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Fuzzy (m, n)-Ideals and n-Interior Ideals in Ordered Semigroups

P. Khamrot[1], A. Iampan[2], T. Gaketem[2,*]

[1] *Department of Mathematics, Faculty of Science and Agricultural Technology, Rajamangala University of Technology Lanna of Phitsanulok, Phitsanulok, Thailand*
[2] *Department of Mathematics, School of Science, University of Phayao, Mae Ka, Mueang, Phayao 56000, Thailand*

Abstract. In 2019, Ahsan et al. developed the concept of fuzzy (m, n)-ideals in semigroups. Later, in 2022 Tiprachot studied n-interior ideals in ordered semigroup, which is a generalization of fuzzy ideals and interior ideals in semigroups. The aim of this paper is to study the concept of fuzzy (m, n)-ideals and n-interior ideals in ordered semigroup and investigate the properties of fuzzy (m, n)-ideals and n-interior ideals.

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1. Introduction

The theory of ordered semigroups originated as a generalization of the semigroup theory. The concept of (m, n)-T. Changphas gave ideals in ordered semigroups in [3] which was obtained by generalizing the idea of (m, n)-ideals in semigroups.

As a theory of fuzzy set it is tool for dealing with possibilities of uncertainty, connected with the imprecision of states, perceptions, and preferences, and was studied by Zadeh in 1965 [14]. It has been applied to many areas, such as medical science, robotics, computer science, information science, control engineering, measure theory, logic, set theory, topology and others. The study of fuzzy sets in semigroups was introduced by Kuroki in 1981. In 2020 Kehayopulu and Tsingelis [5] extended the concept of fuzzy semigroups to the fuzzy ordered semigroups and studied some properties of fuzzy left (right) ideals and fuzzy filters in ordered semigroups. In 2019, Ahsan et al. [9] extended the notion of (m, n)-ideals in semigroups to the notion of fuzzy (m, n)-ideals in semigroups and they characterized (m, n)-regular semigroups by using fuzzy (m, n)-ideals. Tiprachot et al. [12]

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Email addresses: pk_g@rmutl.ac.th (P. Khamrot),
aiyared.ia@up.ac.th (A. Iampan), thiti.ga@up.ac.th (T. Gaketem)

discussed the notion of n-interior ideals as a generalization of interior ideals and characterized many classes of ordered semigroups in terms of (m,n)-ideals and n-interior ideals. In 2023, Tiprachot et al. [13] extend n-interior ideals and (m,n)-ideals to hybrid in ordered semigroups. Recently T. Gaketem and P. Khamrot [6] studied concepts interval valued fuzzy (m,n)-ideals in semigroups. In the same year A. Mahoob et al. [8] gave the concepts of structures of fuzzy (m,n)-quasi ideals in ordered semigroups. Other work of ordered semigroup studied more like fuzzy quasi-ideal [4], fuzzy filters [1, 2, 7] and fuzzy prime [10].

The purpose of this paper is to extend the study of fuzzy (m,n)-ideals in semigroups to ordred semigroups. We prove the properties of fuzzy (m,n)-ideals in ordered semigroup and investigate the properties of minimal fuzzy (m,n) ideals, and fuzzy prime (semiprime) (m,n) ideals in semigroups. The relationship between (m,n) ideals and fuzzy (m,n) ideals in ordered semigroups. Finally, we discuss the properties of fuzzy n-interior ideals and the relationship between n-interior ideals and fuzzy n-interior ideals in ordered semigroups.

2. Preliminaries

In this section, we review some basic concepts that are necessary to understand our next section.

Let $(\mathcal{S},\cdot)$ be a semigroup ordered semigroup and $\mathcal{S},\leq)$ is a partially ordered set. Then $(\mathcal{S},\cdot,\leq)$ is an ordered semigroup if for all $a,b,c\in\mathcal{S}$, we have $a\leq b$ then $ac\leq bc$ and $ca\leq cb$.

For a nonempty subset $\mathcal{X}$ and $\mathcal{Y}$ of ordered semigroup $\mathcal{S}$, we write
$(\mathcal{X}] := \{a\in\mathcal{S} \mid a\leq b \text{ for some } b\in\mathcal{X}\}$ and $\mathcal{X}\mathcal{Y} := \{xy \mid x\in\mathcal{X} \text{ and } y\in\mathcal{Y}\}$.

It is observed that

(1) $\mathcal{X}\subseteq(\mathcal{X}]$,

(2) if $\mathcal{X}\subseteq\mathcal{Y}$, then $(\mathcal{X}]\subseteq(\mathcal{Y}]$,

(3) $((\mathcal{X}]]=(\mathcal{X}]$,

(4) $(\mathcal{X}](\mathcal{Y}]\subseteq(\mathcal{X}\mathcal{Y}]$,

(5) $((\mathcal{X}](\mathcal{Y}]]=(\mathcal{X}\mathcal{Y}]$,

(6) $(\mathcal{X}\cup\mathcal{Y}]=(\mathcal{X}]\cup(\mathcal{Y}]$,

(7) $(\mathcal{X}\cap\mathcal{Y}]=(\mathcal{X}]\cap(\mathcal{Y}]$.

Let $(\mathcal{S},\cdot,\leq)$ be an ordered semigroup, $(\emptyset\neq)\mathcal{K}\subseteq\mathcal{K}$ is called a *subsemigroup* such that $\mathcal{K}^2\subseteq\mathcal{K}$. A *left (right) ideal* of a ordered semigroup $(\mathcal{S},\cdot,\leq)$ is a non-empty set $\mathcal{K}$ of $\mathcal{K}$ such that $\mathcal{S}\mathcal{K}\subseteq\mathcal{K}$ $(\mathcal{K}\mathcal{S}\subseteq\mathcal{K})$ and $(\mathcal{K}]$. By an *ideal* of an ordered semigroup $(\mathcal{S},\cdot,\leq)$, we mean a non-empty set of $\mathcal{S}$ which is both a left and a right ideal of $\mathcal{S}$.

Definition 1. *[3] A subsemigroup $\mathcal{K}$ of an ordered semigroup $(\mathcal{S},\cdot,\leq)$ is called an (m,n)-*ideal *of $\mathcal{S}$ if $\mathcal{K}$ satisfies the following conditions:*

(1) $\mathcal{K}^m\mathcal{S}\mathcal{K}^n \subseteq \mathcal{K}$.

