 0$ such that $\mathbb{B}(0) = \mathbb{B}(1) = 1, \|\mathbb{B}(\varsigma)\| = 1$, and $\beta_{\mathtt{a}} = \beta_{\mathtt{a}}(\mathtt{p},\mathtt{q}) = \int_0^{\mathtt{a}} \flat^{p-1}(1-\flat)^{\mathtt{q}-1}\,\mathrm{d}\flat$ is the beta integral in incomplete sense.

Theorem 6 (see [4]). *Let $\Phi : [\varepsilon_1, \varepsilon_2] \to \mathscr{R}^+_{\mathcal{I}}$ be an interval set-valued map given by $\Phi = [\underline{\Phi}, \bar{\Phi}]$, then we have*

$$
{}^{\mathtt{AB}}_{\varepsilon_1}\mathtt{I}^{\varsigma}_{\mathtt{t}}\{\Phi(\mathtt{t})\} = \left[{}^{\mathtt{AB}}_{\varepsilon_1}\mathtt{I}^{\varsigma}_{\mathtt{t}}\{\underline{\Phi}(\mathtt{t})\}, {}^{\mathtt{AB}}_{\varepsilon_1}\mathtt{I}^{\varsigma}_{\mathtt{t}}\{\overline{\Phi}(\mathtt{t})\}\right]
$$

and

$$
{}^{\mathtt{AB}}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{\Phi(\mathtt{t})\} = \left[{}^{\mathtt{AB}}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{\underline{\Phi}(\mathtt{t})\}, {}^{\mathtt{AB}}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{\overline{\Phi}(\mathtt{t})\}\right].
$$

The following inequalities are frequently used to produce our major results.

Theorem 7 (see [3]). ***(Hölder inequality)**. Let $1 < \mathtt{p}$ and $\frac{1}{\mathtt{p}} + \frac{1}{\mathtt{q}} = 1$. Consider two real-valued functions Φ and $\daleth$ on $[\varepsilon_1, \varepsilon_2]$ with $|\Phi|^{\mathtt{p}}, |\daleth|^{\mathtt{q}}$ are also integrable on $[\varepsilon_1, \varepsilon_2]$, then one has*

$$
\int_{\varepsilon_1}^{\varepsilon_2} |\Phi(\flat)\daleth(\flat)|\mathrm{d}\flat \le \left(\int_{\varepsilon_1}^{\varepsilon_2} |\Phi(\flat)|^{\mathtt{p}}\mathrm{d}\flat\right)^{\frac{1}{\mathtt{p}}} \left(\int_{\varepsilon_1}^{\varepsilon_2} |\daleth(\flat)|^{\mathtt{q}}\mathrm{d}\flat\right)^{\frac{1}{\mathtt{q}}}
$$

Another generalized variant of Hölder's inequality is defined as follows.

Theorem 8 (see [16]). *Let two real-valued functions Φ and $\daleth$ on $[\varepsilon_1, \varepsilon_2]$ and $|\Phi|, |\Phi||\daleth|^{\mathtt{q}}$ are also integrable on $[\varepsilon_1, \varepsilon_2]$, then one has*

$$
\int_{\varepsilon_1}^{\varepsilon_2} |\Phi(\flat)\daleth(\flat)|\mathrm{d}\flat \le \left(\int_{\varepsilon_1}^{\varepsilon_2} |\Phi(\flat)|\mathrm{d}\flat\right)^{1-\frac{1}{\mathtt{q}}} \left(\int_{\varepsilon_1}^{\varepsilon_2} |\Phi(\flat)||\daleth(\flat)|^{\mathtt{q}}\mathrm{d}\flat\right)^{\frac{1}{\mathtt{q}}}.
$$

Theorem 9 (see [16]). ***(Young's inequality)**. Consider $\mathtt{p}, \mathtt{q}$ be positive real numbers satisfying $\frac{1}{\mathtt{p}} + \frac{1}{\mathtt{q}} = 1$. Then if $\Phi, \mathscr{H}$ are nonnegative functions then we have,*

$$
\mathscr{G}\mathscr{H} \le \frac{\Phi^{\mathtt{p}}}{\mathtt{p}} + \frac{\mathscr{H}^{\mathtt{q}}}{\mathtt{q}},
$$

and equality holds iff $\Phi^{\mathtt{p}} = \mathscr{H}^{\mathtt{q}}$.

The following two below Lemmas [29] also play a very crucial role in creating our main findings.

[see [29]]

Let $\Phi : \mathfrak{B}^\circ \subset \mathscr{R} \to \mathscr{R}$ is a differentiable mapping on $\mathfrak{B}^\circ$, where $\varepsilon_1, \varepsilon_2 \in \mathfrak{B}^\circ$, with $\varepsilon_1 < \varepsilon_2$. If $\Phi' \in \mathcal{L}[\varepsilon_1, \varepsilon_2]$ (space of all measurable function), then one has

$$\begin{aligned}\mathfrak{B}_{\mathtt{k}}(\Phi, \varepsilon_1, \varepsilon_2) &= \sum_{\jmath=0}^{\mathtt{k}-1} \frac{1}{2\mathtt{k}} \left[\Phi\left(\frac{(\mathtt{k}-\jmath)\varepsilon_1 + \jmath\varepsilon_2}{\mathtt{k}} \right) + \Phi\left(\frac{(\mathtt{k}-\jmath-1)\varepsilon_1 + (\jmath+1)\varepsilon_2}{\mathtt{k}} \right) \right] \\ &\quad - \frac{1}{\varepsilon_2 - \varepsilon_1} \int_{\varepsilon_1}^{\varepsilon_2} \Phi(\varsigma) d\varsigma \\ &= \sum_{\jmath=0}^{\mathtt{k}-1} \frac{\varepsilon_2 - \varepsilon_1}{2\mathtt{k}^2} \left[\int_0^1 (1 - 2\flat)\Phi' \left(\flat \frac{(\mathtt{k}-\jmath)\varepsilon_1 + \jmath\varepsilon_2}{\mathtt{k}} \right.\right. \\ &\quad \left.\left. + (1-\flat)\frac{(\mathtt{k}-\jmath-1)\varepsilon_1 + (\jmath+1)\varepsilon_2}{\mathtt{k}} \right) d\flat \right].\end{aligned}$$

holds. [see [32]] Let $\varepsilon_1 < \varepsilon_2, \varepsilon_1, \varepsilon_2 \in \mathscr{R}^+, \Phi : \mathscr{R}^+ \to \mathscr{R}^+$ is a differentiable mapping. If $\Phi'' \in \mathcal{L}[\varepsilon_1, \varepsilon_2]$, for each $\varsigma \in (0, 1]$, then one has

$$\begin{aligned}&\frac{1}{\varepsilon_2 - \varepsilon_1} \left[{}^{\mathtt{AB}}\mathtt{I}^{\varsigma}_{\frac{\varepsilon_2+\varepsilon_1}{2}} \{\Phi(\varepsilon_1)\} + {}^{\mathtt{AB}}_{\frac{\varepsilon_2+\varepsilon_1}{2}}\mathtt{I}^{\varsigma}_{\varepsilon_2} \{\Phi(\varepsilon_2)\} \right] \\ &- \frac{1}{(\varepsilon_2 - \varepsilon_1)\mathbb{B}(\varsigma)} [\Phi(\varepsilon_1) + \Phi(\varepsilon_2)] - \frac{(\varepsilon_2 - \varepsilon_1)^{\varsigma-1}}{2^{\varsigma-1}\mathbb{B}(\varsigma)\Gamma(\varsigma)} \Phi\left(\frac{\varepsilon_2 + \varepsilon_1}{2} \right) \\ &= \frac{(\varepsilon_2 - \varepsilon_1)^{\varsigma-1}}{2(\varsigma+1)\mathbb{B}(\varsigma)\Gamma(\varsigma)} \int_0^1 \mathtt{w}^{\varsigma}(\flat) \left[\Phi''(\flat\varepsilon_1 + (1-\flat)\varepsilon_2) + \Phi''(\flat\varepsilon_2 + (1-\flat)\varepsilon_1) \right] d\flat, \\ &\text{where } \mathtt{w}^{\varsigma}(\flat) = \begin{cases} \flat^{\varsigma+1}, \flat \in \left[0, \frac{1}{2}\right), \\ (1-\flat)^{\varsigma+1}, \flat \in \left[\frac{1}{2}, 1\right]. \end{cases}\end{aligned}$$

In [57], the authors introduced these type of inequalities that utilize the $\mathtt{s}$-convexity with the help of Lemma 2.1.

Theorem 10. *Let $\Phi : \mathfrak{B} \subset \mathscr{R} \to \mathscr{R}$ is a differentiable mapping on $\mathfrak{B}^\circ$, where $\varepsilon_1, \varepsilon_2 \in \mathfrak{B}^\circ$, with $\varepsilon_1 < \varepsilon_2$. If $|\Phi'|^{\mathtt{q}}$ is $\mathtt{s}$-convex on $[\varepsilon_1, \varepsilon_2]$ for some $\mathtt{q} > 1$, then one has*

$$\begin{aligned}&|\mathfrak{B}_{\mathtt{k}}(\Phi, \varepsilon_1, \varepsilon_2)| \\ &\leq \sum_{\jmath=0}^{\mathtt{k}-1} \frac{\varepsilon_2 - \varepsilon_1}{2\mathtt{k}^2} \left(\frac{1}{\mathtt{p}+1} \right)^{\frac{1}{\mathtt{p}}} \left(\frac{1}{\mathtt{s}+1} \right)^{\frac{1}{\mathtt{q}}} \\ &\times \left[\left| \Phi'\left(\frac{(\mathtt{k}-\jmath)\varepsilon_1 + \jmath\varepsilon_2}{\mathtt{k}} \right) \right|^{\mathtt{q}} + \left| \Phi'\left(\frac{(\mathtt{k}-\jmath-1)\varepsilon_1 + (\jmath+1)\varepsilon_2}{\mathtt{k}} \right) \right|^{\mathtt{q}} \right]^{\frac{1}{\mathtt{q}}}\end{aligned}$$

holds, where $\frac{1}{\mathtt{p}} + \frac{1}{\mathtt{q}} = 1$.

3. The main results

This section uses the cr-h-Godunova-Levin function to build multiple forms of Hermite-Hadamard inequality, with several particular cases.

Theorem 11. *Let* $\mathsf{h} : (0,1) \to \mathscr{R}^+$ *and* $\mathsf{h} \neq 0$. *Let* $\Phi : [\varepsilon_1, \varepsilon_2] \to \mathscr{R}^+_{\mathcal{I}}$ *is cr-*h*-Godunova-Levin mapping,* $\varepsilon_1, \varepsilon_2 \in \mathscr{R}^+, \varepsilon_1 < \varepsilon_2$. *If* $\Phi \in \mathcal{L}[\varepsilon_1, \varepsilon_2]$, *then the following relation holds true:*

$$\begin{aligned} &\mathsf{h}\left(\frac{1}{2}\right)\frac{(\varepsilon_2-\varepsilon_1)^{\varsigma}}{\mathbb{B}(\varsigma)\Gamma(\varsigma)}\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right)+\frac{1-\varsigma}{\mathbb{B}(\varsigma)}\left[\Phi(\varepsilon_1)+\Phi(\varepsilon_2)\right] \\ &\preceq_{cr}{}^{\mathtt{AB}}_{\varepsilon_1}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{\Phi(\varepsilon_2)\}+{}^{\mathtt{AB}}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{\Phi(\varepsilon_1)\} \\ &\preceq_{cr}\left[\frac{\Phi(\varepsilon_1)+\Phi(\varepsilon_2)}{\mathbb{B}(\varsigma)}\right]\left[1-\varsigma+\frac{\varsigma(\varepsilon_2-\varepsilon_1)^{\varsigma}}{\Gamma(\varsigma)}\right. \\ &\qquad\left.\times\int_0^1 \flat^{\varsigma-1}\left(\frac{1}{\mathsf{h}(\flat)}+\frac{1}{\mathsf{h}(1-\flat)}\right)\mathrm{d}\flat\right], \end{aligned} \tag{3}$$

where $\varsigma \in (0,1)$.

Proof. As $\Phi \in \mathtt{SGX}(\mathsf{h}, [\varepsilon_1, \varepsilon_2], \mathscr{R}^+_{\mathcal{I}})$, we have

$$\Phi\left(\frac{\nu_1+\nu_2}{2}\right) \preceq_{cr} \frac{1}{\left[\mathsf{h}\left(\frac{1}{2}\right)\right]}\left[\Phi(\nu_1)+\Phi(\nu_2)\right],$$

let $\nu_1 = \flat\varepsilon_1 + (1-\flat)\varepsilon_2$, $\nu_2 = \flat\varepsilon_2 + (1-\flat)\varepsilon_1$, the above relation becomes as

$$\mathsf{h}\left(\frac{1}{2}\right)\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right) \preceq_{cr} \left[\Phi(\flat\varepsilon_1+(1-\flat)\varepsilon_2)+\Phi(\flat\varepsilon_2+(1-\flat)\varepsilon_1)\right]. \tag{4}$$

Multiplying by $\flat^{\varsigma-1}$ in (4) and integrating, we have

$$\begin{aligned} \frac{1}{\varsigma}\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right) \preceq_{cr} \frac{1}{\mathsf{h}\left(\frac{1}{2}\right)}&\left[\int_0^1 \flat^{\varsigma-1}\Phi(\flat\varepsilon_1+(1-\flat)\varepsilon_2)\mathrm{d}\flat\right. \\ &\left.+\int_0^1 \flat^{\varsigma-1}\Phi(\flat\varepsilon_2+(1-\flat)\varepsilon_1)\mathrm{d}\flat\right], \end{aligned}$$

that is

$$\begin{aligned} \frac{\mathsf{h}\left(\frac{1}{2}\right)}{\varsigma}\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right) \preceq_{cr} &\int_0^1 \flat^{\varsigma-1}\Phi(\flat\varepsilon_1+(1-\flat)\varepsilon_2)\mathrm{d}\flat \\ &+\int_0^1 \flat^{\varsigma-1}\Phi(\flat\varepsilon_2+(1-\flat)\varepsilon_1)\mathrm{d}\flat. \end{aligned}$$

Multiplying the above relation with $\frac{\varsigma(\varepsilon_2-\varepsilon_1)^\varsigma}{\mathbb{B}(\varsigma)\Gamma(\varsigma)}$ and adding the expression $\frac{1-\varsigma}{\mathbb{B}(\varsigma)}\left[\Phi(\varepsilon_1)+\Phi(\varepsilon_2)\right]$, we get that

$$\begin{aligned}
&\mathsf{h}\left(\frac{1}{2}\right)\frac{(\varepsilon_2-\varepsilon_1)^\varsigma}{\mathbb{B}(\varsigma)\Gamma(\varsigma)}\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right)+\frac{1-\varsigma}{\mathbb{B}(\varsigma)}\left[\Phi(\varepsilon_1)+\Phi(\varepsilon_2)\right]\\
&\preceq_{cr}\frac{\varsigma(\varepsilon_2-\varepsilon_1)^\varsigma}{\mathbb{B}(\varsigma)\Gamma(\varsigma)}\int_0^1 \flat^{\varsigma-1}\Phi(\flat\varepsilon_1+(1-\flat)\varepsilon_2)d\flat\\
&\quad+\frac{\varsigma(\varepsilon_2-\varepsilon_1)^\varsigma}{\mathbb{B}(\varsigma)\Gamma(\varsigma)}\int_0^1 \flat^{\varsigma-1}\Phi(\flat\varepsilon_2+(1-\flat)\varepsilon_1)d\flat+\frac{1-\varsigma}{\mathbb{B}(\varsigma)}\left[\Phi(\varepsilon_1)+\Phi(\varepsilon_2)\right].
\end{aligned}$$

In the final two integrals of the preceding relation, let $\mathsf{a}=\flat\varepsilon_1+(1-\flat)\varepsilon_2$ and $\mathbb{B}=\flat\varepsilon_2+(1-\flat)\varepsilon_1$ respectively, we have

$$\mathsf{h}\left(\frac{1}{2}\right)\frac{(\varepsilon_2-\varepsilon_1)^\varsigma}{\mathbb{B}(\varsigma)\Gamma(\varsigma)}\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right)+\frac{1-\varsigma}{\mathbb{B}(\varsigma)}\left[\Phi(\varepsilon_1)+\Phi(\varepsilon_2)\right]\leq {}^{\mathtt{AB}}_{\varepsilon_1}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{\Phi(\varepsilon_2)\}+{}^{\mathtt{AB}}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{\Phi(\varepsilon_1)\},$$

so the first relation of (3) holds.
Taking into account Definition 4, we have

$$\Phi(\flat\varepsilon_1+(1-\flat)\varepsilon_2)\preceq_{cr}\frac{\Phi(\varepsilon_1)}{\mathsf{h}(\flat)}+\frac{\Phi(\varepsilon_2)}{\mathsf{h}(1-\flat)}.$$

. Multiplying aforementioned result with $\flat^{\varsigma-1}$, and integrating, we have

$$\begin{aligned}
&\int_0^1 \flat^{\varsigma-1}\Phi(\flat\varepsilon_1+(1-\flat)\varepsilon_2)d\flat\\
&\qquad\preceq_{cr}\Phi(\varepsilon_1)\int_0^1\frac{\flat^{\varsigma-1}d\flat}{\mathsf{h}(\flat)}+\Phi(\varepsilon_2)\int_0^1\frac{\flat^{\varsigma-1}d\flat}{\mathsf{h}(1-\flat)}.
\end{aligned}\tag{5}$$

Multiplying both sides of (5) by $\frac{\varsigma(\varepsilon_2-\varepsilon_1)^\varsigma}{\mathbb{B}(\varsigma)\Gamma(\varsigma)}$ and adding the expression $\frac{1-\varsigma}{\mathbb{B}(\varsigma)}\Phi(\varepsilon_2)$ to both sides of the desired relation, we get

$$\begin{aligned}
&\frac{\varsigma(\varepsilon_2-\varepsilon_1)^\varsigma}{\mathbb{B}(\varsigma)\Gamma(\varsigma)}\int_0^1 \flat^{\varsigma-1}\Phi(\flat\varepsilon_1+(1-\flat)\varepsilon_2)d\flat+\frac{1-\varsigma}{\mathbb{B}(\varsigma)}\Phi(\varepsilon_2)\\
&\quad\preceq_{cr}\frac{\varsigma(\varepsilon_2-\varepsilon_1)^\varsigma}{\mathbb{B}(\varsigma)\Gamma(\varsigma)}\left[\Phi(\varepsilon_1)\int_0^1\frac{\flat^{\varsigma-1}d\flat}{\mathsf{h}(\flat)}\right.\\
&\qquad\left.+\Phi(\varepsilon_2)\int_0^1\frac{\flat^{\varsigma-1}d\flat}{\mathsf{h}(1-\flat)}\right]+\frac{1-\varsigma}{\mathbb{B}(\varsigma)}\Phi(\varepsilon_2).
\end{aligned}\tag{6}$$

Making a modification in the previous integral of the preceding relation, $\mathsf{a}=\flat\varepsilon_1+(1-\flat)\varepsilon_2$, then the above relation become as

$$\begin{aligned}
{}^{\mathtt{AB}}_{\varepsilon_1}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{\Phi(\varepsilon_2)\}&\preceq_{cr}\frac{\varsigma(\varepsilon_2-\varepsilon_1)^\varsigma}{\mathbb{B}(\varsigma)\Gamma(\varsigma)}\left[\Phi(\varepsilon_1)\int_0^1\frac{\flat^{\varsigma-1}d\flat}{\mathsf{h}(\flat)}\right.\\
&\qquad\left.+\Phi(\varepsilon_2)\int_0^1\frac{\flat^{\varsigma-1}d\flat}{\mathsf{h}(1-\flat)}\right]+\frac{1-\varsigma}{\mathbb{B}(\varsigma)}\Phi(\varepsilon_2).
\end{aligned}\tag{7}$$

Again by Definition 4, we have

$$\Phi(\flat\varepsilon_1 + (1-\flat)\varepsilon_2) \preceq_{cr} \frac{\Phi(\varepsilon_1)}{\mathsf{h}(\flat)} + \frac{\Phi(\varepsilon_2)}{\mathsf{h}(1-\flat)}.$$

Multiplying aforementioned relation with $\flat^{\varsigma-1}$, and integrating, we have

$$\begin{aligned}\frac{\varsigma(\varepsilon_2-\varepsilon_1)^{\varsigma}}{\mathbb{B}(\varsigma)\Gamma(\varsigma)}\int_0^1 \flat^{\varsigma-1}\Phi(\flat\varepsilon_2+(1-\flat)\varepsilon_1)\mathrm{d}\flat + \frac{1-\varsigma}{\mathbb{B}(\varsigma)}\Phi(\varepsilon_1)\\ \preceq_{cr} \frac{\varsigma(\varepsilon_2-\varepsilon_1)^{\varsigma}}{\mathbb{B}(\varsigma)\Gamma(\varsigma)}\left[\Phi(\varepsilon_2)\int_0^1 \frac{\flat^{\varsigma-1}\mathrm{d}\flat}{\mathsf{h}(\flat)}\right.\\ \left.+\Phi(\varepsilon_1)\int_0^1 \frac{\flat^{\varsigma-1}\mathrm{d}\flat}{\mathsf{h}(1-\flat)}\right] + \frac{1-\varsigma}{\mathbb{B}(\varsigma)}\Phi(\varepsilon_1).\end{aligned}$$

Making a modification in the previous integral of the preceding relation with some dummy variable , $\mathsf{b} = \flat\varepsilon_2 + (1-\flat)\varepsilon_1$, then the above relation becomes

$$\begin{aligned}{}^{\mathtt{AB}}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{\Phi(\varepsilon_1)\} \preceq_{cr} \frac{\varsigma(\varepsilon_2-\varepsilon_1)^{\varsigma}}{\mathbb{B}(\varsigma)\Gamma(\varsigma)}\left[\Phi(\varepsilon_2)\int_0^1 \frac{\flat^{\varsigma-1}\mathrm{d}\flat}{\mathsf{h}(\flat)}\right.\\ \left.+\Phi(\varepsilon_1)\int_0^1 \frac{\flat^{\varsigma-1}\mathrm{d}\flat}{\mathsf{h}(1-\flat)}\right] + \frac{1-\varsigma}{\mathbb{B}(\varsigma)}\Phi(\varepsilon_1).\end{aligned} \tag{8}$$

Adding (7) and (8), we can get that the second relation of (3). This finishes the proof.

Example 2. *Let* $\Phi : [1,4] \to \mathscr{R}^+_{\mathcal{I}}$ *defined as* $\Phi(\mu) = \left[2e^{\mu}+1, 3e^{\mu}+\frac{\sqrt{\mu}}{3}\right]$ *with* $\mathsf{h}(\flat) = \frac{1}{\flat}, \varsigma = \frac{1}{2}$, *then we have*

$$\begin{aligned}&\mathsf{h}\left(\frac{1}{2}\right)\frac{(\varepsilon_2-\varepsilon_1)^{\varsigma}}{\mathbb{B}(\varsigma)\Gamma(\varsigma)}\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right) + \frac{1-\varsigma}{\mathbb{B}(\varsigma)}\left[\Phi(\varepsilon_1)+\Phi(\varepsilon_2)\right]\\ &\approx [81.77390, 103.32659],\\ &{}^{\mathtt{AB}}_{\varepsilon_1}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{\Phi(\varepsilon_2)\} + {}^{\mathtt{AB}}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{\Phi(\varepsilon_1)\}\\ &\approx [90.33565, 131.54364].\end{aligned}$$

and

$$\begin{aligned}&\left[\frac{\Phi(\varepsilon_1)+\Phi(\varepsilon_2)}{\mathbb{B}(\varsigma)}\right]\left[1-\varsigma+\frac{\varsigma(\varepsilon_2-\varepsilon_1)^{\varsigma}}{\Gamma(\varsigma)}\right.\\ &\left.\times\int_0^1 \flat^{\varsigma-1}\left(\frac{1}{\mathsf{h}(\flat)}+\frac{1}{\mathsf{h}(1-\flat)}\right)\mathrm{d}\flat\right]\\ &= \left[e^4+e+\frac{\sqrt{3}\sqrt{\pi}\left(2e^4+2e+2\right)}{3\pi}+1, \left(3e^4+3e+1\right)\left(\frac{1}{2}+\frac{\sqrt{3}\sqrt{\pi}}{3\pi}\right)\right]\\ &\approx [96.30783, 142.81028].\end{aligned}$$

Thus, we have

$$[81.77390, 103.32659] \preceq_{cr} [90.33565, 131.54364] \preceq_{cr} [96.30783, 142.81028].$$

Consequently, Theorem 11 is correct.

The different types of settings allow us to get results for other types of generalized convex mappings, as described in the remark below.

Remark 3. *(i) If $\mathsf{h}(\flat) = \frac{1}{\flat^s}$, then Theorem 11 yields an outcome for the cr-*s*-convex function for* AB *integral operators:*

$$\begin{aligned} &2^s \frac{(\varepsilon_2-\varepsilon_1)^\varsigma}{\mathbb{B}(\varsigma)\Gamma(\varsigma)} \Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right) + \frac{1-\varsigma}{\mathbb{B}(\varsigma)}\left[\Phi(\varepsilon_1)+\Phi(\varepsilon_2)\right] \\ &\leq {}^{\mathsf{AB}}_{\varepsilon_1}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{\Phi(\varepsilon_2)\} + {}^{\mathsf{AB}}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{\Phi(\varepsilon_1)\} \\ &\leq \left[\frac{\Phi(\varepsilon_1)+\Phi(\varepsilon_2)}{\mathbb{B}(\varsigma)}\right]\left[1-\varsigma+\frac{\varsigma(\varepsilon_2-\varepsilon_1)^\varsigma}{\Gamma(\varsigma)(\mathsf{s}+\varsigma)} + \frac{\varsigma(\varepsilon_2-\varepsilon_1)^\varsigma}{\Gamma(\varsigma)}\frac{\Gamma(\varsigma)\Gamma(\mathsf{s}+1)}{\Gamma(\varsigma+\mathsf{s}+2)}\right]. \end{aligned} \tag{9}$$

*(ii) If $\mathsf{h}(\flat) = 1$, then Theorem 11 yields an outcome for the cr-*p*-convex function for* AB *integral operators:*

$$\begin{aligned} &\frac{(\varepsilon_2-\varepsilon_1)^\varsigma}{\mathbb{B}(\varsigma)\Gamma(\varsigma)} \Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right) + \frac{1-\varsigma}{\mathbb{B}(\varsigma)}\left[\Phi(\varepsilon_1)+\Phi(\varepsilon_2)\right] \\ &\qquad \leq {}^{\mathsf{AB}}_{\varepsilon_1}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{\Phi(\varepsilon_2)\} + {}^{\mathsf{AB}}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{\Phi(\varepsilon_1)\} \\ &\qquad \leq \left[\frac{\Phi(\varepsilon_1)+\Phi(\varepsilon_2)}{\mathbb{B}(\varsigma)}\right]\left[1-\varsigma+\frac{2(\varepsilon_2-\varepsilon_1)^\varsigma}{\Gamma(\varsigma)}\right]. \end{aligned} \tag{10}$$

Theorem 12. *Let $\mathsf{h} : (0,1) \to \mathscr{R}^+$ and $\mathsf{h} \neq 0$. Let $\Phi : [\varepsilon_1, \varepsilon_2] \to \mathscr{R}^+_{\mathcal{I}}$ is cr-*h*-Godunova-Levin mapping, $\varepsilon_1, \varepsilon_2 \in \mathscr{R}^+, \varepsilon_1 < \varepsilon_2$ and $\daleth : [\varepsilon_1, \varepsilon_2] \to \mathscr{R}^+$ is symmetric about $\frac{\varepsilon_1+\varepsilon_2}{2}$. If $\Phi \in \mathcal{L}[\varepsilon_1, \varepsilon_2]$, then the following relation holds true:*

$$\begin{aligned} &\frac{\mathsf{h}\left(\frac{1}{2}\right)}{2}\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right)\left[{}^{\mathsf{AB}}_{\varepsilon_1}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{\daleth(\varepsilon_2)\} + {}^{\mathsf{AB}}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{\daleth(\varepsilon_1)\}\right] - \frac{\mathsf{h}\left(\frac{1}{2}\right)}{2}\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right)\frac{1-\varsigma}{\mathbb{B}(\varsigma)}\left[\daleth(\varepsilon_1)+\daleth(\varepsilon_2)\right] \\ &\qquad + \frac{1-\varsigma}{\mathbb{B}(\varsigma)}\left[\Phi(\varepsilon_1)\daleth(\varepsilon_1)+\Phi(\varepsilon_2)\daleth(\varepsilon_2)\right] \leq {}^{\mathsf{AB}}_{\varepsilon_1}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{(\Phi\daleth(\varepsilon_2)\} + {}^{\mathsf{AB}}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{(\Phi\daleth(\varepsilon_1)\} \\ &\quad \preceq_{cr} \frac{\varsigma(\varepsilon_2-\varepsilon_1)^\varsigma}{\mathbb{B}(\varsigma)\Gamma(\varsigma)}\left[\Phi(\varepsilon_1)+\Phi(\varepsilon_2)\right] \times \int_0^1 \flat^{\varsigma-1}\left[\frac{1}{\mathsf{h}(\flat)}+\frac{1}{\mathsf{h}(1-\flat)}\right]\daleth(\flat\varepsilon_2+(1-\flat)\varepsilon_1)\mathrm{d}\flat \\ &\qquad + \frac{1-\varsigma}{\mathbb{B}(\varsigma)}\left[\Phi(\varepsilon_1)\daleth(\varepsilon_1)+\Phi(\varepsilon_2)\daleth(\varepsilon_2)\right], \end{aligned} \tag{11}$$

where $\varsigma \in (0, 1]$.

Proof. As $\Phi \in \mathtt{SGX}(\mathsf{h}, [\varepsilon_1, \varepsilon_2], \mathscr{R}_{\mathcal{I}}^{+})$, we have

$$\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right) \leq \frac{1}{\mathsf{h}\left(\frac{1}{2}\right)}\big[\Phi(\flat\varepsilon_1+(1-\flat)\varepsilon_2)+\Phi(\flat\varepsilon_2+(1-\flat)\varepsilon_1)\big]. \tag{12}$$

Multiplying above relation with $\mathsf{h}\left(\frac{1}{2}\right)\flat^{\varsigma-1}\daleth(\flat\varepsilon_2+(1-\flat)\varepsilon_1)$, and integrating the desired relation over $(0,1)$, we have

$$\begin{aligned}&\mathsf{h}\left(\frac{1}{2}\right)\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right)\int_0^1 \flat^{\varsigma-1}\daleth(\flat\varepsilon_2+(1-\flat)\varepsilon_1)d\flat\\ &\quad\leq \int_0^1 \flat^{\varsigma-1}\big[\Phi(\flat\varepsilon_1+(1-\flat)\varepsilon_2)+\Phi(\flat\varepsilon_2+(1-\flat)\varepsilon_1)\big]\daleth(\flat\varepsilon_2+(1-\flat)\varepsilon_1)d\flat.\end{aligned}$$

Let $\mathtt{u} = \flat\varepsilon_2+(1-\flat)\varepsilon_1$, then the above relation becomes

$$\begin{aligned}&\mathsf{h}\left(\frac{1}{2}\right)\frac{1}{(\varepsilon_2-\varepsilon_1)^{\varsigma}}\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right)\int_{\varepsilon_1}^{\varepsilon_2}(\mathtt{u}-\varepsilon_1)^{\varsigma-1}\daleth(\mathtt{u})d\mathtt{u}\\ &\preceq_{cr}\frac{1}{(\varepsilon_2-\varepsilon_1)^{\varsigma}}\left[\int_{\varepsilon_1}^{\varepsilon_2}(\mathtt{u}-\varepsilon_1)^{\varsigma-1}\Phi(\varepsilon_2+\varepsilon_1-\mathtt{u})\daleth(\mathtt{u})d\mathtt{u}\right.\\ &\qquad\left.+\int_{\varepsilon_1}^{\varepsilon_2}(\mathtt{u}-\varepsilon_1)^{\varsigma-1}\Phi(\mathtt{u})\daleth(\mathtt{u})d\mathtt{u}\right].\end{aligned}$$

Making a modification in the previous integral of the preceding relation, $\mathtt{v} = \varepsilon_2+\varepsilon_1-\mathtt{u}$, from $\daleth(\varepsilon_2+\varepsilon_1-\mathtt{v}) = \daleth(\mathtt{v})$, one has

$$\begin{aligned}&\mathsf{h}\left(\frac{1}{2}\right)\frac{1}{(\varepsilon_2-\varepsilon_1)^{\varsigma}}\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right)\int_{\varepsilon_1}^{\varepsilon_2}(\mathtt{u}-\varepsilon_1)^{\varsigma-1}\daleth(\mathtt{u})d\mathtt{u}\\ &\quad\leq\frac{1}{(\varepsilon_2-\varepsilon_1)^{\varsigma}}\left[\int_{\varepsilon_1}^{\varepsilon_2}(\varepsilon_2-\mathtt{v})^{\varsigma-1}\Phi(\mathtt{v})\daleth(\mathtt{v})d\mathtt{v}+\int_{\varepsilon_1}^{\varepsilon_2}(\mathtt{u}-\varepsilon_1)^{\varsigma-1}\Phi(\mathtt{u})\daleth(\mathtt{u})d\mathtt{u}\right].\end{aligned}$$