(2) $\mathcal{K} = (\mathcal{K}]$, *that is for* $x \in \mathcal{K}$ *and* $y \in \mathcal{S}$, $y \leq x$ *implies* $y \in \mathcal{K}$.

where m, n *are non-negative integers.*

We see that for any $\delta_1, \delta_2 \in [0,1]$, we have

$$\delta_1 \vee \delta_2 = \max\{\delta_1, \delta_2\} \quad \text{and} \quad \delta_1 \wedge \delta_2 = \min\{\delta_1, \delta_2\}.$$

A fuzzy set δ of a non-empty set $\mathcal{T}$ is function from $\mathcal{T}$ into unit closed interval $[0,1]$ of real numbers, i.e., $\delta : \mathcal{T} \to [0,1]$.

For any two fuzzy sets δ and ϑ of a non-empty set $\mathcal{T}$, define $\geq, =, \wedge$ and $\vee$ as follows:

(1) $\delta \geq \vartheta \Leftrightarrow \delta(\mathfrak{e}) \geq \vartheta(\mathfrak{e})$ for all $\mathfrak{e} \in \mathcal{T}$,

(2) $\delta = \vartheta \Leftrightarrow \delta \geq \vartheta$ and $\vartheta \geq \delta$,

(3) $(\delta \wedge \vartheta)(\mathfrak{e}) = \min\{\delta(\mathfrak{e}), \vartheta(\mathfrak{e})\} = \delta(\mathfrak{e}) \wedge \vartheta(\mathfrak{e})$ for all $\mathfrak{e} \in \mathcal{T}$,

(4) $(\delta \vee \vartheta)(\mathfrak{e}) = \max\{\delta(\mathfrak{e}), \vartheta(\mathfrak{e})\} = \delta(\mathfrak{e}) \vee \vartheta(\mathfrak{e})$ for all $\mathfrak{e} \in \mathcal{T}$.

For the symbol $\delta \leq \vartheta$, we mean $\vartheta \geq \delta$.

The following definitions are types of fuzzy substructures of a semigroup.

Definition 2. *[11] A fuzzy set* δ *of a semigroup* $\mathcal{S}$ *is said to be a fuzzy ideal of* $\mathcal{S}$ *if* $\delta(\mathfrak{u}\mathfrak{v}) \geq \delta(\mathfrak{u}) \vee \delta(\mathfrak{v})$ *for all* $\mathfrak{u}, \mathfrak{v} \in \mathcal{S}$.

Definition 3. *[9] A fuzzy subsemigroup of* δ *if a semigroup* $\mathcal{S}$ *is said to be a* fuzzy (m,n)-ideal *of* $\mathcal{S}$ *if for all* $u_1, u_2, ..., u_m, v_1, v_2, ..., v_n, z \in \mathcal{S}$ *and* $m, n \in \mathbb{N}$, *we have*

$$\delta_{\mathcal{S}}(u_1u_2\cdots u_m z v_1 v_2 \cdots v_n) \geq \delta_{\mathcal{S}}(u_1) \wedge \vartheta_{\mathfrak{F}}(u_2) \wedge ... \wedge \delta_{\mathcal{S}}(u_m) \wedge \delta_{\mathcal{S}}(v_1) \wedge \delta_{\mathcal{S}}(v_2) \wedge ... \wedge \delta_{\mathcal{S}}(v_n).$$

For any element k in an ordered semigroup $\mathcal{S}$, define the set F_k by $F_k := \{(y,z) \in \mathcal{S} \times \mathcal{S} \mid k \leq yz\}$.

For two fuzzy sets δ and ϑ on a semigroup $\mathcal{S}$, define the product $\delta \circ \vartheta$ as follows: For all $k \in \mathcal{S}$,

$$(\delta \circ \vartheta)(k) = \begin{cases} \bigvee\limits_{(y,z)\in F_k} \{\delta(y) \wedge \vartheta(z)\} & \text{if} \quad F_k \neq \emptyset, \\ 0 & \text{if} \quad F_k = \emptyset. \end{cases}$$

Definition 4. *Let* $\mathcal{I}$ *be a non-empty set of an ordered semigroup* $\mathcal{S}$. *A* ***characteristic function*** *are respectively defined by*

$$\lambda_{\mathcal{I}} : \mathcal{S} \to [0,1], k \mapsto \lambda_{\mathcal{I}}(u) := \begin{cases} 1 & k \in \mathcal{I}, \\ 0 & k \notin \mathcal{I}, \end{cases}$$

The following definitions are types of fuzzy subsemigroups on ordered semigroups.

Definition 5. *[11] A fuzzy set ξ of an ordered semigroups $\mathcal{S}$ is said to be A fuzzy left (right) ideal of S if $u \leq v$ implies $\delta(u) \geq \delta(y)$ for all $u, v \in \mathcal{S}$ and $\delta(uv) \geq \delta(v)$ $(\delta(uv) \geq \delta(u))$ for all $u, v \in \mathcal{S}$.*

Lemma 1. *Let $\mathcal{K}$ be a nonempty subset of an ordered semigroup $\mathcal{S}$. Then $\mathcal{K}$ is a subsemigroup of $\mathcal{S}$ if and only if the characteristic function $\lambda_{\mathcal{K}}$ is a fuzzy subsemigroup of $\mathcal{S}$.*

3. Fuzzy (m, n)-ideals

In this section, we outline the concept of fuzzy (m, n)-ideals and explore their properties within ordered semigroups.

Definition 6. *A fuzzy subsemigroup δ of an ordered semigroup $\mathcal{S}$ is called a* fuzzy (m, n)-ideal *of $\mathcal{S}$ if*

(1) $\delta(u_1 u_2 \cdots u_m k v_1 v_2 \cdots v_n) \geq \delta(u_1) \wedge \delta(u_2) \wedge \cdots \wedge \delta(u_m) \wedge \delta(v_1) \wedge \delta(v_2) \wedge \cdots \wedge \delta(v_n)$ *for all* $u_1, u_2, \ldots, u_m, k, v_1, v_2, \ldots v_n$ *of* $\mathcal{S}$ *and* $m, n \in \mathbb{N}$.

(2) $u \leq v$ *implies* $\delta(u) \geq \delta(v)$ *for all* $u, v \in \mathcal{S}$.