Multiplying above relation with $\frac{\varsigma(\varepsilon_2-\varepsilon_1)^{\varsigma}}{\mathbb{B}(\varsigma)\Gamma(\varsigma)}$ and adding the expression $\frac{1-\varsigma}{\mathbb{B}(\varsigma)}\big[\Phi(\varepsilon_1)+\Phi(\varepsilon_2)\big]$ to both sides of the desired results, we get that

$$\begin{aligned}&\mathsf{h}\left(\frac{1}{2}\right)\frac{\varsigma}{\mathbb{B}(\varsigma)\Gamma(\varsigma)}\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right)\int_{\varepsilon_1}^{\varepsilon_2}(\mathtt{u}-\varepsilon_1)^{\varsigma-1}\daleth(\mathtt{u})d\mathtt{u}+\frac{1-\varsigma}{\mathbb{B}(\varsigma)}\big[\Phi(\varepsilon_1)\daleth(\varepsilon_1)+\Phi(\varepsilon_2)\daleth(\varepsilon_2)\big]\\ &\preceq_{cr}\frac{\varsigma}{\mathbb{B}(\varsigma)\Gamma(\varsigma)}\left[\int_{\varepsilon_1}^{\varepsilon_2}(\varepsilon_2-\mathtt{v})^{\varsigma-1}\Phi(\mathtt{v})\daleth(\mathtt{v})d\mathtt{v}+\int_{\varepsilon_1}^{\varepsilon_2}(\mathtt{u}-\varepsilon_1)^{\varsigma-1}\Phi(\mathtt{u})\daleth(\mathtt{u})d\mathtt{u}\right]\\ &\quad+\frac{1-\varsigma}{\mathbb{B}(\varsigma)}\big[\Phi(\varepsilon_1)\daleth(\varepsilon_1)+\Phi(\varepsilon_2)\daleth(\varepsilon_2)\big].\end{aligned}$$

From this, it can be follows as

$$\begin{aligned}&\mathsf{h}\left(\frac{1}{2}\right)\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right){}^{\mathtt{AB}}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{\daleth(\varepsilon_1)\}\\ &\qquad-\mathsf{h}\left(\frac{1}{2}\right)\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right)\frac{1-\varsigma}{\mathbb{B}(\varsigma)}\daleth(\varepsilon_1)+\frac{1-\varsigma}{\mathbb{B}(\varsigma)}\big[\Phi(\varepsilon_1)\daleth(\varepsilon_1)+\Phi(\varepsilon_2)\daleth(\varepsilon_2)\big]\\ &\quad\preceq_{cr}{}_{\varepsilon_1}^{\mathtt{AB}}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{(\Phi\daleth(\varepsilon_2)\}+{}^{\mathtt{AB}}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{(\Phi\daleth(\varepsilon_1)\}.\end{aligned} \tag{13}$$

Similarly, multiplying $\mathsf{h}\left(\frac{1}{2}\right)\flat^{\varsigma-1}\daleth(\flat\varepsilon_1+(1-\flat)\varepsilon_2)$ on both sides of (12) and integrating, we have

$$\begin{aligned}&\mathsf{h}\left(\frac{1}{2}\right)\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right)\int_0^1 \flat^{\varsigma-1}\daleth(\flat\varepsilon_1+(1-\flat)\varepsilon_2)d\flat\\ &\quad\leq \int_0^1 \flat^{\varsigma-1}\big[\Phi(\flat\varepsilon_1+(1-\flat)\varepsilon_2)+\Phi(\flat\varepsilon_2+(1-\flat)\varepsilon_1)\big]\daleth(\flat\varepsilon_1+(1-\flat)\varepsilon_2)d\flat.\end{aligned}$$

Let $\mathtt{u}=\flat\varepsilon_1+(1-\flat)\varepsilon_2$, then the above result becomes

$$\begin{aligned}&\mathsf{h}\left(\frac{1}{2}\right)\frac{1}{(\varepsilon_2-\varepsilon_1)^{\varsigma}}\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right)\int_{\varepsilon_1}^{\varepsilon_2}(\varepsilon_2-\mathrm{u})^{\varsigma-1}\daleth(\mathrm{u})d\mathrm{u}\\ &\preceq_{cr}\frac{1}{(\varepsilon_2-\varepsilon_1)^{\varsigma}}\left[\int_{\varepsilon_1}^{\varepsilon_2}(\varepsilon_2-\mathrm{u})^{\varsigma-1}\Phi(\mathrm{u})\daleth(\mathrm{u})d\mathrm{u}\right.\\ &\quad\left.+\int_{\varepsilon_1}^{\varepsilon_2}(\varepsilon_2-\mathrm{u})^{\varsigma-1}\Phi(\varepsilon_2+\varepsilon_1-\mathrm{u})\daleth(\mathrm{u})d\mathrm{u}\right].\end{aligned}$$

Making a modification in the previous integral of the preceding relation, $\mathtt{v}=\varepsilon_2+\varepsilon_1-\mathtt{u}$, $\daleth(\varepsilon_2+\varepsilon_1-\mathtt{v})=\daleth(\mathtt{v})$, we have

$$\begin{aligned}&\mathsf{h}\left(\frac{1}{2}\right)\frac{1}{(\varepsilon_2-\varepsilon_1)^{\varsigma}}\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right)\int_{\varepsilon_1}^{\varepsilon_2}(\varepsilon_2-\mathrm{u})^{\varsigma-1}\daleth(\mathrm{u})d\mathrm{u}\\ &\quad\leq\frac{1}{(\varepsilon_2-\varepsilon_1)^{\varsigma}}\left[\int_{\varepsilon_1}^{\varepsilon_2}(\varepsilon_2-\mathrm{u})^{\varsigma-1}\Phi(\mathrm{u})\daleth(\mathrm{u})d\mathrm{u}+\int_{\varepsilon_1}^{\varepsilon_2}(\mathrm{v}-\varepsilon_1)^{\varsigma-1}\Phi(\mathrm{v})\daleth(\mathrm{v})d\mathrm{v}\right].\end{aligned}$$

Multiplying above relation with $\frac{\varsigma(\varepsilon_2-\varepsilon_1)^{\varsigma}}{\mathbb{B}(\varsigma)\Gamma(\varsigma)}$ and adding the expression $\frac{1-\varsigma}{\mathbb{B}(\varsigma)}\big[\Phi(\varepsilon_1)+\Phi(\varepsilon_2)\big]$ to both sides of the desired result, we get that

$$\begin{aligned}&\mathsf{h}\left(\frac{1}{2}\right)\frac{\varsigma}{\mathbb{B}(\varsigma)\Gamma(\varsigma)}\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right)\int_{\varepsilon_1}^{\varepsilon_2}(\varepsilon_2-\mathrm{u})^{\varsigma-1}\daleth(\mathrm{u})d\mathrm{u}\\ &\quad+\frac{1-\varsigma}{\mathbb{B}(\varsigma)}\big[\Phi(\varepsilon_1)\daleth(\varepsilon_1)+\Phi(\varepsilon_2)\daleth(\varepsilon_2)\big]\\ &\preceq_{cr}\frac{\varsigma}{\mathbb{B}(\varsigma)\Gamma(\varsigma)}\left[\int_{\varepsilon_1}^{\varepsilon_2}(\varepsilon_2-\mathrm{u})^{\varsigma-1}\Phi(\mathrm{u})\daleth(\mathrm{u})d\mathrm{u}+\int_{\varepsilon_1}^{\varepsilon_2}(\mathrm{v}-\varepsilon_1)^{\varsigma-1}\Phi(\mathrm{v})\daleth(\mathrm{v})d\mathrm{v}\right]\\ &\quad+\frac{1-\varsigma}{\mathbb{B}(\varsigma)}\big[\Phi(\varepsilon_1)\daleth(\varepsilon_1)+\Phi(\varepsilon_2)\daleth(\varepsilon_2)\big].\end{aligned}$$

From this, it can be follows that

$$\begin{aligned}&\mathsf{h}\left(\frac{1}{2}\right)\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right){}^{\mathtt{AB}}_{\varepsilon_1}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{\daleth(\varepsilon_2)\}\\ &\quad-\mathsf{h}\left(\frac{1}{2}\right)\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right)\frac{1-\varsigma}{\mathbb{B}(\varsigma)}\daleth(\varepsilon_2)+\frac{1-\varsigma}{\mathbb{B}(\varsigma)}\big[\Phi(\varepsilon_1)\daleth(\varepsilon_1)+\Phi(\varepsilon_2)\daleth(\varepsilon_2)\big]\\ &\preceq_{cr}{}^{\mathtt{AB}}_{\varepsilon_1}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{(\Phi\daleth(\varepsilon_2)\}+{}^{\mathtt{AB}}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{(\Phi\daleth(\varepsilon_1)\}.\end{aligned}\tag{14}$$

Adding (13) and (14), we can get the first relation in (11). Now again taking into account Definition 4, we have

$$\Phi(\flat\varepsilon_1+(1-\flat)\varepsilon_2) \preceq_{cr} \frac{\Phi(\varepsilon_1)}{\mathsf{h}(\flat)}+\frac{\Phi(\varepsilon_2)}{\mathsf{h}(1-\flat)},$$
$$\Phi(\flat\varepsilon_2+(1-\flat)\varepsilon_1) \preceq_{cr} \frac{\Phi(\varepsilon_2)}{\mathsf{h}(\flat)}+\frac{\Phi(\varepsilon_1)}{\mathsf{h}(1-\flat)},$$

adding the above two relations yields that

$$\Phi(\flat\varepsilon_1+(1-\flat)\varepsilon_2)+\Phi(\flat\varepsilon_2+(1-\flat)\varepsilon_1) \leq \left[\frac{1}{\mathsf{h}(\flat)}+\frac{1}{\mathsf{h}(1-\flat)}\right]\big[\Phi(\varepsilon_1)+\Phi(\varepsilon_2)\big].$$

Multiplying aforementioned result with $\flat^{\varsigma-1}\gimel(\flat\varepsilon_2+(1-\flat)\varepsilon_1)$ and integrating, we have

$$\int_0^1 \flat^{\varsigma-1}\big[\Phi(\flat\varepsilon_1+(1-\flat)\varepsilon_2)+\Phi(\flat\varepsilon_2+(1-\flat)\varepsilon_1)\big]\gimel(\flat\varepsilon_2+(1-\flat)\varepsilon_1)\mathrm{d}\flat$$
$$\leq \big[\Phi(\varepsilon_1)+\Phi(\varepsilon_2)\big]\int_0^1 \flat^{\varsigma-1}\left[\frac{1}{\mathsf{h}(\flat)}+\frac{1}{\mathsf{h}(1-\flat)}\right]\gimel(\flat\varepsilon_2+(1-\flat)\varepsilon_1)\mathrm{d}\flat.$$

Making a modification in the previous integral of the preceding relation, $\mathtt{v}=\varepsilon_2+\varepsilon_1-\mathtt{u}$, from $\gimel(\varepsilon_2+\varepsilon_1-\mathtt{v})=\gimel(\mathtt{v})$, we have

$$\frac{1}{(\varepsilon_2-\varepsilon_1)^{\varsigma}}\left[\int_{\varepsilon_1}^{\varepsilon_2}(\varepsilon_2-\mathtt{v})^{\varsigma-1}\Phi(\mathtt{v})\gimel(\mathtt{v})\mathrm{d}\mathtt{v}+\int_{\varepsilon_1}^{\varepsilon_2}(\mathtt{u}-\varepsilon_1)^{\varsigma-1}\Phi(\mathtt{u})\gimel(\mathtt{u})\mathrm{d}\mathtt{u}\right]$$
$$\leq \big[\Phi(\varepsilon_1)+\Phi(\varepsilon_2)\big]\int_0^1 \flat^{\varsigma-1}\left[\frac{1}{\mathsf{h}(\flat)}+\frac{1}{\mathsf{h}(1-\flat)}\right]\gimel(\flat\varepsilon_2+(1-\flat)\varepsilon_1)\mathrm{d}\flat.$$

Multiplying above relation with $\frac{\varsigma(\varepsilon_2-\varepsilon_1)^{\varsigma}}{\mathbb{B}(\varsigma)\Gamma(\varsigma)}$ and adding the expression $\frac{1-\varsigma}{\mathbb{B}(\varsigma)}\big[\Phi(\varepsilon_1)+\Phi(\varepsilon_2)\big]$, we have

$$\frac{\varsigma}{\mathbb{B}(\varsigma)\Gamma(\varsigma)}\left[\int_{\varepsilon_1}^{\varepsilon_2}(\varepsilon_2-\mathtt{v})^{\varsigma-1}\Phi(\mathtt{v})\gimel(\mathtt{v})\mathrm{d}\mathtt{v}+\int_{\varepsilon_1}^{\varepsilon_2}(\mathtt{u}-\varepsilon_1)^{\varsigma-1}\Phi(\mathtt{u})\gimel(\mathtt{u})\mathrm{d}\mathtt{u}\right]$$
$$+\frac{1-\varsigma}{\mathbb{B}(\varsigma)}\big[\Phi(\varepsilon_1)\gimel(\varepsilon_1)+\Phi(\varepsilon_2)\gimel(\varepsilon_2)\big]$$
$$\preceq_{cr}\frac{\varsigma(\varepsilon_2-\varepsilon_1)^{\varsigma}}{\mathbb{B}(\varsigma)\Gamma(\varsigma)}\big[\Phi(\varepsilon_1)+\Phi(\varepsilon_2)\big]$$
$$\times\int_0^1 \flat^{\varsigma-1}\left[\frac{1}{\mathsf{h}(\flat)}+\frac{1}{\mathsf{h}(1-\flat)}\right]\gimel(\flat\varepsilon_2+(1-\flat)\varepsilon_1)\mathrm{d}\flat$$
$$+\frac{1-\varsigma}{\mathbb{B}(\varsigma)}\big[\Phi(\varepsilon_1)\gimel(\varepsilon_1)+\Phi(\varepsilon_2)\gimel(\varepsilon_2)\big],$$

that is

$${}^{\mathtt{AB}}_{\varepsilon_1}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{(\Phi\gimel(\varepsilon_2)\}+{}^{\mathtt{AB}}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{(\Phi\gimel(\varepsilon_1)\}$$

$$\begin{aligned}&\preceq_{cr}\frac{\varsigma(\varepsilon_2-\varepsilon_1)^{\varsigma}}{\mathbb{B}(\varsigma)\Gamma(\varsigma)}\left[\Phi(\varepsilon_1)+\Phi(\varepsilon_2)\right]\\&\quad\times\int_0^1 \flat^{\varsigma-1}\left[\frac{1}{\mathsf{h}(\flat)}+\frac{1}{\mathsf{h}(1-\flat)}\right]\daleth(\flat\varepsilon_2+(1-\flat)\varepsilon_1)\mathrm{d}\flat\\&\quad+\frac{1-\varsigma}{\mathbb{B}(\varsigma)}\left[\Phi(\varepsilon_1)\daleth(\varepsilon_1)+\Phi(\varepsilon_2)\daleth(\varepsilon_2)\right],\end{aligned}$$

so the second relation holds true in (11). This concludes the proof.

Example 3. *Let* $\Phi:[1,4]\to\mathscr{R}_{\mathcal{I}}^{+}$ *defined as* $\Phi(\mu)=\left[2e^{\mu}+1,3e^{\mu}+\frac{\sqrt{\mu}}{3}\right]$ *with* $\mathsf{h}_1(\flat)=\frac{1}{\flat},\eta=\frac{1}{2}$ *and a real-valued symmetric functions are defined as* $\daleth(\eth)=\eth-1$ *for* $\eth\in\left[1,\frac{5}{2}\right]$ *and* $\daleth(\eth)=-\eth+4$ *for* $\eth\in\left[\frac{5}{2},4\right]$*, then we consider*

$$\begin{aligned}&\frac{\mathsf{h}\left(\frac{1}{2}\right)}{2}\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right)\left[{}^{\mathtt{AB}}_{\varepsilon_1}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{\daleth(\varepsilon_2)\}+{}^{\mathtt{AB}}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{\daleth(\varepsilon_1)\}\right]\\&-\frac{\mathsf{h}\left(\frac{1}{2}\right)}{2}\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right)\frac{1-\varsigma}{\mathbb{B}(\varsigma)}\left[\daleth(\varepsilon_1)+\daleth(\varepsilon_2)\right]+\frac{1-\varsigma}{\mathbb{B}(\varsigma)}\left[\Phi(\varepsilon_1)\daleth(\varepsilon_1)+\Phi(\varepsilon_2)\daleth(\varepsilon_2)\right]\\&=\Phi\left(\frac{5}{2}\right)\left[\frac{1}{2}\daleth\left(\frac{5}{2}\right)+\frac{1}{2\sqrt{\pi}}\int_1^{\frac{5}{2}}(\mu-1)\left(\frac{5}{2}-\mu\right)^{\frac{-1}{2}}+\frac{1}{2}\daleth\left(\frac{5}{2}\right)+\frac{1}{2\sqrt{\pi}}\int_{\frac{5}{2}}^{4}(-\mu+4)\left(\mu-\frac{5}{2}\right)^{\frac{-1}{2}}\right]\mathsf{d}\mu\\&\approx[73.10130,106.84792],\end{aligned}$$

and

$$\begin{aligned}&{}^{\mathtt{AB}}_{\varepsilon_1}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{(\Phi\daleth(\varepsilon_2)\}+{}^{\mathtt{AB}}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{(\Phi\daleth(\varepsilon_1)\}\\&=\left[\frac{1}{2}\Phi\left(\frac{3}{2}\right)+\frac{1}{2\sqrt{\pi}}\int_1^{\frac{5}{2}}\left[2e^{\mu}+1,3e^{\mu}+\frac{\sqrt{\mu}}{3}\right](\mu-1)\left(\frac{5}{2}-\mu\right)^{\frac{-1}{2}}+\frac{1}{2}\Phi\left(\frac{3}{2}\right)\right.\\&\quad\left.+\frac{1}{2\sqrt{\pi}}\int_{\frac{5}{2}}^{4}\left[2e^{\mu}+1,3e^{\mu}+\frac{\sqrt{\mu}}{3}\right](-\mu+4)\left(\mu-\frac{5}{2}\right)^{\frac{-1}{2}}\right]\mathsf{d}\mu\\&\approx[81.16120,111.35182].\end{aligned}$$

Finally, we have

$$\begin{aligned}&\frac{\varsigma(\varepsilon_2-\varepsilon_1)^{\varsigma}}{\mathbb{B}(\varsigma)\Gamma(\varsigma)}\left[\Phi(\varepsilon_1)+\Phi(\varepsilon_2)\right]\\&\times\int_0^1 \flat^{\varsigma-1}\left[\frac{1}{\mathsf{h}(\flat)}+\frac{1}{\mathsf{h}(1-\flat)}\right]\daleth(\flat\varepsilon_2+(1-\flat)\varepsilon_1)\mathrm{d}\flat\\&+\frac{1-\varsigma}{\mathbb{B}(\varsigma)}\left[\Phi(\varepsilon_1)\daleth(\varepsilon_1)+\Phi(\varepsilon_2)\daleth(\varepsilon_2)\right]\\&\approx[85.14621,115.36241].\end{aligned}$$

This implies that

$$[73.10130, 106.84792] \preceq_{cr} [81.16120, 111.35182] \preceq_{cr} [85.14621, 115.36241].$$

Consequently, Theorem 12 is valid.

Remark 4. *(i) If* $\mathsf{h}(\flat) = \frac{1}{\flat^s}$*, then Theorem 12 yields an outcome for the cr-*s*-convex function for* AB *integral operators:*

$$\begin{aligned}
&2^{s-1}\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right)\left[{}^{\mathtt{AB}}_{\varepsilon_1}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{\beth(\varepsilon_2)\} + {}^{\mathtt{AB}}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{\beth(\varepsilon_1)\}\right] \\
&\quad - 2^{s-1}\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right)\frac{1-\varsigma}{\mathbb{B}(\varsigma)}\left[\beth(\varepsilon_1)+\beth(\varepsilon_2)\right] \\
&\quad + \frac{1-\varsigma}{\mathbb{B}(\varsigma)}\left[\Phi(\varepsilon_1)\beth(\varepsilon_1)+\Phi(\varepsilon_2)\beth(\varepsilon_2)\right] \\
&\preceq_{cr} {}^{\mathtt{AB}}_{\varepsilon_1}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{(\Phi\beth(\varepsilon_2)\} + {}^{\mathtt{AB}}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{(\Phi\beth(\varepsilon_1)\} \\
&\preceq_{cr} \frac{\varsigma(\varepsilon_2-\varepsilon_1)^{\varsigma}}{\mathbb{B}(\varsigma)\Gamma(\varsigma)}\left[\Phi(\varepsilon_1)+\Phi(\varepsilon_2)\right] \\
&\quad \times \int_0^1 \flat^{\varsigma-1}\left[\flat^s + (1-\flat)^s\right]\beth(\flat\varepsilon_2+(1-\flat)\varepsilon_1)\mathrm{d}\flat \\
&\quad + \frac{1-\varsigma}{\mathbb{B}(\varsigma)}\left[\Phi(\varepsilon_1)\beth(\varepsilon_1)+\Phi(\varepsilon_2)\beth(\varepsilon_2)\right].
\end{aligned} \tag{15}$$

(ii) If $\mathsf{h}(\flat) = 1$*, then Theorem 12 yields an outcome for the cr-*p*-convex function for* AB *integral operators:*

$$\begin{aligned}
&\frac{1}{2}f\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right)\left[{}^{\mathtt{AB}}_{\varepsilon_1}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{\beth(\varepsilon_2)\} + {}^{\mathtt{AB}}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{\beth(\varepsilon_1)\}\right] \\
&\quad - \frac{1}{2}\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right)\frac{1-\varsigma}{\mathbb{B}(\varsigma)}\left[\beth(\varepsilon_1)+\beth(\varepsilon_2)\right] \\
&\quad + \frac{1-\varsigma}{\mathbb{B}(\varsigma)}\left[\Phi(\varepsilon_1)\beth(\varepsilon_1)+\Phi(\varepsilon_2)\beth(\varepsilon_2)\right] \\
&\preceq_{cr} {}^{\mathtt{AB}}_{\varepsilon_1}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{(\Phi\beth(\varepsilon_2)\} + {}^{\mathtt{AB}}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{(\Phi\beth(\varepsilon_1)\} \\
&\preceq_{cr} \frac{2\varsigma(\varepsilon_2-\varepsilon_1)^{\varsigma}}{\mathbb{B}(\varsigma)\Gamma(\varsigma)}\left[\Phi(\varepsilon_1)+\Phi(\varepsilon_2)\right] \\
&\quad \times \int_0^1 \flat^{\varsigma-1}\beth(\flat\varepsilon_2+(1-\flat)\varepsilon_1)\mathrm{d}\flat \\
&\quad + \frac{1-\varsigma}{\mathbb{B}(\varsigma)}\left[\Phi(\varepsilon_1)\beth(\varepsilon_1)+\Phi(\varepsilon_2)\beth(\varepsilon_2)\right].
\end{aligned}$$

Using Holder and Young inequalities, we present a novel refinement of (H-H) fractional integral inequalities when the function Φ is twice differentiable and belongs to the class of cr-Godunova-Levin mappings, based on the identity in Lemma 2.1.

Theorem 13. *Let* $\mathsf{h} : (0,1) \to \mathscr{R}^+$ *and* $\mathsf{h} \neq 0$. *Let* $\Phi : [\varepsilon_1, \varepsilon_2] \to \mathscr{R}^+_{\mathcal{I}}$ *is cr-*h*-Godunova-Levin mapping,* $\varepsilon_1, \varepsilon_2 \in \mathscr{R}^+, \varepsilon_1 < \varepsilon_2$. *If* $\Phi'' \in \mathcal{L}[\varepsilon_1, \varepsilon_2]$ *and* $|\Phi''|$ *is also cr-*h*-Godunova-Levin function, then the following double relation hold true:*

$$\begin{aligned}
&\frac{1}{\varepsilon_2 - \varepsilon_1}\left[{}^{\mathtt{AB}}\mathtt{I}^{\varsigma}_{\frac{\varepsilon_2+\varepsilon_1}{2}}\{\Phi(\varepsilon_1)\} + {}^{\mathtt{AB}}_{\frac{\varepsilon_2+\varepsilon_1}{2}}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{\Phi(\varepsilon_2)\}\right] \\
&- \frac{1}{(\varepsilon_2 - \varepsilon_1)\mathbb{B}(\varsigma)}\left[\Phi(\varepsilon_1) + \Phi(\varepsilon_2)\right] - \frac{(\varepsilon_2 - \varepsilon_1)^{\varsigma-1}}{2^{\varsigma-1}\mathbb{B}(\varsigma)\Gamma(\varsigma)}\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right) \\
&\preceq_{cr} \frac{(\varepsilon_2 - \varepsilon_1)^{\varsigma-1}}{(\varsigma+1)\mathbb{B}(\varsigma)\Gamma(\varsigma)}\left[|\Phi''(\varepsilon_1)| + |\Phi''(\varepsilon_2)|\right] \\
&\times \int_0^{\frac{1}{2}} \flat^{\varsigma+1}\left[\frac{1}{\mathsf{h}(\flat)} + \frac{1}{\mathsf{h}(1-\flat)}\right] d\flat,
\end{aligned} \tag{16}$$

where $\varsigma \in (0,1]$.

Proof. Firstly, from Lemma 2.1, we have

$$\begin{aligned}
&\frac{1}{\varepsilon_2 - \varepsilon_1}\left[{}^{\mathtt{AB}}\mathtt{I}^{\varsigma}_{\frac{\varepsilon_2+\varepsilon_1}{2}}\{\Phi(\varepsilon_1)\} + {}^{\mathtt{AB}}_{\frac{\varepsilon_2+\varepsilon_1}{2}}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{\Phi(\varepsilon_2)\}\right] \\
&- \frac{1}{(\varepsilon_2 - \varepsilon_1)\mathbb{B}(\varsigma)}\left[\Phi(\varepsilon_1) + \Phi(\varepsilon_2)\right] - \frac{(\varepsilon_2 - \varepsilon_1)^{\varsigma-1}}{2^{\varsigma-1}\mathbb{B}(\varsigma)\Gamma(\varsigma)}\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right) \\
&\preceq_{cr} \frac{(\varepsilon_2 - \varepsilon_1)^{\varsigma-1}}{2(\varsigma+1)\mathbb{B}(\varsigma)\Gamma(\varsigma)} \times \int_0^1 |\mathtt{w}^{\varsigma}(\flat)|\left[|\Phi''(\flat\varepsilon_1 + (1-\flat)\varepsilon_2)| + |\Phi''(\flat\varepsilon_2 + (1-\flat)\varepsilon_1)|\right] d\flat.
\end{aligned} \tag{17}$$

As $|\Phi''|$ is cr-h-Godunova-Levin function, we have

$$\begin{aligned}
&\frac{1}{\varepsilon_2 - \varepsilon_1}\left[{}^{\mathtt{AB}}\mathtt{I}^{\varsigma}_{\frac{\varepsilon_2+\varepsilon_1}{2}}\{\Phi(\varepsilon_1)\} + {}^{\mathtt{AB}}_{\frac{\varepsilon_2+\varepsilon_1}{2}}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{\Phi(\varepsilon_2)\}\right] \\
&- \frac{1}{(\varepsilon_2 - \varepsilon_1)\mathbb{B}(\varsigma)}\left[\Phi(\varepsilon_1) + \Phi(\varepsilon_2)\right] - \frac{(\varepsilon_2 - \varepsilon_1)^{\varsigma-1}}{2^{\varsigma-1}\mathbb{B}(\varsigma)\Gamma(\varsigma)}\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right) \\
&\leq \frac{(\varepsilon_2 - \varepsilon_1)^{\varsigma-1}}{2(\varsigma+1)\mathbb{B}(\varsigma)\Gamma(\varsigma)} \times \left\{\int_0^{\frac{1}{2}} \flat^{\varsigma+1}\left[\frac{|\Phi''(\varepsilon_1)|}{\mathsf{h}(\flat)} + \frac{|\Phi''(\varepsilon_2)|}{\mathsf{h}(1-\flat)}\right] d\flat \right. \\
&+ \int_{\frac{1}{2}}^1 (1-\flat)^{\varsigma+1}\left[\frac{|\Phi''(\varepsilon_1)|}{\mathsf{h}(\flat)} + \frac{|\Phi''(\varepsilon_2)|}{\mathsf{h}(1-\flat)}\right] d\flat \\
&+ \int_0^{\frac{1}{2}} \flat^{\varsigma+1}\left[\frac{|\Phi''(\varepsilon_2)|}{\mathsf{h}(\flat)} + \frac{|\Phi''(\varepsilon_1)|}{\mathsf{h}(1-\flat)}\right] d\flat \\
&\left. + \int_{\frac{1}{2}}^1 (1-\flat)^{\varsigma+1}\left[\frac{|\Phi''(\varepsilon_2)|}{\mathsf{h}(\flat)} + \frac{|\Phi''(\varepsilon_1)|}{\mathsf{h}(1-\flat)}\right] d\flat \right\} \\
&= \frac{(\varepsilon_2 - \varepsilon_1)^{\varsigma-1}}{(\varsigma+1)\mathbb{B}(\varsigma)\Gamma(\varsigma)}
\end{aligned}$$

$$\times\left\{\int_0^{\frac{1}{2}} \flat^{\varsigma+1}\left[\frac{|\Phi''(\varepsilon_1)|}{\mathsf{h}(\flat)}+\frac{|\Phi''(\varepsilon_2)|}{\mathsf{h}(1-\flat)}\right]d\flat\right.$$

$$\left.+\int_0^{\frac{1}{2}} \flat^{\varsigma+1}\left[\frac{|\Phi''(\varepsilon_2)|}{\mathsf{h}(\flat)}+\frac{|\Phi''(\varepsilon_1)|}{\mathsf{h}(1-\flat)}\right]d\flat\right\}$$

$$=\frac{(\varepsilon_2-\varepsilon_1)^{\varsigma-1}}{(\varsigma+1)\mathbb{B}(\varsigma)\Gamma(\varsigma)}\big[|\Phi''(\varepsilon_1)|+|\Phi''(\varepsilon_2)|\big]\int_0^{\frac{1}{2}} \flat^{\varsigma+1}\left[\frac{1}{\mathsf{h}(\flat)}+\frac{1}{\mathsf{h}(1-\flat)}\right]d\flat.$$

Remark 5. *(i) If* $\mathsf{h}(\flat)=\frac{1}{\flat^s}$*, then Theorem 13 yields an outcome for the cr-*s*-convex function for* AB *integral operators:*

$$\frac{1}{\varepsilon_2-\varepsilon_1}\left[{}^{\mathtt{AB}}\mathrm{I}^{\varsigma}_{\frac{\varepsilon_2+\varepsilon_1}{2}}\{\Phi(\varepsilon_1)\}+{}^{\mathtt{AB}}_{\frac{\varepsilon_2+\varepsilon_1}{2}}\mathrm{I}^{\varsigma}_{\varepsilon_2}\{\Phi(\varepsilon_2)\}\right]$$
$$-\frac{1}{(\varepsilon_2-\varepsilon_1)\mathbb{B}(\varsigma)}\big[\Phi(\varepsilon_1)+\Phi(\varepsilon_2)\big]-\frac{(\varepsilon_2-\varepsilon_1)^{\varsigma-1}}{2^{\varsigma-1}\mathbb{B}(\varsigma)\Gamma(\varsigma)}\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right)$$
$$\leq\frac{(\varepsilon_2-\varepsilon_1)^{\varsigma-1}}{(\varsigma+1)\mathbb{B}(\varsigma)\Gamma(\varsigma)}\big[|\Phi''(\varepsilon_1)|+|\Phi''(\varepsilon_2)|\big]\left[\frac{(\frac{1}{2})^{\varsigma+\mathsf{s}+2}}{\varsigma+\mathsf{s}+2}+\beta_{\frac{1}{2}}(\varsigma+2,\mathsf{s}+1)\right].$$