Example 1. *Consider the ordered semigroup $\mathcal{S} = \{w, x, y, z\}$ with the following Cayley table:*

$\cdot$	w	x	y	z
w	w	w	w	w
x	w	w	z	w
y	w	w	w	w
z	w	w	w	w

and $\leq: \{(w, w), (x, x), (y, y), (z, z)\}$. *Define a function $\delta : \mathcal{S} \to [0, 1]$ by $\delta(w) = 0.4$, $\delta(x) = 0.4$, $\delta(y) = 0$, $\delta(z) = 0$. Then δ is a fuzzy (m, n)-ideal of $\mathcal{S}$.*

Theorem 1. *Let $\{\delta_i \mid i \in \mathcal{J}\}$ be a family of fuzzy (m, n)-ideals of an ordered semigroup $\mathcal{S}$ with $\delta(u) \geq \delta(v)$ whenever $u \leq v$. Then $\bigwedge_{i \in \mathcal{F}} \vartheta_i$ is a fuzzy (m, n)-ideal of $\mathcal{S}$.*

Proof. Let $u, v \in \mathcal{S}$. Then,

$$\bigwedge_{i \in \mathcal{J}} \delta_i(uv) \geq \bigwedge_{i \in \mathcal{J}} \{\delta_i(u) \wedge \delta(v)\} = \bigwedge_{i \in \mathcal{J}} \delta_i(u) \wedge \bigwedge_{i \in \mathcal{J}} \delta_i(v).$$

Thus, $\bigwedge_{i \in \mathcal{J}} \delta_i$ is a fuzzy subsemigroup of $\mathcal{S}$.
Let $u_1, u_2, \ldots, u_m, k, v_1, v_2, \ldots v_n \in \mathcal{S}$. Then,

$$\bigwedge_{i \in \mathcal{J}} \delta_i(u_1 u_2 \cdots u_m k v_1 v_2 \cdots v_n)$$

$$\geq \bigwedge_{i\in\mathcal{J}} \{\delta_i(u_1) \wedge \delta_i(u_2) \cdots \wedge \delta_i(u_n) \wedge \delta_i(v_1) \wedge \delta_i(v_2) \ldots \delta_i(v_n)\}$$
$$= \bigwedge_{i\in\mathcal{J}} \delta_i(u_1) \wedge \bigwedge_{i\in\mathcal{J}} \delta_i(u_2) \cdots \wedge \bigwedge_{i\in\mathcal{J}} \delta_i(u_n) \wedge \bigwedge_{i\in\mathcal{J}} \delta_i(v_1) \wedge \bigwedge_{i\in\mathcal{J}} \delta_i(v_2) \ldots \bigwedge_{i\in\mathcal{J}} \delta_i(v_n).$$

Thus, $\bigwedge_{i\in\mathcal{J}} \delta_i$ is a fuzzy (m,n)-ideal of $\mathcal{S}$.

Theorem 2. *Let $\mathcal{K}$ be a nonempty subset of an ordered semigroup $\mathcal{S}$ and m,n are positive integers. Then $\mathcal{K}$ is an (m,n)-ideal of $\mathcal{S}$ if and only if the characteristic function $\lambda_{\mathcal{K}}$ is a fuzzy (m,n)-ideal of $\mathcal{S}$.*

Proof. Suppose that $\mathcal{K}$ is an (m,n)-ideal of $\mathcal{S}$. Then, $\mathcal{K}$ is a subsemigroup of $\mathcal{S}$. By Lemma 1, $\lambda_{\mathcal{K}}$ is a fuzzy subsemigroup of $\mathcal{S}$.

Let $u_1, u_2, \ldots u_m, k, v_1, v_2, \ldots, v_n \in \mathcal{S}$. Then the following cases:

Case 1 If $u_i, v_j \in \mathcal{K}$ for all $i \in \{1,2,\ldots,m\}$ and $j \in \{1,2,\ldots,n\}$, then $u_1u_2\cdots u_mkv_1v_2\cdots v_n \in \mathcal{K}^m\mathcal{S}\mathcal{K}^n$. Thus, $\lambda_K(u_1u_2\cdots u_mkv_1v_2\cdots v_n) = 1$, $\lambda_{\mathcal{K}}(u_i) = 1$ for all $i \in \{1,2,\ldots,m\}$ and $\lambda_{\mathcal{K}}(r_j) = 1$ for all $j \in \{1,2,\ldots,n\}$. So, we have

$$\lambda_{\mathcal{K}}(u_1u_2\cdots u_mkv_1v_2\cdots v_n) \geq \lambda_{\mathcal{K}}(u_1) \wedge \lambda_K(u_2) \wedge \cdots \wedge \lambda_{\mathcal{K}}(u_m) \wedge \cdots \wedge \lambda_{\mathcal{K}}(v_1) \wedge \lambda_{\mathcal{K}}(r_2) \wedge \cdots \wedge \lambda_{\mathcal{K}}(r_n).$$

Case 2 If $e_i \notin \mathcal{K}$ or $r_j \notin \mathcal{K}$ for some $i \in \{1,2,\ldots,m\}$ and $j \in \{1,2,\ldots,n\}$, then

$$\lambda_{\mathcal{K}}(u_1u_2\cdots u_mkv_1v_2\cdots v_n) \geq \lambda_{\mathcal{K}}(u_1) \wedge \lambda_K(u_2) \wedge \cdots \wedge \lambda_{\mathcal{K}}(u_m) \wedge \cdots \wedge \lambda_{\mathcal{K}}(v_1) \wedge \lambda_{\mathcal{K}}(r_2) \wedge \cdots \wedge \lambda_{\mathcal{K}}(r_n).$$

Let $u, v \in \mathcal{S}$ such that $u \leq v$ and $u \in \mathcal{K}$. Then $\lambda_{\mathcal{K}}(u) = 1$. Thus, $\lambda_{\mathcal{K}}(u) \geq \lambda_{\mathcal{K}}(v)$.

Therefore, $\lambda_{\mathcal{K}}$ is a fuzzy (m,n)-ideal of $\mathcal{S}$.

Conversely, suppose that $\lambda_{\mathcal{K}}$ is a fuzzy (m,n)-ideal of $\mathcal{S}$. Then $\lambda_{\mathcal{K}}$ is a fuzzy subsemigroup of $\mathcal{S}$. By Lemma 1, $\mathcal{K}$ is a subsemigroup of $\mathcal{S}$.

Let $u_1, u_2, \ldots u_m, k, v_1, v_2, \ldots, v_n \in \mathcal{K}^m\mathcal{S}\mathcal{K}^n$. Then $\lambda_K(u_i) = 1$ and $\lambda_{\mathcal{K}}(v_j) = 1$ for some $i \in \{1,2,\ldots m\}$ and $j \in \{1,2,\ldots,n\}$. By assumption, $\lambda_{\mathcal{K}}(u_1u_2\cdots u_mkv_1v_2\cdots v_n) \geq \lambda_{\mathcal{K}}(u_1) \wedge \lambda_K(u_2) \wedge \cdots \wedge \lambda_K(u_m) \wedge \cdots \wedge \lambda_{\mathcal{K}}(v_1) \wedge \lambda_{\mathcal{K}}(v_2) \wedge \cdots \wedge \lambda_{\mathcal{K}}(v_n)$.
Thus, $\lambda_{\mathcal{K}}(u_1u_2\cdots u_mkv_1v_2\cdots v_n) = 1$. It impiles that, $e_1e_2\cdots e_mkv_1v_2\cdots v_n \in \mathcal{K}$. Hence, $\mathcal{K}^m\mathcal{S}\mathcal{K}^n \subseteq \mathcal{K}$.

Let $u \in \mathcal{K}$ such that $v \leq u$ and $v \in \mathcal{S}$. Then $\lambda_{\mathcal{K}}(u) \geq \lambda_{\mathcal{K}}(v) \geq 1$. Thus, $v \in \mathcal{K}$. Therefore, $\mathcal{K}$ is an (m,n)-ideal of $\mathcal{S}$.

Let δ be a fuzzy set and $t \in [0,1]$. Define the set $U_t := \{e \in \mathcal{S} \mid \delta(e) \geq t\}$ is called an t-level subset of fuzzy set of δ.

Lemma 2. *A fuzzy set δ is a fuzzy subsemigroup of a semigroup $\mathcal{S}$ if and only if the level set U_t is a subsemigroup of $\mathcal{S}$ for all $t \in [0,1]$.*

Proof. Let δ be a fuzzy subsemigroup of $\mathcal{S}$ and $u, v \in U_t$. Then $\delta(u_1) \geq t$, $\delta(v) \geq t$. By assumption, $\delta(uv) \geq \delta(u) \wedge \delta(ev)$. Thus, $\delta(uv) \geq \delta(u) \wedge \delta(v) \geq t$. It impiles that, $uv \in U_t$. Hence, U_t is a subsemigroup of $\mathcal{S}$.

Conversely, suppose that U_t is a subsemigroup of $\mathcal{S}$ and $u, v \in \mathcal{S}$.
If $u, v \in U_t$, then $\delta(u) \geq t$, $\delta(v) \geq t$. Thus, $\delta(uv) \geq \delta(u) \wedge \delta(v)$.
If $u \notin U_t$ or $v \notin U_t$, then $\delta(uv) \geq \delta(u) \wedge \delta(v)$.
Hence, δ be a fuzzy subsemigroup of $\mathcal{S}$.

Theorem 3. *A fuzzy set δ is a fuzzy (m,n)-ideal of an ordered semigroup $\mathcal{S}$ if and only if the level set U_t is an (m,n)-ideal of $\mathcal{S}$ for all $t \in [0,1]$.*

Proof. Let δ be a fuzzy (m,n)-ideal of $\mathcal{S}$. Then δ is a fuzzy subsemigroup of $\mathcal{S}$. By Lemma 2, U_t is a subsemigroup of $\mathcal{S}$. Let $u_1, u_2, \dots u_m, k, v_1, v_2, \dots, v_n \in U_t$. Then $\delta(u_i) \geq t$, $\delta(v_j) \geq t$ for some $i \in \{1, 2, \dots, m\}$ and $j \in \{1, 2, \dots, m\}$. By assumption, $\delta(u_1 u_2 \cdots u_m k v_1 v_2 \cdots v_n) \geq \delta(u_1) \wedge \delta(u_2) \wedge \cdots \wedge \delta(u_m) \wedge \cdots \wedge \delta(v_1) \wedge \delta(v_2) \wedge \cdots \wedge \delta(v_n)$. Thus, $\delta(u_1 u_2 \cdots u_m k v_1 v_2 \cdots v_n) \geq t$. It impiles that, $u_1 u_2 \cdots u_m k v_1 v_2 \cdots v_n \in U_t$.

Let $u, v \in \mathcal{S}$ such that $u \leq v$ and $v \in U_t$. Then $\delta(u) \geq \delta(v) \geq t$. Thus, $u \in U_t$. Hence, U_t is an (m,n)-ideal of $\mathcal{S}$.

Conversely, suppose that U_t is an (m,n)-ideal of $\mathcal{S}$. Then U_t is a subsemigroup of $\mathcal{S}$. By Lemma 2, δ is a fuzzy subsemigroup of a semigroup $\mathcal{S}$.

Let $u, v \in \mathcal{S}$ such that $u \leq v$. We choose $\delta(v) = t$. Thus, $v \in U_t$. By assumption, $u \in U_t$. Then $\delta(u) \geq t = \delta(v)$.

If δ is not a fuzzy (m,n)-ideal of $\mathcal{S}$, then there exists $u_i, k, v_j \in \mathcal{S}$ such that $\delta(u_1 u_2 \cdots u_m k v_1 v_2 \cdots v_n) < \delta(u_1) \wedge \delta(u_2) \wedge \cdots \wedge \delta(u_m) \wedge \cdots \wedge \delta(v_1) \wedge \delta(v_2) \wedge \cdots \wedge \delta(v_n)$. By assumption, we have $u_1 u_2 \cdots u_m k v_1 v_2 \cdots v_n \in U_t$. Thus, $\delta(u_1 u_2 \cdots u_m k v_1 v_2 \cdots v_n) \geq \delta(u_1) \wedge \delta(u_2) \wedge \cdots \wedge \delta(u_m) \wedge \cdots \wedge \delta(v_1) \wedge \delta(v_2) \wedge \cdots \wedge \delta(v_n)$. It is a contradiction. Hence, δ be a fuzzy (m,n)-ideal of $\mathcal{S}$.

Definition 7. *An (m,n)-ideal $\mathcal{K}$ of a ordered semigroup $\mathcal{S}$ is called*

(1) *a* minimal *if for every (m,n)-ideal of $\mathcal{J}$ of $\mathcal{S}$ such that $\mathcal{J} \subseteq \mathcal{K}$, we have $\mathcal{J} = \mathcal{K}$.*

(2) *a* maximal *if for every (m,n)-ideal of $\mathcal{J}$ of $\mathcal{S}$ such that $\mathcal{K} \subseteq \mathcal{J}$, we have $\mathcal{J} = \mathcal{K}$.*

Definition 8. *A fuzzy (m,n)-ideal δ of an ordered semigroup $\mathcal{S}$ is*

(1) *a* minimal *if for all fuzzy (m,n)-ideal ξ of $\mathcal{S}$ such that $\xi \leq \delta$, then $\xi = \delta$.*

(2) *a* maximal *if for all fuzzy (m,n)-ideal ξ of $\mathcal{S}$ such that $\delta \leq \xi$, then $\xi = \delta$.*

Theorem 4. *A non-empty subset $\mathcal{K}$ of an ordered semigroup $\mathcal{S}$. Then the following statements ture*

(1) *$\mathcal{K}$ is a minimal (m,n)-ideal if and only if λ_K is a minimal fuzzy (m,n)-ideal.*

(2) *$\mathcal{K}$ is a maximal (m,n)-ideal if and only if λ_K is a maximal fuzzy (m,n)-ideal.*

Proof.