(ii) If $\mathsf{h}(\flat)=1$*, then Theorem 13 yields an outcome for the cr-*p*-convex function for* AB *integral operators:*

$$\frac{1}{\varepsilon_2-\varepsilon_1}\left[{}^{\mathtt{AB}}\mathrm{I}^{\varsigma}_{\frac{\varepsilon_2+\varepsilon_1}{2}}\{\Phi(\varepsilon_1)\}+{}^{\mathtt{AB}}_{\frac{\varepsilon_2+\varepsilon_1}{2}}\mathrm{I}^{\varsigma}_{\varepsilon_2}\{\Phi(\varepsilon_2)\}\right]$$
$$-\frac{1}{(\varepsilon_2-\varepsilon_1)\mathbb{B}(\varsigma)}\big[\Phi(\varepsilon_1)+\Phi(\varepsilon_2)\big]-\frac{(\varepsilon_2-\varepsilon_1)^{\varsigma-1}}{2^{\varsigma-1}\mathbb{B}(\varsigma)\Gamma(\varsigma)}\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right)$$
$$\leq\frac{\left(\frac{1}{2}\right)^{\varsigma+1}}{(\varsigma+1)(\varsigma+2)}\frac{(\varepsilon_2-\varepsilon_1)^{\varsigma-1}}{\mathbb{B}(\varsigma)\Gamma(\varsigma)}\big[|\Phi''(\varepsilon_1)|+|\Phi''(\varepsilon_2)|\big].$$

Theorem 14. *Let* $\mathsf{h}:(0,1)\to\mathscr{R}^+$ *and* $\mathsf{h}\neq 0$*. Let* $\Phi:[\varepsilon_1,\varepsilon_2]\to\mathscr{R}^+_{\mathcal{I}}$ *is cr-*h*-Godunova-Levin mapping,* $\varepsilon_1,\varepsilon_2\in\mathscr{R}^+,\varepsilon_1<\varepsilon_2$*. If* $\Phi''\in\mathcal{L}[\varepsilon_1,\varepsilon_2]$ *and* $|\Phi''|$ *is also cr-*h*-Godunova-Levin function, then the following double relation holds true:*

$$\frac{1}{\varepsilon_2-\varepsilon_1}\left[{}^{\mathtt{AB}}\mathrm{I}^{\varsigma}_{\frac{\varepsilon_2+\varepsilon_1}{2}}\{\Phi(\varepsilon_1)\}+{}^{\mathtt{AB}}_{\frac{\varepsilon_2+\varepsilon_1}{2}}\mathrm{I}^{\varsigma}_{\varepsilon_2}\{\Phi(\varepsilon_2)\}\right]$$
$$-\frac{1}{(\varepsilon_2-\varepsilon_1)\mathbb{B}(\varsigma)}\big[\Phi(\varepsilon_1)+\Phi(\varepsilon_2)\big]-\frac{(\varepsilon_2-\varepsilon_1)^{\varsigma-1}}{2^{\varsigma-1}\mathbb{B}(\varsigma)\Gamma(\varsigma)}\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right)$$
$$\preceq_{cr}\frac{(\varepsilon_2-\varepsilon_1)^{\varsigma-1}}{(\varsigma+1)\mathbb{B}(\varsigma)\Gamma(\varsigma)}\left(\frac{\left(\frac{1}{2}\right)^{\varsigma\mathsf{p}+2\mathsf{p}}}{\varsigma\mathsf{p}+\mathsf{p}+1}\right)^{\frac{1}{\mathsf{p}}}\big[|\Phi''(\varepsilon_1)|+|\Phi''(\varepsilon_2)|\big]$$
$$\times\left[\left(\int_0^1\frac{d\flat}{\mathsf{h}(\flat)}\right)^{\frac{1}{\mathsf{q}}}+\left(\int_0^1\frac{d\flat}{\mathsf{h}(1-\flat)}\right)^{\frac{1}{\mathsf{q}}}\right], \tag{18}$$

where $\varsigma \in (0,1], \frac{1}{\mathtt{p}} + \frac{1}{\mathtt{q}} = 1.$

Proof. According to Lemma 2.1 and taking into account Hölder's inequality and apply it to relation (17), we have

$$\begin{aligned}&\frac{1}{\varepsilon_2-\varepsilon_1}\left[{}^{\mathtt{AB}}\mathtt{I}^{\varsigma}_{\frac{\varepsilon_2+\varepsilon_1}{2}}\{\Phi(\varepsilon_1)\} + {}^{\mathtt{AB}}_{\frac{\varepsilon_2+\varepsilon_1}{2}}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{\Phi(\varepsilon_2)\}\right]\\&-\frac{1}{(\varepsilon_2-\varepsilon_1)\mathbb{B}(\varsigma)}\big[\Phi(\varepsilon_1)+\Phi(\varepsilon_2)\big] - \frac{(\varepsilon_2-\varepsilon_1)^{\varsigma-1}}{2^{\varsigma-1}\mathbb{B}(\varsigma)\Gamma(\varsigma)}\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right)\\&\preceq_{cr} \frac{(\varepsilon_2-\varepsilon_1)^{\varsigma-1}}{2(\varsigma+1)\mathbb{B}(\varsigma)\Gamma(\varsigma)}\left(\int_0^1 |\mathtt{w}^{\varsigma}(\flat)|^{\mathtt{p}} d\flat\right)^{\frac{1}{\mathtt{p}}}\left[\left(\int_0^1 |\Phi''(\flat\varepsilon_1+(1-\flat)\varepsilon_2)|^{\mathtt{q}} d\flat\right)^{\frac{1}{\mathtt{q}}}\right.\\&\left.+\left(\int_0^1 |\Phi''(\flat\varepsilon_2+(1-\flat)\varepsilon_1)|^{\mathtt{q}} d\flat\right)^{\frac{1}{\mathtt{q}}}\right]. \qquad (19)\end{aligned}$$

As $|\Phi''|^{\mathtt{q}}$ is an cr-h-Godunova-Levin mapping, we have

$$\begin{aligned}&\frac{1}{\varepsilon_2-\varepsilon_1}\left[{}^{\mathtt{AB}}\mathtt{I}^{\varsigma}_{\frac{\varepsilon_2+\varepsilon_1}{2}}\{\Phi(\varepsilon_1)\} + {}^{\mathtt{AB}}_{\frac{\varepsilon_2+\varepsilon_1}{2}}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{\Phi(\varepsilon_2)\}\right]\\&-\frac{1}{(\varepsilon_2-\varepsilon_1)\mathbb{B}(\varsigma)}\big[\Phi(\varepsilon_1)+\Phi(\varepsilon_2)\big] - \frac{(\varepsilon_2-\varepsilon_1)^{\varsigma-1}}{2^{\varsigma-1}\mathbb{B}(\varsigma)\Gamma(\varsigma)}\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right)\\&\preceq_{cr} \frac{(\varepsilon_2-\varepsilon_1)^{\varsigma-1}}{(\varsigma+1)\mathbb{B}(\varsigma)\Gamma(\varsigma)}\left(\frac{\left(\frac{1}{2}\right)^{\varsigma\mathtt{p}+2\mathtt{p}}}{\varsigma\mathtt{p}+\mathtt{p}+1}\right)^{\frac{1}{\mathtt{p}}}\\&\times\left\{\left[\int_0^1\left(\frac{|\Phi''(\varepsilon_1)|^{\mathtt{q}}}{\mathsf{h}(\flat)}+\frac{|\Phi''(\varepsilon_2)|^{\mathtt{q}}}{\mathsf{h}(1-\flat)}\right)d\flat\right]^{\frac{1}{\mathtt{q}}}\right.\\&\left.+\left[\int_0^1\left(\frac{|\Phi''(\varepsilon_2)|^{\mathtt{q}}}{\mathsf{h}(\flat)}+\frac{|\Phi''(\varepsilon_1)|^{\mathtt{q}}}{\mathsf{h}(1-\flat)}\right)d\flat\right]^{\frac{1}{\mathtt{q}}}\right\}.\end{aligned}$$

Then, we apply the fact that

$$\sum_{\mathtt{k}=1}^{\varepsilon_2}(\mathtt{u}_{\mathtt{k}}+\mathtt{v}_{\mathtt{k}})^{\varepsilon_1} \le \sum_{\mathtt{k}=1}^{\varepsilon_2}\mathtt{u}_{\mathtt{k}}{}^{\varepsilon_1} + \sum_{\mathtt{k}=1}^{\varepsilon_2}\mathtt{v}_{\mathtt{k}}{}^{\varepsilon_1},$$

for $0<\varepsilon_1<1, \mathtt{u}_1,\mathtt{u}_2,\cdots,\mathtt{u}_{\varepsilon_2}\ge 0, \mathtt{v}_1,\mathtt{v}_2,\cdots,\mathtt{v}_{\varepsilon_2}\ge 0$. This further implies as follows:

$$\begin{aligned}&\frac{1}{\varepsilon_2-\varepsilon_1}\left[{}^{\mathtt{AB}}\mathtt{I}^{\varsigma}_{\frac{\varepsilon_2+\varepsilon_1}{2}}\{\Phi(\varepsilon_1)\} + {}^{\mathtt{AB}}_{\frac{\varepsilon_2+\varepsilon_1}{2}}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{\Phi(\varepsilon_2)\}\right]\\&-\frac{1}{(\varepsilon_2-\varepsilon_1)\mathbb{B}(\varsigma)}\big[\Phi(\varepsilon_1)+\Phi(\varepsilon_2)\big] - \frac{(\varepsilon_2-\varepsilon_1)^{\varsigma-1}}{2^{\varsigma-1}\mathbb{B}(\varsigma)\Gamma(\varsigma)}\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right)\\&\preceq_{cr} \frac{(\varepsilon_2-\varepsilon_1)^{\varsigma-1}}{(\varsigma+1)\mathbb{B}(\varsigma)\Gamma(\varsigma)}\left(\frac{\left(\frac{1}{2}\right)^{\varsigma\mathtt{p}+2\mathtt{p}}}{\varsigma\mathtt{p}+\mathtt{p}+1}\right)^{\frac{1}{\mathtt{p}}}\end{aligned}$$

$$\times\left[\left(\int_0^1 \frac{|\Phi''(\varepsilon_1)|^{\mathsf{q}}}{\mathsf{h}(\flat)}d\flat\right)^{\frac{1}{\mathsf{q}}}+\left(\int_0^1 \frac{|\Phi''(\varepsilon_2)|^{\mathsf{q}}}{\mathsf{h}(1-\flat)}d\flat\right)^{\frac{1}{\mathsf{q}}}\right.$$
$$\left.+\left(\int_0^1 \frac{|\Phi''(\varepsilon_2)|^{\mathsf{q}}}{\mathsf{h}(\flat)}d\flat\right)^{\frac{1}{\mathsf{q}}}+\left(\int_0^1 \frac{|\Phi''(\varepsilon_1)|^{\mathsf{q}}}{\mathsf{h}(1-\flat)}d\flat\right)^{\frac{1}{\mathsf{q}}}\right]$$
$$=\frac{(\varepsilon_2-\varepsilon_1)^{\varsigma-1}}{(\varsigma+1)\mathbb{B}(\varsigma)\Gamma(\varsigma)}\left(\frac{\left(\frac{1}{2}\right)^{\varsigma\mathsf{p}+2\mathsf{p}}}{\varsigma\mathsf{p}+\mathsf{p}+1}\right)^{\frac{1}{\mathsf{p}}}\left[|\Phi''(\varepsilon_1)|+|\Phi''(\varepsilon_2)|\right]$$
$$\times\left[\left(\int_0^1 \frac{d\flat}{\mathsf{h}(\flat)}\right)^{\frac{1}{\mathsf{q}}}+\left(\int_0^1 \frac{d\flat}{\mathsf{h}(1-\flat)}\right)^{\frac{1}{\mathsf{q}}}\right].$$

The finishes the proof.

Remark 6. *(i) If $\mathsf{h}(\flat)=\frac{1}{\flat^s}$, then Theorem 14 yields an outcome for the cr-s-convex function for* AB *integral operators:*

$$\frac{1}{\varepsilon_2-\varepsilon_1}\left[{}^{\mathtt{AB}}\mathtt{I}^{\varsigma}_{\frac{\varepsilon_2+\varepsilon_1}{2}}\{\Phi(\varepsilon_1)\}+{}^{\mathtt{AB}}_{\frac{\varepsilon_2+\varepsilon_1}{2}}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{\Phi(\varepsilon_2)\}\right]$$
$$-\frac{1}{(\varepsilon_2-\varepsilon_1)\mathbb{B}(\varsigma)}\left[\Phi(\varepsilon_1)+\Phi(\varepsilon_2)\right]-\frac{(\varepsilon_2-\varepsilon_1)^{\varsigma-1}}{2^{\varsigma-1}\mathbb{B}(\varsigma)\Gamma(\varsigma)}\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right)$$
$$\preceq_{cr}\frac{(\varepsilon_2-\varepsilon_1)^{\varsigma-1}}{(\varsigma+1)\mathbb{B}(\varsigma)\Gamma(\varsigma)}\left(\frac{\left(\frac{1}{2}\right)^{\varsigma+1}}{\varsigma+2}\right)^{\frac{1}{\mathsf{p}}}\left[|\Phi''(\varepsilon_1)|^{\mathsf{q}}+|\Phi''(\varepsilon_2)|^{\mathsf{q}}\right]^{\frac{1}{\mathsf{q}}}$$
$$\times\left[\frac{\left(\frac{1}{2}\right)^{\varsigma+\mathsf{s}+2}}{\varsigma+\mathsf{s}+2}+\beta_{\frac{1}{2}}(\varsigma+2,\mathsf{s}+1)\right]^{\frac{1}{\mathsf{q}}}. \tag{20}$$

(ii) If $\mathsf{h}(\flat)=$, then Theorem 14 yields an outcome for the cr-p-convex function for AB *integral operators:*

$$\frac{1}{\varepsilon_2-\varepsilon_1}\left[{}^{\mathtt{AB}}\mathtt{I}^{\varsigma}_{\frac{\varepsilon_2+\varepsilon_1}{2}}\{\Phi(\varepsilon_1)\}+{}^{\mathtt{AB}}_{\frac{\varepsilon_2+\varepsilon_1}{2}}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{\Phi(\varepsilon_2)\}\right]$$
$$-\frac{1}{(\varepsilon_2-\varepsilon_1)\mathbb{B}(\varsigma)}\left[\Phi(\varepsilon_1)+\Phi(\varepsilon_2)\right]-\frac{(\varepsilon_2-\varepsilon_1)^{\varsigma-1}}{2^{\varsigma-1}\mathbb{B}(\varsigma)\Gamma(\varsigma)}\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right)$$
$$\preceq_{cr}\frac{(\varepsilon_2-\varepsilon_1)^{\varsigma-1}}{(\varsigma+1)\mathbb{B}(\varsigma)\Gamma(\varsigma)}\frac{\left(\frac{1}{2}\right)^{\varsigma+1}}{\varsigma+2}\left[|\Phi''(\varepsilon_1)|^{\mathsf{q}}+|\Phi''(\varepsilon_2)|^{\mathsf{q}}\right]^{\frac{1}{\mathsf{q}}}. \tag{21}$$

Theorem 15. *Let $\mathsf{h}:(0,1)\to\mathscr{R}^+$ and $\mathsf{h}\neq 0$. Let $\Phi:[\varepsilon_1,\varepsilon_2]\to\mathscr{R}^+_{\mathcal{I}}$ is cr-h-Godunova-Levin mapping, $\varepsilon_1,\varepsilon_2\in\mathscr{R}^+,\varepsilon_1<\varepsilon_2$. If $\Phi''\in\mathcal{L}[\varepsilon_1,\varepsilon_2]$ and $|\Phi''|$ is also cr-h-Godunova-Levin function, then the following double relation holds true:*

$$\frac{1}{\varepsilon_2-\varepsilon_1}\left[{}^{\mathtt{AB}}\mathtt{I}^{\varsigma}_{\frac{\varepsilon_2+\varepsilon_1}{2}}\{\Phi(\varepsilon_1)\}+{}^{\mathtt{AB}}_{\frac{\varepsilon_2+\varepsilon_1}{2}}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{\Phi(\varepsilon_2)\}\right]$$

$$
\begin{aligned}
&- \frac{1}{(\varepsilon_2-\varepsilon_1)\mathbb{B}(\varsigma)}\left[\Phi(\varepsilon_1)+\Phi(\varepsilon_2)\right] - \frac{(\varepsilon_2-\varepsilon_1)^{\varsigma-1}}{2^{\varsigma-1}\mathbb{B}(\varsigma)\Gamma(\varsigma)}\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right)\\
&\preceq_{cr} \frac{(\varepsilon_2-\varepsilon_1)^{\varsigma-1}}{(\varsigma+1)\mathbb{B}(\varsigma)\Gamma(\varsigma)}\left(\frac{\left(\frac{1}{2}\right)^{(\varsigma+1)\left(\frac{\mathtt{q}-\mathtt{p}}{\mathtt{q}-1}\right)}(\mathtt{q}-1)}{(\varsigma+1)(\mathtt{q}-\mathtt{p})+\mathtt{q}-1}\right)^{1-\frac{1}{\mathtt{q}}}\\
&\times\left[|\Phi''(\varepsilon_1)|^{\mathtt{q}}+|\Phi''(\varepsilon_2)|^{\mathtt{q}}\right]^{\frac{1}{\mathtt{q}}}\left[\int_0^{\frac{1}{2}}\frac{\flat^{\varsigma\mathtt{p}+\mathtt{p}}d\flat}{\mathtt{h}(\flat)}+\int_0^{\frac{1}{2}}\frac{\flat^{\varsigma\mathtt{p}+\mathtt{p}}d\flat}{\mathtt{h}(1-\flat)}\right]^{\frac{1}{\mathtt{q}}}, \qquad (22)
\end{aligned}
$$

where $\varsigma\in(0,1], \mathtt{q}\geq\mathtt{p}>1$.