(1) Let $\mathcal{K}$ be a minimal (m,n)-ideal of $\mathcal{S}$. Then $\mathcal{K}$ is an (m,n)-ideal of $\mathcal{S}$. Thus, by Theorem 2, $\lambda_{\mathcal{K}}$ is a fuzzy (m,n)-ideal of $\mathcal{S}$. Let $\mathcal{J}$ be an (m,n)-ideal of $\mathcal{S}$ such that $\mathcal{J} \subseteq \mathcal{K}$. Then by Theorem 2, $\lambda_{\mathcal{J}}$ is a fuzzy (m,n)-ideal of $\mathcal{S}$ and $\lambda_{\mathcal{J}} \leq \lambda_{\mathcal{K}}$. Since $\mathcal{K}$ is a minimal (m,n)-ideal of $\mathcal{S}$ we have $\mathcal{J} = \mathcal{K}$. Thus, $\lambda_{\mathcal{J}} = \lambda_{\mathcal{K}}$. Hence, $\lambda_{\mathcal{K}}$ is minimal fuzzy (m,n)-ideal of $\mathcal{S}$.

Conversely, λ_K is minimal fuzzy (m,n)-ideal of $\mathcal{S}$. Then λ_K is a fuzzy (m,n)-ideal of $\mathcal{S}$. Thus, by Theorem 2, $\mathcal{K}$ is an (m,n)-ideal of $\mathcal{S}$. Let $\lambda_{\mathcal{J}}$ be a fuzzy (m,n)-ideal of $\mathcal{S}$ such that $\lambda_{\mathcal{J}} \leq \lambda_{\mathcal{K}}$. Then by Theorem 2, $\mathcal{J}$ is an (m,n)-ideal of $\mathcal{S}$ such that $\mathcal{J} \subseteq \mathcal{K}$. Since $\lambda_{\mathcal{K}}$ is minimal fuzzy (m,n)-ideal of $\mathcal{S}$ we have $\lambda_{\mathcal{J}} = \lambda_{\mathcal{K}}$. Thus, $\mathcal{J} = \mathcal{K}$. Hence, $\mathcal{K}$ is a minimal (m,n)-ideal of $\mathcal{S}$.

(2) If follows from (1).

Next, we give the relationship between prime, semiprime (m,n)-ideals and prime, semiprime fuzzy (m,n)-ideals.

Definition 9. *Let $\mathcal{K}$ be an (m,n)-ideal of an ordered semigroup $\mathcal{S}$ is called*

(1) prime *if $uv \in \mathcal{K}$ implies $u \in \mathcal{K}$ or $v \in \mathcal{K}$ for all $e,h \in \mathcal{S}$,*

(2) semiprime *if $u^2 \in \mathcal{K}$ implies $u \in \mathcal{K}$ for all $u \in \mathcal{S}$.*

Definition 10. *Let δ be a fuzzy (m,n)-ideal of a ordered semigroup is called*

(1) prime *if $\delta(uv) \leq \delta(u) \vee \delta(v)$ for all $u,v \in \mathcal{S}$,*

(2) semiprime *if $\delta(u^2) \leq \delta(u)$ for all $u \in \mathcal{S}$.*

Remark 1. *Every prime (m,n)-ideal is semiprime (m,n)-ideal in an ordered semigroup.*

Theorem 5. *Let $\mathcal{K}$ be a non-empty subset of an ordered semigroup $\mathcal{S}$. Then $\mathcal{K}$ is a prime (m,n)-ideal of $\mathcal{S}$ if and only if $\lambda_{\mathcal{K}}$ is a prime fuzzy (m,n)-ideal of $\mathcal{S}$.*

Proof. Suppose that $\mathcal{K}$ is a prime (m,n)-ideal of $\mathcal{S}$. Then $\mathcal{K}$ is an (m,n)-ideal of $\mathcal{S}$. Thus, by Theorem 2 $\lambda_{\mathcal{K}}$ is a fuzzy (m,n)-ideal of $\mathcal{S}$. Let $u,v \in \mathcal{S}$.

Case 1: If $uv \in \mathcal{K}$, then $u \in \mathcal{K}$ or $v \in \mathcal{K}$. Thus, $\lambda_{\mathcal{K}}(uv) = 1 = \lambda_{\mathcal{K}}(u)$ and $\lambda_{\mathcal{K}}(uv) = -1 = \lambda_{\mathcal{K}}(u)$ or $\lambda_{\mathcal{K}}(v) = 1 = \lambda_K(uv)$. Hence, $\lambda_{\mathcal{K}}(uv) \leq \lambda_{\mathcal{K}}(u) \vee \lambda_{\mathcal{K}}(v)$.

Case 2: If $uv \notin \mathcal{K}$, then $\lambda\mathcal{K}(uv) = 0$. Thus, $\lambda_{\mathcal{K}}(uv) \leq \lambda_{\mathcal{K}}(u) \vee \lambda_{\mathcal{K}}(v)$.

Therefore, $\lambda_{\mathcal{K}}$ is a prime fuzzy (m,n)-ideal of $\mathcal{S}$.

Conversely, suppose that $\lambda_{\mathcal{K}}$ is a prime fuzzy (m,n)-ideal of $\mathcal{S}$. Then $\lambda_{\mathcal{K}}$ is a fuzzy (m,n)-ideal of $\mathcal{S}$. Thus, by Theorem 2, $\mathcal{K}$ is an (m,n)-ideal of $\mathcal{S}$. Let $u,v \in \mathcal{S}$ with $uv \in \mathcal{K}$. Then, $\lambda_{\mathcal{K}}(uv) = 1$. If $u \notin \mathcal{K}$ and $v \notin \mathcal{K}$, then $\lambda_{\mathcal{K}}(u) = 0 = \lambda_{\mathcal{K}}(v)$. By assumption, $\lambda_{\mathcal{K}}(uv) \leq \lambda_{\mathcal{K}}(u) \vee \lambda_{\mathcal{K}}(v)$. Thus,$\lambda_{\mathcal{K}}(uv) = 0$. It is a contradiction, so $u \in \mathcal{K}$ or $v \in \mathcal{K}$. Hence, $\mathcal{K}$ is a prime (m,n)-ideal of $\mathcal{S}$.

Theorem 6. *Let $\mathcal{K}$ be a non-empty subset of an ordered semigroup $\mathcal{S}$. Then $\mathcal{K}$ is a semiprime (m,n)-ideal of $\mathcal{K}$ if and only if $\lambda_{\mathcal{K}}$ is a semiprime fuzzy (m,n)-ideal of $\mathcal{S}$.*

Proof. It follows from Theorem 5.

4. Fuzzy n-interior ideals

Before, we will review the definition of n-interior ideals in ordered semigroups.

Definition 11. *[12] A subsemigroup $\mathcal{K}$ of an ordered semigroup $\mathcal{S}$ is said to be an* n-interior ideal *of $\mathcal{S}$ if $\mathcal{S}\mathcal{K}^n\mathcal{S} \subseteq \mathcal{K}$ where n is an integer and $\mathcal{K} = (\mathcal{K}]$, that is for $x \in \mathcal{K}$ and $y \in \mathcal{S}$, $y \leq x$ implies $y \in \mathcal{K}$.*

Definition 12. *A fuzzy subsemigroup δ in an ordered semigroup $\mathcal{S}$ is called* fuzzy n-interior ideal *of $\mathcal{S}$ if*

(1) $\delta(uk_i^n v) \geq \delta(k_1) \wedge \delta(k_2) \wedge \cdots \wedge \delta(k_n)$

(2) $u \leq v$ *imples* $\delta(u) \geq \delta(v)$

for all $u, k_i^n, v \in \mathcal{S}$ and where $i \in \{1, 2, \ldots, n\}$.

Example 2. *Consider the ordered semigroup $\mathcal{S} = \{u, v, w, x, y, z\}$ with the following Cayley table:*

$\cdot$	u	v	w	x	y	z
u	u	u	w	u	u	u
v	u	u	u	u	u	v
w	u	u	u	u	u	v
z	u	u	u	u	u	x
y	u	x	x	u	u	x
z	u	x	x	x	y	z

and $\leq$: $\{(u,u), (v,v), (w,w), (x,x), (y,y), (z,z)\}$. Define a function $\delta : \mathcal{S} \to [0,1]$ by $\delta(u) = 0.6$, $\delta(v) = 0.2$, $\delta(w) = 0.4$, $\delta(x) = 0.5$ $\delta(y) = 0.3$ $\delta(z) = 0.1$. Then δ is a fuzzy n-interior ideal of $\mathcal{S}$.

Theorem 7. *Let $\{\delta_i \mid i \in \mathcal{J}\}$ be a family of fuzzy n-interior ideals of an ordered semigroup $\mathcal{S}$ with $\delta(u) \geq \delta(v)$ whenever $u \leq v$. Then $\bigwedge_{i \in \mathcal{F}} \vartheta_i$ is a fuzzy n-interior ideal of $\mathcal{S}$.*

Proof. Let $u, v \in \mathcal{S}$. Then,

$$\bigwedge_{i \in \mathcal{J}} \delta_i(uv) \geq \bigwedge_{i \in \mathcal{J}} \{\delta_i(u) \wedge \delta_i(v)\} = \bigwedge_{i \in \mathcal{J}} \delta_i(u) \wedge \bigwedge_{i \in \mathcal{J}} \delta_i(v).$$

Thus, $\bigwedge_{i \in \mathcal{J}} \delta_i$ is a fuzzy subsemigroup of $\mathcal{S}$.
Let $u, k_i^n, v \in \mathcal{S}$ for all $i \in \{1, 2, \ldots, n\}$. Then,

$$\bigwedge_{i \in \mathcal{J}} \delta_i(uk_i^n v) \geq \bigwedge_{i \in \mathcal{J}} \{\delta_i(k_1) \wedge \delta_i(k_2) \cdots \wedge \delta_i(k_n)\} = \bigwedge_{i \in \mathcal{J}} \delta_i(k_1) \wedge \bigwedge_{i \in \mathcal{J}} \delta_i(k_2) \cdots \wedge \bigwedge_{i \in \mathcal{J}} \delta_i(k_n).$$

Thus, $\bigwedge_{i \in \mathcal{J}} \delta_i$ is a fuzzy n-interior ideal of $\mathcal{S}$.

Theorem 8. *Let $\mathcal{K}$ be an ideal of a semigroup $\mathcal{S}$ and m, n are positive integers. Then $\mathcal{K}$ is an n-interior ideal of $\mathcal{S}$ if and only if the characteristic function $\lambda_{\mathcal{K}}$ is a fuzzy n-interior ideal of $\mathcal{S}$.*

Proof. Suppose that $\mathcal{K}$ is an n-interior ideal of $\mathcal{S}$. Then $\mathcal{K}$ is a subsemigroup of $\mathcal{S}$. Thus, by Theorem 1, $\lambda_{\mathcal{K}}$ is a BF subsemigroup of $\mathcal{E}$. Let $h, r_i, k \in \mathcal{E}$ where $i \in \{1, 2, \ldots, n\}$.

If $r_i \in \mathcal{K}$ for all $i \in \{1, 2, \ldots, n\}$, then $hr_i^n k \in \mathcal{K}$. Thus, $\lambda_{\mathcal{K}}(r_i) = \lambda_{\mathcal{K}}(hr_i^n k) = 1$ for all $i \in \{1, 2, \ldots, n\}$. Hence, $\lambda_{\mathcal{K}}(hr_i^n k) \geq \lambda_{\mathcal{K}}(r_1) \wedge \lambda_{\mathcal{K}}(r_2) \wedge \cdots \wedge \lambda_{\mathcal{K}}(r_n)$.

If $r_i \notin \mathcal{K}$ for some $i \in \{1, 2, \ldots, n\}$, then $\lambda_{\mathcal{K}}(r_i) = 0$ for some $i \in \{1, 2, \ldots, n\}$. Thus, $\lambda_{\mathcal{K}}(hr_i^n k) \geq \lambda_{\mathcal{K}}(r_1) \wedge \lambda_{\mathcal{K}}(r_2) \wedge \cdots \wedge \lambda_{\mathcal{K}}(r_n)$.

Therefore, $\lambda_{\mathcal{K}}$ is a fuzzy n-interior ideal of $\mathcal{S}$.