Proof. By using the Holder's inequality and taking into account relation (17), we have

$$
\begin{aligned}
&\frac{1}{\varepsilon_2-\varepsilon_1}\left[{}^{\mathtt{AB}}\mathtt{I}^{\varsigma}_{\frac{\varepsilon_2+\varepsilon_1}{2}}\{\Phi(\varepsilon_1)\}+{}^{\mathtt{AB}}_{\frac{\varepsilon_2+\varepsilon_1}{2}}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{\Phi(\varepsilon_2)\}\right]\\
&- \frac{1}{(\varepsilon_2-\varepsilon_1)\mathbb{B}(\varsigma)}\left[\Phi(\varepsilon_1)+\Phi(\varepsilon_2)\right] - \frac{(\varepsilon_2-\varepsilon_1)^{\varsigma-1}}{2^{\varsigma-1}\mathbb{B}(\varsigma)\Gamma(\varsigma)}\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right) = \frac{(\varepsilon_2-\varepsilon_1)^{\varsigma-1}}{2(\varsigma+1)\mathbb{B}(\varsigma)\Gamma(\varsigma)}\\
&\times\int_0^1|\mathtt{w}^{\varsigma}(\flat)|^{\frac{\mathtt{q}-\mathtt{p}}{\mathtt{q}}}\cdot|\mathtt{w}^{\varsigma}(\flat)|^{\frac{\mathtt{p}}{\mathtt{q}}}\left[|\Phi''(\flat\varepsilon_1+(1-\flat)\varepsilon_2)|+|\Phi''(\flat\varepsilon_2+(1-\flat)\varepsilon_1)|\right]d\flat\\
&\preceq_{cr}\frac{(\varepsilon_2-\varepsilon_1)^{\varsigma-1}}{2(\varsigma+1)\mathbb{B}(\varsigma)\Gamma(\varsigma)}\left(\int_0^1|\mathtt{w}^{\varsigma}(\flat)|^{\frac{\mathtt{q}-\mathtt{p}}{\mathtt{q}-1}}d\flat\right)^{1-\frac{1}{\mathtt{q}}}\\
&\times\left[\left(\int_0^1|\mathtt{w}^{\varsigma}(\flat)|^{\mathtt{p}}|\Phi''(\flat\varepsilon_1+(1-\flat)\varepsilon_2)|^{\mathtt{q}}d\flat\right)^{\frac{1}{\mathtt{q}}}\right.\\
&\left.+\left(\int_0^1|\mathtt{w}^{\varsigma}(\flat)|^{\mathtt{p}}|\Phi''(\flat\varepsilon_2+(1-\flat)\varepsilon_1)|^{\mathtt{q}}d\flat\right)^{\frac{1}{\mathtt{q}}}\right].
\end{aligned}
$$

As $|\Phi''|^{\mathtt{q}}$ is cr-h-Godunova-Levin mapping, one has

$$
\begin{aligned}
&\frac{1}{\varepsilon_2-\varepsilon_1}\left[{}^{\mathtt{AB}}\mathtt{I}^{\varsigma}_{\frac{\varepsilon_2+\varepsilon_1}{2}}\{\Phi(\varepsilon_1)\}+{}^{\mathtt{AB}}_{\frac{\varepsilon_2+\varepsilon_1}{2}}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{\Phi(\varepsilon_2)\}\right]\\
&- \frac{1}{(\varepsilon_2-\varepsilon_1)\mathbb{B}(\varsigma)}\left[\Phi(\varepsilon_1)+\Phi(\varepsilon_2)\right] - \frac{(\varepsilon_2-\varepsilon_1)^{\varsigma-1}}{2^{\varsigma-1}\mathbb{B}(\varsigma)\Gamma(\varsigma)}\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right) = \frac{(\varepsilon_2-\varepsilon_1)^{\varsigma-1}}{2(\varsigma+1)\mathbb{B}(\varsigma)\Gamma(\varsigma)}\\
&\times\int_0^1|\mathtt{w}^{\varsigma}(\flat)|^{\frac{\mathtt{q}-\mathtt{p}}{\mathtt{q}}}\cdot|\mathtt{w}^{\varsigma}(\flat)|^{\frac{\mathtt{p}}{\mathtt{q}}}\left[|\Phi''(\flat\varepsilon_1+(1-\flat)\varepsilon_2)|+|\Phi''(\flat\varepsilon_2+(1-\flat)\varepsilon_1)|\right]d\flat\\
&\preceq_{cr}\frac{(\varepsilon_2-\varepsilon_1)^{\varsigma-1}}{2(\varsigma+1)\mathbb{B}(\varsigma)\Gamma(\varsigma)}\left(\int_0^1|\mathtt{w}^{\varsigma}(\flat)|^{\frac{\mathtt{q}-\mathtt{p}}{\mathtt{q}-1}}d\flat\right)^{1-\frac{1}{\mathtt{q}}}\times\left[\left(\int_0^1\frac{|\mathtt{w}^{\varsigma}(\flat)|^{\mathtt{p}}|\Phi''(\varepsilon_1)|^{\mathtt{q}}d\flat}{\mathtt{h}(\flat)}\right.\right.\\
&\left.\left.+\int_0^1\frac{|\mathtt{w}^{\varsigma}(\flat)|^{\mathtt{p}}|\Phi''(\varepsilon_2)|^{\mathtt{q}}d\flat}{\mathtt{h}(1-\flat)}\right)^{\frac{1}{\mathtt{q}}}+\left(\int_0^1\frac{|\mathtt{w}^{\varsigma}(\flat)|^{\mathtt{p}}|\Phi''(\varepsilon_2)|^{\mathtt{q}}d\flat}{\mathtt{h}(\flat)}+\int_0^1\frac{|\mathtt{w}^{\varsigma}(\flat)|^{\mathtt{p}}|\Phi''(\varepsilon_1)|^{\mathtt{q}}d\flat}{\mathtt{h}(1-\flat)}\right)^{\frac{1}{\mathtt{q}}}\right]
\end{aligned}
$$

$$=\frac{(\varepsilon_2-\varepsilon_1)^{\varsigma-1}}{2(\varsigma+1)\mathbb{B}(\varsigma)\Gamma(\varsigma)}\left(\frac{\left(\frac{1}{2}\right)^{(\varsigma+1)\left(\frac{\mathtt{q}-\mathtt{p}}{\mathtt{q}-1}\right)}(\mathtt{q}-1)}{(\varsigma+1)(\mathtt{q}-\mathtt{p})+\mathtt{q}-1}\right)^{1-\frac{1}{\mathtt{q}}}\left\{\left[|\Phi''(\varepsilon_1)|^{\mathtt{q}}\left(\int_0^{\frac{1}{2}}\frac{\flat^{\varsigma\mathtt{p}+\mathtt{p}}d\flat}{\mathsf{h}(\flat)}\right.\right.\right.$$

$$\left.+\int_{\frac{1}{2}}^1\frac{(1-\flat)^{\varsigma\mathtt{p}+\mathtt{p}}d\flat}{\mathsf{h}(\flat)}\right)+|\Phi''(\varepsilon_2)|^{\mathtt{q}}\left.\left(\int_0^{\frac{1}{2}}\frac{\flat^{\varsigma\mathtt{p}+\mathtt{p}}d\flat}{\mathsf{h}(1-\flat)}+\int_{\frac{1}{2}}^1\frac{(1-\flat)^{\varsigma\mathtt{p}+\mathtt{p}}d\flat}{\mathsf{h}(1-\flat)}\right)\right]^{\frac{1}{\mathtt{q}}}$$

$$+\left[|\Phi''(\varepsilon_2)|^{\mathtt{q}}\left(\int_0^{\frac{1}{2}}\frac{\flat^{\varsigma\mathtt{p}+\mathtt{p}}d\flat}{\mathsf{h}(\flat)}+\int_{\frac{1}{2}}^1\frac{(1-\flat)^{\varsigma\mathtt{p}+\mathtt{p}}d\flat}{\mathsf{h}(\flat)}\right)+|\Phi''(\varepsilon_1)|^{\mathtt{q}}\left(\int_0^{\frac{1}{2}}\frac{\flat^{\varsigma\mathtt{p}+\mathtt{p}}d\flat}{\mathsf{h}(1-\flat)}\right.\right.$$

$$\left.\left.\left.+\int_{\frac{1}{2}}^1\frac{(1-\flat)^{\varsigma\mathtt{p}+\mathtt{p}}d\flat}{\mathsf{h}(1-\flat)}\right)\right]^{\frac{1}{\mathtt{q}}}\right\}=\frac{(\varepsilon_2-\varepsilon_1)^{\varsigma-1}}{(\varsigma+1)\mathbb{B}(\varsigma)\Gamma(\varsigma)}\left(\frac{\left(\frac{1}{2}\right)^{(\varsigma+1)\left(\frac{\mathtt{q}-\mathtt{p}}{\mathtt{q}-1}\right)}(\mathtt{q}-1)}{(\varsigma+1)(\mathtt{q}-\mathtt{p})+\mathtt{q}-1}\right)^{1-\frac{1}{\mathtt{q}}}$$

$$\times\left[|\Phi''(\varepsilon_1)|^{\mathtt{q}}+|\Phi''(\varepsilon_2)|^{\mathtt{q}}\right]^{\frac{1}{\mathtt{q}}}\left[\int_0^{\frac{1}{2}}\frac{\flat^{\varsigma\mathtt{p}+\mathtt{p}}d\flat}{\mathsf{h}(\flat)}+\int_0^{\frac{1}{2}}\frac{\flat^{\varsigma\mathtt{p}+\mathtt{p}}d\flat}{\mathsf{h}(1-\flat)}\right]^{\frac{1}{\mathtt{q}}}.$$

Remark 7. *(i) If* $\mathsf{h}(\flat)=\frac{1}{\flat^s}$*, then Theorem 15 yields an outcome for the cr-*s*-convex function for* AB *integral operators:*

$$\frac{1}{\varepsilon_2-\varepsilon_1}\left[{}^{\mathtt{AB}}\mathrm{I}^{\varsigma}_{\frac{\varepsilon_2+\varepsilon_1}{2}}\{\Phi(\varepsilon_1)\}+{}^{\mathtt{AB}}_{\frac{\varepsilon_2+\varepsilon_1}{2}}\mathrm{I}^{\varsigma}_{\varepsilon_2}\{\Phi(\varepsilon_2)\}\right]$$
$$-\frac{1}{(\varepsilon_2-\varepsilon_1)\mathbb{B}(\varsigma)}\left[\Phi(\varepsilon_1)+\Phi(\varepsilon_2)\right]-\frac{(\varepsilon_2-\varepsilon_1)^{\varsigma-1}}{2^{\varsigma-1}\mathbb{B}(\varsigma)\Gamma(\varsigma)}\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right)$$
$$\preceq_{cr}\frac{(\varepsilon_2-\varepsilon_1)^{\varsigma-1}}{(\varsigma+1)\mathbb{B}(\varsigma)\Gamma(\varsigma)}\left(\frac{\left(\frac{1}{2}\right)^{(\varsigma+1)\left(\frac{\mathtt{q}-\mathtt{p}}{\mathtt{q}-1}\right)}(\mathtt{q}-1)}{(\varsigma+1)(\mathtt{q}-\mathtt{p})+\mathtt{q}-1}\right)^{1-\frac{1}{\mathtt{q}}}$$
$$\times\left[|\Phi''(\varepsilon_1)|^{\mathtt{q}}+|\Phi''(\varepsilon_2)|^{\mathtt{q}}\right]^{\frac{1}{\mathtt{q}}}\left[\frac{\left(\frac{1}{2}\right)^{\varsigma\mathtt{p}+\mathtt{p}+\mathtt{s}+1}}{\varsigma\mathtt{p}+\mathtt{p}+\mathtt{s}+1}+\beta_{\frac{1}{2}}(\varsigma\mathtt{p}+\mathtt{p}+1,\mathtt{s}+1)\right]^{\frac{1}{\mathtt{q}}}.$$

(ii) If $\mathsf{h}(\flat)=1$*, then Theorem 15 yields an outcome for the cr-*$\mathtt{p}$*-convex function for* AB *integral operators:*

$$\frac{1}{\varepsilon_2-\varepsilon_1}\left[{}^{\mathtt{AB}}\mathrm{I}^{\varsigma}_{\frac{\varepsilon_2+\varepsilon_1}{2}}\{\Phi(\varepsilon_1)\}+{}^{\mathtt{AB}}_{\frac{\varepsilon_2+\varepsilon_1}{2}}\mathrm{I}^{\varsigma}_{\varepsilon_2}\{\Phi(\varepsilon_2)\}\right]$$
$$-\frac{1}{(\varepsilon_2-\varepsilon_1)\mathbb{B}(\varsigma)}\left[\Phi(\varepsilon_1)+\Phi(\varepsilon_2)\right]-\frac{(\varepsilon_2-\varepsilon_1)^{\varsigma-1}}{2^{\varsigma-1}\mathbb{B}(\varsigma)\Gamma(\varsigma)}\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right)$$
$$\preceq_{cr}\frac{(\varepsilon_2-\varepsilon_1)^{\varsigma-1}}{(\varsigma+1)\mathbb{B}(\varsigma)\Gamma(\varsigma)}\left(\frac{\left(\frac{1}{2}\right)^{(\varsigma+1)\left(\frac{\mathtt{q}-\mathtt{p}}{\mathtt{q}-1}\right)}(\mathtt{q}-1)}{(\varsigma+1)(\mathtt{q}-\mathtt{p})+\mathtt{q}-1}\right)^{1-\frac{1}{\mathtt{q}}}\times\left(\frac{\left(\frac{1}{2}\right)^{\varsigma\mathtt{p}+\mathtt{p}}}{\varsigma\mathtt{p}+\mathtt{p}+1}\right)^{\frac{1}{\mathtt{q}}}\left[|\Phi''(\varepsilon_1)|^{\mathtt{q}}+|\Phi''(\varepsilon_2)|^{\mathtt{q}}\right]^{\frac{1}{\mathtt{q}}}.$$

Theorem 16. *Let* $\mathsf{h} : (0,1) \to \mathscr{R}^+$ *and* $\mathsf{h} \neq 0$. *Let* $\Phi : [\varepsilon_1, \varepsilon_2] \to \mathscr{R}_{\mathcal{I}}^+$ *is cr-*h*-Godunova-Levin mapping,* $\varepsilon_1, \varepsilon_2 \in \mathscr{R}^+, \varepsilon_1 < \varepsilon_2$. *If* $|\Phi''| \in \mathcal{L}[\varepsilon_1, \varepsilon_2]$ *and* $|\Phi''|$ *is also cr-*h*-Godunova-Levin function, then the following double relation holds true:*

$$\begin{aligned}
&\frac{1}{\varepsilon_2 - \varepsilon_1}\left[{}^{\mathtt{AB}}\mathtt{I}^{\varsigma}_{\frac{\varepsilon_2+\varepsilon_1}{2}}\{\Phi(\varepsilon_1)\} + {}^{\mathtt{AB}}_{\frac{\varepsilon_2+\varepsilon_1}{2}}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{\Phi(\varepsilon_2)\}\right] \\
&- \frac{1}{(\varepsilon_2 - \varepsilon_1)\mathbb{B}(\varsigma)}\big[\Phi(\varepsilon_1) + \Phi(\varepsilon_2)\big] - \frac{(\varepsilon_2 - \varepsilon_1)^{\varsigma-1}}{2^{\varsigma-1}\mathbb{B}(\varsigma)\Gamma(\varsigma)}\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right) \\
&\preceq_{cr} \frac{(\varepsilon_2 - \varepsilon_1)^{\varsigma-1}}{2(\varsigma+1)\mathbb{B}(\varsigma)\Gamma(\varsigma)}\left\{\frac{\left(\frac{1}{2}\right)^{\varsigma\mathtt{p}+\mathtt{p}-1}}{(\varsigma\mathtt{p}+\mathtt{p}+1)p}\right. \\
&\left.+\frac{1}{\mathtt{q}}\big[|\Phi''(\varepsilon_1)|^{\mathtt{q}} + |\Phi''(\varepsilon_2)|^{\mathtt{q}}\big]\int_0^1\left(\frac{1}{\mathsf{h}(\flat)} + \frac{1}{\mathsf{h}(1-\flat)}\right)\mathrm{d}\flat\right\},
\end{aligned}$$

where $\varsigma \in (0,1]$.