Conversely, suppose that $\lambda_{\mathcal{K}}$ is a fuzzy n-interior ideal of $\mathcal{S}$. Then $\lambda_{\mathcal{K}}$ is a fuzzy subsemigroup of $\mathcal{S}$. Thus, by Theorem 1, $\mathcal{K}$ is a subsemigroup of $\mathcal{S}$. Let $r_i^n \in \mathcal{S}\mathcal{K}^n\mathcal{S}$ where n is an integer and for all $i \in \{1, 2, \ldots, n\}$. Then $\lambda_{\mathcal{K}}(r_i^n) = 1$ for all $i \in \{1, 2, \ldots, n\}$. By assumption, $\lambda_{\mathcal{K}}(hr_i^n k) \geq \lambda_{\mathcal{K}}(r_1) \wedge \lambda_{\mathcal{K}}(r_2) \wedge \cdots \wedge \lambda_{\mathcal{K}}(r_n)$. Thus, $\lambda_{\mathcal{K}}(hr_i^n k) = 1$ for all $i \in \{1, 2, \ldots, n\}$. Hence, $r_i^n \in \mathcal{K}$ for all $i \in \{1, 2, \ldots, n\}$. Therefore, $\mathcal{K}$ is an n-interior ideal of $\mathcal{S}$.

Theorem 9. *A fuzzy set δ is a fuzzy n-interior ideal of a semigroup $\mathcal{S}$ if and only if the level set U_t is an n-interior ideal of $\mathcal{S}$ for all $t \in [0, 1]$.*

Proof. Let δ be a fuzzy n-interior ideal of $\mathcal{S}$. Then δ is a fuzzy subsemigroup of $\mathcal{S}$. By Lemma 2, U_t is a subsemigroup of $\mathcal{S}$. Let $r_1, r_2, \ldots r_m, k, h \in U_{\vartheta}^{(s,t)}$. Then $\delta(r_i) \geq t$ for some $i \in \{1, 2, \ldots, n\}$. By assumption, $\delta(hr_i^n k) \geq \delta(r_1) \wedge \delta(r_2) \wedge \cdots \wedge \delta(r_n)$. Thus, $\delta^P(hr_i^n k) \geq t$. It impiles that, $r_i^n \in U_t$. Hence, U_t is an n-interior ideal of $\mathcal{S}$.

Conversely, suppose that U_t is an n-interior ideal of $\mathcal{S}$. Then U_t is a subsemigroup of $\mathcal{S}$. By Lemma 2, δ is a fuzzy subsemigroup of $\mathcal{S}$. If δ is not a fuzzy n- interior ideal of E, then there exists $r_i, k, h \in \mathcal{S}$ such that $\delta(hr_i^n k) < \delta(r_1) \wedge \delta(r_2) \wedge \cdots \wedge \delta(r_n)$. By assumption, we have $hr_i^n k \in U_t$. Thus, $\delta(hr_i^n k) \geq \delta(r_1) \wedge \delta(r_2) \wedge \cdots \wedge \delta(r_n)$. It is a contradiction. Hence, δ is a fuzzy n-interior ideal of $\mathcal{S}$.

Definition 13. *An n-interior ideal $\mathcal{K}$ of a semigroup $\mathcal{S}$ is called a* minimal *if for every n-interior ideal of $\mathcal{J}$ of $\mathcal{S}$ such that $\mathcal{J} \subseteq \mathcal{K}$, we have $\mathcal{J} = \mathcal{K}$.*

Definition 14. *A fuzzy n-interior ideal δ of a semigroup $\mathcal{S}$ is a* minimal *if for all fuzzy n-interior ideal ξ of $\mathcal{S}$ such that $\xi \leq \delta$, then $\xi = \delta$.*

Theorem 10. *A non-empty subset $\mathcal{K}$ of a semigroup $\mathcal{S}$ is a minimal n-interior ideal if and only if $\lambda_{\mathcal{K}}$ is a minimal fuzzy n-interior ideal $\mathcal{S}$.*

Proof. Let $\mathcal{K}$ be a minimal n-interior ideal of $\mathcal{S}$. Then $\mathcal{K}$ is an n-interior ideal of $\mathcal{S}$. Thus, by Theorem 8, λ_K is a fuzzy n-interior ideal of E. Let $\mathcal{J}$ be an n-interior ideal of $\mathcal{E}$ such that $\mathcal{J} \subseteq \mathcal{K}$. Then by Theorem 8, $\lambda_{\mathcal{J}}$ is a fuzzy n-interior ideal of $\mathcal{S}$ and $\lambda_{\mathcal{J}} \leq \lambda_{\mathcal{K}}$. Since $\mathcal{K}$ is a minimal n-interior ideal of $\mathcal{S}$ we have $\mathcal{J} = \mathcal{K}$. Thus, $\lambda_{\mathcal{J}} = \lambda_{\mathcal{K}}$. Hence, $\lambda_{\mathcal{K}}$ is minimal fuzzy n-interior ideal of $\mathcal{S}$.

Conversely, $\lambda_{\mathcal{K}}$ is minimal fuzzy n-interior ideal of $\mathcal{S}$. Then $\lambda_{\mathcal{K}}$ is a fuzzy n-interior ideal of $\mathcal{S}$. Thus, by Theorem 8, $\mathcal{K}$ is an n-interior ideal of $\mathcal{S}$. Let $\lambda_{\mathcal{J}}$ be a fuzzy n-interior ideal of $\mathcal{S}$ such that $\lambda_{\mathcal{J}} \leq \lambda_{\mathcal{K}}$. Then by Theorem 8, $\mathcal{J}$ is an n-interior ideal of $\mathcal{S}$ such that $\mathcal{J} \subseteq \mathcal{K}$. Since $\lambda_{\mathcal{K}}$ is minimal fuzzy n-interior ideal of $\mathcal{S}$ we have $\lambda_{\mathcal{J}} = \lambda_{\mathcal{K}}$. Thus, $\mathcal{J} = \mathcal{K}$. Hence, $\mathcal{K}$ is a minimal n-interior ideal of $\mathcal{S}$.

Definition 15. *An n-interior ideal $\mathcal{K}$ of a semigroup $\mathcal{S}$ is called a* maximal*if for every n-interior ideal of $\mathcal{J}$ of $\mathcal{E}$ such that $\mathcal{K} \subseteq \mathcal{J}$, we have $\mathcal{J} = \mathcal{K}$.*

Definition 16. *A fuzzy n-interior ideal δ of a semigroup $\mathcal{S}$ is a* maximal *if for all fuzzy n-interior ideal ξ of $\mathcal{E}$ such that $\delta \leq \xi$, then $\xi = \delta$.*

Theorem 11. *A non-empty subset $\mathcal{K}$ of a semigroup $\mathcal{S}$ is a maximal n-interior ideal if and only if $\lambda_{\mathcal{K}}$ is a maximal fuzzy n-interior ideal $\mathcal{S}$.*

Proof. Let $\mathcal{K}$ be a maximal n-interior ideal of $\mathcal{S}$. Then $\mathcal{K}$ is an n-interior ideal of $\mathcal{S}$. Thus, by Theorem 8, λ_K is a fuzzy n-interior ideal of E. Let $\mathcal{J}$ be an n-interior ideal of $\mathcal{E}$ such that $\mathcal{K} \subseteq \mathcal{J}$. Then by Theorem 8, $\lambda_{\mathcal{J}}$ is a fuzzy n-interior ideal of $\mathcal{S}$ and $\lambda_{\mathcal{K}} \leq \lambda_{\mathcal{J}}$. Since $\mathcal{K}$ is a maximal n-interior ideal of $\mathcal{S}$ we have $\mathcal{J} = \mathcal{K}$. Thus, $\lambda_{\mathcal{J}} = \lambda_{\mathcal{K}}$. Hence, $\lambda_{\mathcal{K}}$ is maximal fuzzy n-interior ideal of $\mathcal{S}$.

Conversely, $\lambda_{\mathcal{K}}$ is maximal fuzzy n-interior ideal of $\mathcal{S}$. Then $\lambda_{\mathcal{K}}$ is a fuzzy n-interior ideal of $\mathcal{S}$. Thus, by Theorem 8, $\mathcal{K}$ is an n-interior ideal of $\mathcal{S}$. Let $\lambda_{\mathcal{J}}$ be a fuzzy n-interior ideal of $\mathcal{S}$ such that $\lambda_{\mathcal{K}} \leq \lambda_{\mathcal{J}}$. Then by Theorem 8, $\mathcal{J}$ is an n-interior ideal of $\mathcal{S}$ such that $\mathcal{K} \subseteq \mathcal{J}$. Since $\lambda_{\mathcal{K}}$ is maximal fuzzy n-interior ideal of $\mathcal{S}$ we have $\lambda_{\mathcal{J}} = \lambda_{\mathcal{K}}$. Thus, $\mathcal{J} = \mathcal{K}$. Hence, $\mathcal{K}$ is a maximal n-interior ideal of $\mathcal{S}$.

Next, we give the relationship between prime, semiprime n-interior ideals and prime, semiprime fuzzy n-interior ideals.

Definition 17. *Let $\mathcal{K}$ be an n-interior ideal of a semigroup $\mathcal{S}$ is called*

(1) prime *if $eh \in \mathcal{K}$ implies $e \in \mathcal{K}$ or $h \in \mathcal{K}$ for all $e, h \in \mathcal{S}$,*

(2) semiprime *if $e^2 \in \mathcal{K}$ implies $e \in \mathcal{K}$ for all $e \in \mathcal{S}$.*

Definition 18. *Let δ be a fuzzy n-interior ideal of a semigroup $\mathcal{S}$ is called*

(1) prime *if $\delta(eh) \leq \delta(e) \vee \delta(h)$ for all $e, h \in \mathcal{S}$,*

(2) semiprime *if $\delta(e^2) \leq \delta(e)$ for all $e \in \mathcal{S}$.*

Remark 2. *Every prime n-interior ideal is semiprime n-interior ideal in a semigroup.*

Theorem 12. *Let $\mathcal{K}$ be a non-empty subset of a semigroup $\mathcal{S}$. Then the following statement holds:*

(1) *$\mathcal{K}$ is a prime n-interior ideal of $\mathcal{S}$ if and only if $\lambda_{\mathcal{K}}$ is a prime fuzzy n-interior ideal of $\mathcal{S}$.*

(2) *$\mathcal{K}$ is a semiprime n-interior ideal of $\mathcal{S}$ if and only if $\lambda_{\mathcal{K}}$ is a semiprime fuzzy n-interior ideal of $\mathcal{S}$.*

Proof.

(1) Suppose that $\mathcal{K}$ is a prime n-interior ideal of $\mathcal{S}$. Then $\mathcal{K}$ is an n-interior ideal of $\mathcal{S}$. Thus, by Theorem 8 $\lambda_{\mathcal{K}}$ is a fuzzy n-interior ideal of $\mathcal{S}$. Let $e, h \in \mathcal{S}$.

Case 1: If $eh \in \mathcal{K}$, then $e \in \mathcal{K}$ or $h \in \mathcal{K}$. Thus, $\lambda_{\mathcal{K}}(eh) = 1 = \lambda_{\mathcal{K}}(e)$ and $\lambda_{\mathcal{K}}(eh) = 1$. Hence, $\lambda_{\mathcal{K}}(eh) \leq \lambda_{\mathcal{K}}(e) \vee \lambda_{\mathcal{K}}(h)$.

Case 2: If $eh \notin K$, then $\lambda_{\mathcal{K}}(eh) = 0$. Thus, $\lambda_{\mathcal{K}}(eh) \leq \lambda_{\mathcal{K}}(e) \vee \lambda_{\mathcal{K}}(h)$.

Therefore, $\lambda_{\mathcal{K}}$ is a prime fuzzy n-interior ideal of $\mathcal{S}$.

Conversely, suppose that $\lambda_{\mathcal{K}}$ is a prime fuzzy n-interior ideal of $\mathcal{S}$. Then $\lambda_{\mathcal{K}}$ is a fuzzy n-interior ideal of $\mathcal{S}$. Thus, by Theorem 8, $\mathcal{K}$ is an n-interior ideal of $\mathcal{S}$. Let $e, h \in \mathcal{S}$ with $eh \in \mathcal{K}$. Then, $\lambda_{\mathcal{K}}(eh) = 1$. If $e \notin \mathcal{K}$ and $h \notin \mathcal{K}$, then $\lambda_{\mathcal{K}}(e) = 0 = \lambda_{\mathcal{K}}(h)$. By assumption, $\lambda_{\mathcal{K}}(eh) \leq \lambda_{\mathcal{K}}(e) \vee \lambda_{\mathcal{K}}(h)$. Thus,$\lambda_{\mathcal{K}}(eh) = 0.$. It is a contradiction, so $e \in \mathcal{K}$ or $h \in \mathcal{K}$. Hence, $\mathcal{K}$ is a prime n-interior ideal of $\mathcal{S}$.

(2) It follows from (1).

5. Conclusion

In this paper, we introduce the concept of bipolar fuzzy (m, n)-ideals in semigroups and investigate their properties. Additionally, we establish the relationship between (m, n)-ideals and fuzzy (m, n)-ideals. Furthermore, we define bipolar fuzzy n-interior ideals in semigroup and prove the relationship between n-interior ideals and fuzzy n-interior ideals. In the future, we plan to explore bipolar (m, n)-ideals and n-interior ideals in ordered semigroups or within the algebraic context.

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