Proof. By using the Holder's inequality and taking into account result (17), based on the Young's result: $\mathtt{ab} \leq \frac{1}{\mathtt{p}}\mathtt{a}^{\mathtt{p}} + \frac{1}{\mathtt{q}}\mathtt{b}^{\mathtt{q}}$, we obtain

$$\begin{aligned}
&\frac{1}{\varepsilon_2 - \varepsilon_1}\left[{}^{\mathtt{AB}}\mathtt{I}^{\varsigma}_{\frac{\varepsilon_2+\varepsilon_1}{2}}\{\Phi(\varepsilon_1)\} + {}^{\mathtt{AB}}_{\frac{\varepsilon_2+\varepsilon_1}{2}}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{\Phi(\varepsilon_2)\}\right] \\
&- \frac{1}{(\varepsilon_2 - \varepsilon_1)\mathbb{B}(\varsigma)}\big[\Phi(\varepsilon_1) + \Phi(\varepsilon_2)\big] - \frac{(\varepsilon_2 - \varepsilon_1)^{\varsigma-1}}{2^{\varsigma-1}\mathbb{B}(\varsigma)\Gamma(\varsigma)}\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right) \\
&\preceq_{cr} \frac{(\varepsilon_2 - \varepsilon_1)^{\varsigma-1}}{2(\varsigma+1)\mathbb{B}(\varsigma)\Gamma(\varsigma)}\left[\frac{2}{\mathtt{p}}\int_0^1 |\mathtt{w}^{\varsigma}(\flat)|^{\mathtt{p}}\mathrm{d}\flat\right. \\
&\left.+\frac{1}{\mathtt{q}}\left(\int_0^1 |\Phi''(\flat\varepsilon_1 + (1-\flat)\varepsilon_2)|^{\mathtt{q}}\mathrm{d}\flat + \int_0^1 |\Phi''(\flat\varepsilon_2 + (1-\flat)\varepsilon_1)|^{\mathtt{q}}\mathrm{d}\flat\right)\right].
\end{aligned}$$

As $|\Phi''|^{\mathtt{q}}$ is cr-h-Godunova-Levin, one has

$$\begin{aligned}
&\frac{1}{\varepsilon_2 - \varepsilon_1}\left[{}^{\mathtt{AB}}\mathtt{I}^{\varsigma}_{\frac{\varepsilon_2+\varepsilon_1}{2}}\{\Phi(\varepsilon_1)\} + {}^{\mathtt{AB}}_{\frac{\varepsilon_2+\varepsilon_1}{2}}\mathtt{I}^{\varsigma}_{\varepsilon_2}\{\Phi(\varepsilon_2)\}\right] \\
&- \frac{1}{(\varepsilon_2 - \varepsilon_1)\mathbb{B}(\varsigma)}\big[\Phi(\varepsilon_1) + \Phi(\varepsilon_2)\big] - \frac{(\varepsilon_2 - \varepsilon_1)^{\varsigma-1}}{2^{\varsigma-1}\mathbb{B}(\varsigma)\Gamma(\varsigma)}\Phi\left(\frac{\varepsilon_2+\varepsilon_1}{2}\right) \\
&\preceq_{cr} \frac{(\varepsilon_2 - \varepsilon_1)^{\varsigma-1}}{2(\varsigma+1)\mathbb{B}(\varsigma)\Gamma(\varsigma)}\left\{\frac{2}{\mathtt{p}}\int_0^1 |\mathtt{w}^{\varsigma}(\flat)|^{\mathtt{p}}\mathrm{d}\flat\right. \\
&+\frac{1}{\mathtt{q}}\left[\int_0^1 \frac{|\Phi''(\varepsilon_1)|^{\mathtt{q}}\mathrm{d}\flat}{\mathsf{h}(\flat)} + \int_0^1 \frac{|\Phi''(\varepsilon_2)|^{\mathtt{q}}\mathrm{d}\flat}{\mathsf{h}(1-\flat)}\right. \\
&\left.\left.+\int_0^1 \frac{|\Phi''(\varepsilon_2)|^{\mathtt{q}}\mathrm{d}\flat}{\mathsf{h}(\flat)} + \int_0^1 \frac{|\Phi''(\varepsilon_1)|^{\mathtt{q}}\mathrm{d}\flat}{\mathsf{h}(1-\flat)}\right]\right\} \\
&= \frac{(\varepsilon_2 - \varepsilon_1)^{\varsigma-1}}{2(\varsigma+1)\mathbb{B}(\varsigma)\Gamma(\varsigma)}\left\{\frac{\left(\frac{1}{2}\right)^{\varsigma\mathtt{p}+\mathtt{p}-1}}{(\varsigma\mathtt{p}+\mathtt{p}+1)p} + \frac{1}{\mathtt{q}}\big[|\Phi''(\varepsilon_1)|^{\mathtt{q}} + |\Phi''(\varepsilon_2)|^{\mathtt{q}}\big]\right.
\end{aligned}$$

$$\times \int_0^1 \left(\frac{1}{\mathsf{h}(\flat)} + \frac{1}{\mathsf{h}(1-\flat)} \right) d\flat \Bigg\}.$$

The proof is completed.

Remark 8. *(i) If* $\mathsf{h}(\flat) = \frac{1}{\flat^{\mathsf{s}}}$*, then Theorem 16 yields an outcome for the cr-*s*-convex function for* AB *integral operators:*

$$\frac{1}{\varepsilon_2 - \varepsilon_1} \left[{}^{\mathtt{AB}}\mathrm{I}^{\varsigma}_{\frac{\varepsilon_2+\varepsilon_1}{2}} \{\Phi(\varepsilon_1)\} + {}^{\mathtt{AB}}_{\frac{\varepsilon_2+\varepsilon_1}{2}}\mathrm{I}^{\varsigma}_{\varepsilon_2} \{\Phi(\varepsilon_2)\} \right]$$
$$- \frac{1}{(\varepsilon_2 - \varepsilon_1)\mathbb{B}(\varsigma)} \left[\Phi(\varepsilon_1) + \Phi(\varepsilon_2) \right] - \frac{(\varepsilon_2 - \varepsilon_1)^{\varsigma-1}}{2^{\varsigma-1}\mathbb{B}(\varsigma)\Gamma(\varsigma)} \Phi\left(\frac{\varepsilon_2 + \varepsilon_1}{2} \right)$$
$$\preceq_{cr} \frac{(\varepsilon_2 - \varepsilon_1)^{\varsigma-1}}{(\varsigma+1)\mathbb{B}(\varsigma)\Gamma(\varsigma)} \left\{ \frac{\left(\frac{1}{2}\right)^{\varsigma \mathtt{p}+\mathtt{p}}}{(\varsigma \mathtt{p} + \mathtt{p} + 1)p} + \frac{1}{q(\mathsf{s}+1)} \left[|\Phi''(\varepsilon_1)|^{\mathtt{q}} + |\Phi''(\varepsilon_2)|^{\mathtt{q}} \right] \right\}.$$

(ii) If $\mathsf{h}(\flat) = 1$*, then Theorem 16 yields an outcome for the cr-*p*-convex function for* AB *integral operators:*

$$\frac{1}{\varepsilon_2 - \varepsilon_1} \left[{}^{\mathtt{AB}}\mathrm{I}^{\varsigma}_{\frac{\varepsilon_2+\varepsilon_1}{2}} \{\Phi(\varepsilon_1)\} + {}^{\mathtt{AB}}_{\frac{\varepsilon_2+\varepsilon_1}{2}}\mathrm{I}^{\varsigma}_{\varepsilon_2} \{\Phi(\varepsilon_2)\} \right]$$
$$- \frac{1}{(\varepsilon_2 - \varepsilon_1)\mathbb{B}(\varsigma)} \left[\Phi(\varepsilon_1) + \Phi(\varepsilon_2) \right] - \frac{(\varepsilon_2 - \varepsilon_1)^{\varsigma-1}}{2^{\varsigma-1}\mathbb{B}(\varsigma)\Gamma(\varsigma)} \Phi\left(\frac{\varepsilon_2 + \varepsilon_1}{2} \right)$$
$$\preceq_{cr} \frac{(\varepsilon_2 - \varepsilon_1)^{\varsigma-1}}{(\varsigma+1)\mathbb{B}(\varsigma)\Gamma(\varsigma)} \left\{ \frac{\left(\frac{1}{2}\right)^{\varsigma \mathtt{p}+\mathtt{p}}}{(\varsigma \mathtt{p} + \mathtt{p} + 1)p} + \frac{1}{\mathtt{q}} \left[|\Phi''(\varepsilon_1)|^{\mathtt{q}} + |\Phi''(\varepsilon_2)|^{\mathtt{q}} \right] \right\}.$$

Theorem 17. *Let* $\mathsf{h} : (0,1) \to \mathscr{R}^+$ *and* $\mathsf{h} \neq 0$*. Let* $\Phi : [\varepsilon_1, \varepsilon_2] \to \mathscr{R}^+_{\mathcal{I}}$ *is cr-*h*-Godunova-Levin mapping,* $\varepsilon_1, \varepsilon_2 \in \mathscr{R}^+, \varepsilon_1 < \varepsilon_2$ *and* $[\mathsf{h}(\flat)]^{\mathtt{q}} \in \mathcal{L}_1[0,1], \Phi \in \mathcal{L}_1[\varepsilon_1, \varepsilon_2]$*. If* $|\Phi'|$ *is an cr-*h*-Godunova-Levin mapping on* $[\varepsilon_1, \varepsilon_2]$*, then the following relation*

$$|\mathfrak{B}_{\mathtt{k}}(\Phi, \varepsilon_1, \varepsilon_2)| = \sum_{\jmath=0}^{\mathtt{k}-1} \frac{\varepsilon_2 - \varepsilon_1}{2\mathtt{k}^2} \left[\left(\frac{1}{2} \right)^{\frac{\mathtt{q}-1}{\mathtt{q}}} \left(\int_0^1 \frac{|1-2\flat|}{\mathsf{h}(\flat)} \left| \Phi'\left(\frac{(\mathtt{k}-\jmath)\varepsilon_1 + \jmath\varepsilon_2}{\mathtt{k}} \right) \right|^{\mathtt{q}} d\flat \right.\right.$$
$$\left.\left. + \int_0^1 \frac{|1-2\flat|}{\mathsf{h}(1-\flat)} \left| \Phi'\left(\frac{(\mathtt{k}-\jmath-1)\varepsilon_1 + (\jmath+1)\varepsilon_2}{\mathtt{k}} \right) \right|^{\mathtt{q}} d\flat \right)^{\frac{1}{\mathtt{q}}} \right]$$

holds, where $1 < \mathtt{p}$ *and* $\frac{1}{\mathtt{p}} + \frac{1}{\mathtt{q}} = 1$.

Proof. Let $\mathtt{q} \geq 1$ and using identity from Lemma 2.1 and taking into account Powermean inequality, then we have

$$
\begin{aligned}
&|\mathfrak{B}_{\mathtt{k}}(\Phi,\varepsilon_1,\varepsilon_2)| \\
&\preceq_{cr} \sum_{\jmath=0}^{\mathtt{k}-1} \frac{\varepsilon_2-\varepsilon_1}{2\mathtt{k}^2}\left(\int_0^1 \left|(1-2\flat)\Phi'\left(\flat\frac{(\mathtt{k}-\jmath)\varepsilon_1+\jmath\varepsilon_2}{\mathtt{k}}+(1-\flat)\frac{(\mathtt{k}-\jmath-1)\varepsilon_1+(\jmath+1)\varepsilon_2}{\mathtt{k}}\right)\right| d\flat\right) \\
&\preceq_{cr} \sum_{\jmath=0}^{\mathtt{k}-1} \frac{\varepsilon_2-\varepsilon_1}{2\mathtt{k}^2}\left(\int_0^1 |1-2\flat| d\flat\right)^{1-\frac{1}{\mathtt{q}}} \\
&\quad\times\left(\int_0^1 |1-2\flat|\left|\Phi'\left(\flat\frac{(\mathtt{k}-\jmath)\varepsilon_1+\jmath\varepsilon_2}{\mathtt{k}}+(1-\flat)\frac{(\mathtt{k}-\jmath-1)\varepsilon_1+(\jmath+1)\varepsilon_2}{\mathtt{k}}\right)\right|^{\mathtt{q}} d\flat\right)^{\frac{1}{\mathtt{q}}}.
\end{aligned}
$$

As $|\Phi'|^{\mathtt{q}}$ is cr-h-Godunova-Levin function, we have

$$
\begin{aligned}
&|\mathfrak{B}_{\mathtt{k}}(\Phi,\varepsilon_1,\varepsilon_2)| \\
&\preceq_{cr} \sum_{\jmath=0}^{\mathtt{k}-1} \frac{\varepsilon_2-\varepsilon_1}{2\mathtt{k}^2}\left[\int_0^1 |1-2\flat| d\flat\right]^{1-\frac{1}{\mathtt{q}}}\left[\int_0^1 |1-2\flat|\left(\frac{1}{\mathsf{h}(\flat)}\left|\Phi'\left(\frac{(\mathtt{k}-\jmath)\varepsilon_1+\jmath\varepsilon_2}{\mathtt{k}}\right)\right|^{\mathtt{q}}\right.\right. \\
&\left.\left.+\frac{1}{\mathsf{h}(1-\flat)}\cdot\left|\Phi'\left(\frac{(\mathtt{k}-\jmath-1)\varepsilon_1+(\jmath+1)\varepsilon_2}{\mathtt{k}}\right)\right|^{\mathtt{q}}\right) d\flat\right]^{\frac{1}{\mathtt{q}}} \\
&= \sum_{\jmath=0}^{\mathtt{k}-1} \frac{\varepsilon_2-\varepsilon_1}{2\mathtt{k}^2}\left(\int_0^1 |1-2\flat| d\flat\right)^{1-\frac{1}{\mathtt{q}}}\left(\int_0^1 \frac{|1-2\flat|}{\mathsf{h}(\flat)}\left|\Phi'\left(\frac{(\mathtt{k}-\jmath)\varepsilon_1+\jmath\varepsilon_2}{\mathtt{k}}\right)\right|^{\mathtt{q}} d\flat\right. \\
&\left.+\int_0^1 \frac{|1-2\flat|}{\mathsf{h}(1-\flat)}\left|\Phi'\left(\frac{(\mathtt{k}-\jmath-1)\varepsilon_1+(\jmath+1)\varepsilon_2}{\mathtt{k}}\right)\right|^{\mathtt{q}} d\flat\right)^{\frac{1}{\mathtt{q}}} \\
&= \sum_{\jmath=0}^{\mathtt{k}-1} \frac{\varepsilon_2-\varepsilon_1}{2\mathtt{k}^2}\left[\left(\frac{1}{2}\right)^{\frac{\mathtt{q}-1}{\mathtt{q}}}\left(\int_0^1 \frac{|1-2\flat|}{\mathsf{h}(\flat)}\cdot\left|\Phi'\left(\frac{(\mathtt{k}-\jmath)\varepsilon_1+\jmath\varepsilon_2}{\mathtt{k}}\right)\right|^{\mathtt{q}} d\flat\right.\right. \\
&\left.\left.+\int_0^1 \frac{|1-2\flat|}{\mathsf{h}(1-\flat)}\left|\Phi'\left(\frac{(\mathtt{k}-\jmath-1)\varepsilon_1+(\jmath+1)\varepsilon_2}{\mathtt{k}}\right)\right|^{\mathtt{q}} d\flat\right)^{\frac{1}{\mathtt{q}}}\right].
\end{aligned}
$$

Corollary 1. *Setting* $\mathsf{h}(\flat)=\frac{1}{\flat}$ *and* $\underline{\Phi}=\overline{\Phi}$ *in Theorem 17, we get*

$$
|\mathfrak{B}_{\mathtt{k}}(\Phi,\varepsilon_1,\varepsilon_2)| = \sum_{\jmath=0}^{\mathtt{k}-1} \frac{\varepsilon_2-\varepsilon_1}{\mathtt{k}^2(2)^{2+\frac{1}{\mathtt{q}}}}\left(\left|\Phi'\left(\frac{(\mathtt{k}-\jmath)\varepsilon_1+\jmath\varepsilon_2}{z}\right)\right|^{\mathtt{q}}+\left|\Phi'\left(\frac{(\mathtt{k}-\jmath-1)\varepsilon_1+(\jmath+1)\varepsilon_2}{\mathtt{k}}\right)\right|^{\mathtt{q}}\right)^{\frac{1}{\mathtt{q}}}
$$

which has been obtained by authors in [29].

Corollary 2. *Setting* $\mathsf{h}(\flat) = \frac{1}{\flat^{s}}$ *and* $\underline{\Phi} = \overline{\Phi}$ *in Theorem 17, we get*

$$|\mathfrak{B}_{\mathtt{k}}(\Phi, \varepsilon_1, \varepsilon_2)| \preceq_{cr} \sum_{\varepsilon=0}^{\mathtt{k}-1} \frac{\varepsilon_2 - \varepsilon_1}{\mathtt{k}^2 2^{2-\frac{1}{\mathtt{q}}}} \left(\frac{1}{2^s(\mathtt{s}+1)(\mathtt{s}+2)} + \frac{s}{(\mathtt{s}+1)(\mathtt{s}+2)} \right)^{\frac{1}{\mathtt{q}}}$$
$$\times \left[\left| \Phi' \left(\frac{(\mathtt{k}-\varepsilon)\varepsilon_1 + \varepsilon r}{\mathtt{k}} \right) \right|^{\mathtt{q}} + \left| \Phi' \left(\frac{(\mathtt{k}-\varepsilon-1)\varepsilon_1 + (\varepsilon+1)\varepsilon_2}{\mathtt{k}} \right) \right|^{\mathtt{q}} \right]^{\frac{1}{\mathtt{q}}},$$

which has been proved by authors in [57].

Theorem 18. *Let* $\mathsf{h} : (0,1) \to \mathscr{R}^+$ *and* $\mathsf{h} \neq 0$*. Let* $\Phi : [\varepsilon_1, \varepsilon_2] \to \mathscr{R}^+_{\mathcal{I}}$ *is cr-*h*-Godunova-Levin mapping,* $\varepsilon_1, \varepsilon_2 \in \mathscr{R}^+, \varepsilon_1 < \varepsilon_2$ *and* $[\mathsf{h}(\flat)]^{\mathtt{q}} \in \mathcal{L}_1[0,1], \Phi \in \mathcal{L}_1[\varepsilon_1, \varepsilon_2]$*. If* $|\Phi'|$ *is also cr-*h*-Godunova-Levin mapping on* $[\varepsilon_1, \varepsilon_2]$*, then the following relation*

$$|\mathfrak{B}_{\mathtt{k}}(\Phi, \varepsilon_1, \varepsilon_2)| \preceq_{cr} \sum_{\jmath=0}^{\mathtt{k}-1} \frac{\varepsilon_2 - \varepsilon_1}{2\mathtt{k}^2} \left[\left(\frac{1}{1+\mathtt{p}} \right)^{\frac{1}{\mathtt{p}}} \right.$$
$$\left. \times \left(\int_0^1 \left(\frac{1}{\mathsf{h}(\flat)} \left| \Phi' \left(\frac{(\mathtt{k}-\jmath)\varepsilon_1 + \jmath\varepsilon_2}{\mathtt{k}} \right) \right|^{\mathtt{q}} + \frac{1}{\mathsf{h}(1-\flat)} \left| \Phi' \left(\frac{(\mathtt{k}-\jmath-1)\varepsilon_1 + (\jmath+1)\varepsilon_2}{\mathtt{k}} \right) \right|^{\mathtt{q}} \right) d\flat \right)^{\frac{1}{\mathtt{q}}} \right]$$

holds, where $\frac{1}{\mathtt{q}} + \frac{1}{\mathtt{p}} = 1$.

Proof. Assume that $1 < \mathtt{p}$. Taking into account Lemma 2.1 and the Hölder inequality, one has

$$|\mathfrak{B}_{\mathtt{k}}(\Phi, \varepsilon_1, \varepsilon_2)| \preceq_{cr} \sum_{\jmath=0}^{\mathtt{k}-1} \frac{\varepsilon_2 - \varepsilon_1}{2\mathtt{k}^2} \left[\left(\int_0^1 \left| (1-2\flat)\Phi' \left(\flat \frac{(\mathtt{k}-\jmath)\varepsilon_1 + \jmath\varepsilon_2}{\mathtt{k}} + (1-\flat) \frac{(\mathtt{k}-\jmath-1)\varepsilon_1 + (\jmath+1)\varepsilon_2}{\mathtt{k}} \right) \right| d\flat \right) \right]$$
$$\preceq_{cr} \sum_{\jmath=0}^{\mathtt{k}-1} \frac{\varepsilon_2 - \varepsilon_1}{2\mathtt{k}^2} \left[\left(\int_0^1 |1-2\flat|^{\mathtt{p}} d\flat \right)^{\frac{1}{\mathtt{p}}} \right.$$
$$\left. \times \left(\int_0^1 \left| \Phi' \left(\flat \frac{(\mathtt{k}-\jmath)\varepsilon_1 + \jmath b}{\mathtt{k}} + (1-\flat) \frac{(\mathtt{k}-\jmath-1)\varepsilon_1 + (\jmath+1)b}{\mathtt{k}} \right) \right|^{\mathtt{q}} d\flat \right)^{\frac{1}{\mathtt{q}}} \right].$$

As $|\Phi'|^{\mathtt{q}}$ is cr-h-Godunova-Levin mapping, one has

$$\int_0^1 \left| \Phi' \left(\flat \frac{(\mathtt{k}-\jmath)\varepsilon_1 + \jmath\varepsilon_2}{\mathtt{k}} + (1-\flat) \frac{(\mathtt{k}-\jmath-1)\varepsilon_1 + (\jmath+1)\varepsilon_2}{\mathtt{k}} \right) \right| d\flat$$
$$\preceq_{cr} \int_0^1 \left(\frac{1}{\mathsf{h}(\flat)} \left| \Phi' \left(\frac{(\mathtt{k}-\jmath)\varepsilon_1 + \jmath\varepsilon_2}{\mathtt{k}} \right) \right|^{\mathtt{q}} + \frac{1}{\mathsf{h}(1-\flat)} \left| \Phi' \left(\frac{(\mathtt{k}-\jmath-1)\varepsilon_1 + (\jmath+1)\varepsilon_2}{\mathtt{k}} \right) \right| \right) d\flat.$$

Therefore, we deduce

$$
\begin{aligned}
&|\mathfrak{B}_{\mathtt{k}}(\Phi,\varepsilon_1,\varepsilon_2)| \\
&\preceq_{cr} \sum_{\jmath=0}^{\mathtt{k}-1}\frac{\varepsilon_2-\varepsilon_1}{2\mathtt{k}^2}\left[\left(\int_0^1|1-2\flat|^{\mathtt{p}}d\flat\right)^{\frac{1}{\mathtt{p}}}\right.\\
&\quad\left.\times\left(\int_0^1\left(\frac{1}{\mathsf{h}(\flat)}\left|\Phi'\left(\frac{(\mathtt{k}-\jmath)\varepsilon_1+\jmath\varepsilon_2}{\mathtt{k}}\right)\right|^{\mathtt{q}}+\frac{1}{\mathsf{h}(1-\flat)}\left|\Phi'\left(\frac{(\mathtt{k}-\jmath-1)\varepsilon_1+(\jmath+1)\varepsilon_2}{\mathtt{k}}\right)\right|^{\mathtt{q}}\right)d\flat\right)^{\frac{1}{\mathtt{q}}}\right]\\
&\preceq_{cr} \sum_{\jmath=0}^{\mathtt{k}-1}\frac{\varepsilon_2-\varepsilon_1}{2\mathtt{k}^2}\left[\left(\frac{1}{1+\mathtt{p}}\right)^{\frac{1}{\mathtt{p}}}\right.\\
&\quad\left.\times\left(\int_0^1\left(\frac{1}{\mathsf{h}(\flat)}\left|\Phi'\left(\frac{(\mathtt{k}-\jmath)\varepsilon_1+\jmath\varepsilon_2}{\mathtt{k}}\right)\right|^{\mathtt{q}}+\frac{1}{\mathsf{h}(1-\flat)}\left|\Phi'\left(\frac{(\mathtt{k}-\jmath-1)\varepsilon_1+(\jmath+1)\varepsilon_2}{\mathtt{k}}\right)\right|^{\mathtt{q}}\right)d\flat\right)^{\frac{1}{\mathtt{q}}}\right].
\end{aligned}
$$

Remark 9. *Setting* $\mathsf{h}(\flat)=\flat^{-\mathtt{s}}$ *and* $\underline{\Phi}=\overline{\Phi}$ *in Theorem 18, then we get Theorem 6 in [57].*

4. Applications to special means

The following section relates some of our main results with special means, and illustrates some of their applications. Let $\varepsilon_1,\varepsilon_2\in\mathscr{R}$,

(i) The arithmetic mean:

$$\mathtt{A}=\mathtt{A}(\varepsilon_1,\varepsilon_2):=\frac{\varepsilon_1+\varepsilon_2}{2},\varepsilon_1,\varepsilon_2\geq 0.$$

(ii) The harmonic mean:

$$\mathtt{H}=\mathtt{H}(\varepsilon_1,\varepsilon_2):=\frac{2\varepsilon_1\varepsilon_2}{\varepsilon_1+\varepsilon_2},\varepsilon_1,\varepsilon_2>0.$$

(iii) The logarithmic mean:

$$\mathtt{L}=\mathtt{L}(\varepsilon_1,\varepsilon_2):=\begin{cases}\varepsilon_1, & \text{if } \varepsilon_1=\varepsilon_2\\ \frac{\varepsilon_2-\varepsilon_1}{\ln\varepsilon_2-\ln\varepsilon_1}, & \text{if } \varepsilon_1\neq\varepsilon_2,\end{cases}\quad \varepsilon_1,\varepsilon_2>0.$$

(iv) The $\mathtt{p}$-logarithmic mean:

$$\mathtt{L}_{\mathtt{p}}=\mathtt{L}_{\mathtt{p}}(\varepsilon_1,\varepsilon_2):=\begin{cases}\varepsilon_1, & \text{if } \varepsilon_1=\varepsilon_2\\ \left[\frac{r^{\mathtt{p}+1}-\varepsilon_1{}^{\mathtt{p}+1}}{(\mathtt{p}+1)(\varepsilon_2-\varepsilon_1)}\right]^{\frac{1}{\mathtt{p}}}, & \text{if } \varepsilon_1\neq\varepsilon_2,\end{cases}\quad \mathtt{p}\in\mathscr{R}\backslash\{-1,0\},\varepsilon_1,\varepsilon_2>0.$$

Let $\varepsilon_1,\varepsilon_2\in\mathscr{R}, 0<\varepsilon_1<\varepsilon_2$, and $\mathtt{m}\in\mathsf{N},\mathtt{m}\geq 2$. Then, the following

$$\left|\sum_{\jmath=0}^{\mathtt{k}-1}\frac{1}{\mathtt{k}\jmath}\mathtt{A}\left(\left(\frac{(\mathtt{k}-\jmath)\varepsilon_1+\jmath\varepsilon_2}{\mathtt{k}}\right)^{\mathtt{m}},\left(\frac{(\mathtt{k}-\jmath-1)\varepsilon_1+(\jmath+1)\varepsilon_2}{\mathtt{k}}\right)^{\mathtt{m}}\right)-\mathtt{L}_{\mathtt{m}}{}^{\mathtt{m}}(\varepsilon_1,\varepsilon_2)\right|$$
$$\preceq_{cr}\sum_{\jmath=0}^{\mathtt{k}-1}\frac{(\varepsilon_2-\varepsilon_1)\mathtt{m}}{2^{2-\frac{1}{\mathtt{q}}}\mathtt{k}^2}\left[\left(\int_0^1\frac{|1-2\flat|}{\mathsf{h}(\flat)}d\flat\right)\left(\frac{(\mathtt{k}-\jmath)\varepsilon_1+\jmath\varepsilon_2}{\mathtt{k}}\right)^{(\mathtt{m}-1)\mathtt{q}}\right.$$
$$\left.+\left(\int_0^1\frac{|1-2\flat|}{\mathsf{h}(1-\flat)}d\flat\right)\left(\frac{(\mathtt{k}-\jmath-1)\varepsilon_1+(\jmath+1)\varepsilon_2}{\mathtt{k}}\right)^{(\mathtt{m}-1)\mathtt{q}}\right)\Bigg]^{\frac{1}{\mathtt{q}}}$$

holds, for all $1\preceq_{cr}\mathtt{q}$.

Proof.

This proof is proven using Theorem 17 with the following settings $\Phi(\flat)=\flat^{\mathtt{m}},\flat\in[\varepsilon_1,\varepsilon_2],\mathtt{m}\in\mathsf{N},\mathtt{m}\geq 2$.

Let $\varepsilon_1,\varepsilon_2\in\mathscr{R},0<\varepsilon_1<\varepsilon_2$, and $\mathtt{m}\in\mathsf{N},\mathtt{m}\geq 2$. Then, the following

$$\left|\sum_{\jmath=0}^{\mathtt{k}-1}\frac{1}{\mathtt{k}}\mathtt{A}\left(\left(\frac{(\mathtt{k}-\jmath)\varepsilon_1+\jmath\varepsilon_2}{\mathtt{k}}\right)^{\mathtt{m}},\left(\frac{(\mathtt{k}-\jmath-1)\varepsilon_1+(\jmath+1)\varepsilon_2}{\mathtt{k}}\right)^{\mathtt{m}}\right)-\mathtt{L}_{\mathtt{m}}{}^{\mathtt{m}}(\varepsilon_1,\varepsilon_2)\right|$$
$$\preceq_{cr}\sum_{\jmath=0}^{\mathtt{k}-1}\frac{(\varepsilon_2-\varepsilon_1)\mathtt{m}}{2^{2-\frac{1}{\mathtt{q}}}\mathtt{k}^2}\left(\frac{1}{\mathtt{p}+1}\right)^{\frac{1}{\mathtt{p}}}$$
$$\times\left[\left(\int_0^1\frac{d\flat}{\mathsf{h}(\flat)}\right)\left(\frac{(\mathtt{k}-\jmath)\varepsilon_1+\jmath\varepsilon_2}{\mathtt{k}}\right)^{(\mathtt{m}-1)\mathtt{q}}+\left(\int_0^1\frac{d\flat}{\mathsf{h}(1-\flat)}\right)\left(\frac{(\mathtt{k}-\jmath-1)\varepsilon_1+(\jmath+1)\varepsilon_2}{\mathtt{k}}\right)^{(\mathtt{m}-1)\mathtt{q}}\right)\Bigg]^{\frac{1}{\mathtt{q}}}$$

holds, for all $1\preceq_{cr}\mathtt{q}$.

Proof. This proof is proven using Theorem 18 with the following settings $\Phi(\flat)=\flat^{\mathtt{m}},\flat\in[\varepsilon_1,\varepsilon_2],\mathtt{m}\in\mathsf{N},\mathtt{m}\geq 2$.

5. Conclusion

The primary contribution in this paper to present the different variants of double inequalities by using fractional integral operators involving special functions. Recently, authors in [4, 11, 13] developed several relevant results by using classical integral operators and partial standard order relation. Additionally, we use a number of other well-known inequalities, including Holder's, Young's, and Minkowski's, to extend these upper bounds for Hermite-Hadamard inequality. Furthermore, these types of results involving Godunova-Levin mappings and fractional operators are not initiated, and we believe that our study opens up a whole new path for other relevant classes of Godunova-Levin functions, including s-Godunova-Levin, tgs-Godunova-Levin, Harmonic, and various others.

In the future, we suggest that readers extend these inequalities to probabilistic and fractional stochastic settings, where randomness is incorporated into the functions themselves or into the bounds, so they can use them more effectively in risk analysis, finance, or uncertainty quantification. Furthermore, generalizing Hermite-Hadamard inequalities to multiple variables, especially in convexity spaces like convex domains in $\mathtt{R}^n$, may yield inequalities useful in multivariate optimization, machine learning, and control theory.

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