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Numerical Radius Inequalities Involving 2×2 Block Matrices

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Abstract. In this paper, we give several upper and lower bounds for the numerical radius of 2×2 block matrices. Several special cases of our results are given.
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1. Introduction

Let $M_n(\mathbb{C})$ denote the space of all $n \times n$ complex matrices. The spectral norm of a matrix $A \in \mathbb{M}_n(\mathbb{C})$ is defined by

$$\|A\| = \max_{\|x\|=1} \{\|Ax\| : x \in \mathbb{C}^n\}.$$

The numerical radius of a matrix $A \in \mathbb{M}_n(\mathbb{C})$ is defined by

$$\omega(A) = \max_{\|x\|=1} \{|\langle Ax, x\rangle| : x \in \mathbb{C}^n\}.$$

In [22], the author proved that the numerical radius of a matrix $A \in M_n(\mathbb{C})$ can be formulated as

$$w\left(A\right) = \max_{\theta \in \mathbb{R}} \left\| \mathrm{Re}\left(e^{i\theta} A\right) \right\|,$$

where $\mathrm{Re}\left(e^{i\theta} A\right)$ denotes the real part of the matrix $e^{i\theta} A$. Clearly, we always have

$$w(A) \leq \|A\| \tag{1}$$

for any $A \in \mathbb{M}_n(\mathbb{C})$.

Many generalizations and recent related results of the numerical radius $w(\cdot)$ were discussed by many authors, some of these results can be found in [3], [7], [10], [12], [11], [9], [8], [13], [14], [15], [19], and [20].

Some basic properties of the numerical radii and the spectral norms of matrices that we need in our paper are the following: For $A, B \in \mathbb{M}_n(\mathbb{C})$, we have the following relations:

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(i) $w\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = \max\{w(A), w(B))\}$

(ii) $w(A^*) = w(A)$

(iii) $\|A^*A\| = \|AA^*\| = \|A\|^2$

(iv) $\left\|\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right\| = \left\|\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right\| = \max\{\|A\|, \|B\|\}.$

Recent results concerning inequalities can be found in [1], [2], [4], [6], [5], and [21].

2. Main results

We start with the following theorem.

Theorem 1. *Let $A, B, C, D \in \mathbb{M}_n(\mathbb{C})$. Then*

$$w\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \le \sqrt{\max\{\|A^*A + C^*C\|, \|B^*B + D^*D\|\} + \|A^*B + C^*D\|}. \tag{2}$$

Proof. We have

$$\begin{aligned}
&w\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \\
&\le \left\|\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right\| \text{ (by inequality (1))} \\
&= \sqrt{\left\|\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right\|^2} \\
&= \sqrt{\left\|\begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right\|} \\
&= \sqrt{\left\|\begin{bmatrix} A^*A + C^*C & A^*B + C^*D \\ B^*A + D^*C & B^*B + D^*D \end{bmatrix}\right\|} \qquad (3) \\
&= \sqrt{\left\|\begin{bmatrix} A^*A + C^*C & 0 \\ 0 & B^*B + D^*D \end{bmatrix} + \begin{bmatrix} 0 & A^*B + C^*D \\ B^*A + D^*C & 0 \end{bmatrix}\right\|} \qquad (4) \\
&\le \sqrt{\left\|\begin{bmatrix} A^*A + C^*C & 0 \\ 0 & B^*B + D^*D \end{bmatrix}\right\| + \left\|\begin{bmatrix} 0 & A^*B + C^*D \\ B^*A + D^*C & 0 \end{bmatrix}\right\|} \\
&\qquad \text{(by the triangle inequality)} \\
&= \sqrt{\max\{\|A^*A + C^*C\|, \|B^*B + D^*D\|\} + \|A^*B + C^*D\|},
\end{aligned}$$

as required.

Based on Theorem 1 and its proof, we have several corollaries.

Corollary 1. *Let* $A, B \in \mathbb{M}_n(\mathbb{C})$*. Then*

$$w\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right) \leq \sqrt{\max\{\|A\|^2, \|B\|^2\} + \|A^*B\|}. \tag{5}$$

Proof. The result follows by letting $C = D = 0$ in inequality (2).

To state our next corollary, we need the following lemma [16].

Lemma 1. *Let* $A, B \in \mathbb{M}_n(\mathbb{C})$ *be normal. Then*

$$\|A + B\| \leq \||A| + |B|\|,$$

where $|T|$ *is the absolute value of* $T \in \mathbb{M}_n(\mathbb{C})$ *which is defined by* $|T| = (T^*T)^{1/2}$.

Corollary 2. *Let* $A, B, C, D \in \mathbb{M}_n(\mathbb{C})$*. Then*

$$w\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leq \sqrt{\max\{\|A^*A + C^*C + |B^*A + D^*C|\|, \|B^*B + D^*D + |A^*B + C^*D|\|\}}.$$

Proof. By inequality (4), we have

$$\begin{aligned}
w\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)
&\leq \sqrt{\left\|\begin{bmatrix} A^*A + C^*C & 0 \\ 0 & B^*B + D^*D \end{bmatrix} + \begin{bmatrix} 0 & A^*B + C^*D \\ B^*A + D^*C & 0 \end{bmatrix}\right\|} \\
&\leq \sqrt{\left\|\begin{bmatrix} A^*A + C^*C & 0 \\ 0 & B^*B + D^*D \end{bmatrix} + \left|\begin{bmatrix} 0 & A^*B + C^*D \\ B^*A + D^*C & 0 \end{bmatrix}\right|\right\|} \\
&\qquad \text{(by Lemma 1)} \\
&= \sqrt{\left\|\begin{bmatrix} A^*A + C^*C & 0 \\ 0 & B^*B + D^*D \end{bmatrix} + \begin{bmatrix} |B^*A + D^*C| & 0 \\ 0 & |A^*B + C^*D| \end{bmatrix}\right\|} \\
&= \sqrt{\left\|\begin{bmatrix} A^*A + C^*C + |B^*A + D^*C| & 0 \\ 0 & B^*B + D^*D + |A^*B + C^*D| \end{bmatrix}\right\|} \\
&= \sqrt{\max\{\|A^*A + C^*C + |B^*A + D^*C|\|, \|B^*B + D^*D + |A^*B + C^*D|\|\}},
\end{aligned}$$

as required.

Letting $C = D = 0$ in Corollary 2, we have the following result.

Corollary 3. *Let* $A, B \in \mathbb{M}_n(\mathbb{C})$*. Then*

$$w^2\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right) \leq \max\{\|A^*A + |B^*A|\|, \|B^*B + |A^*B|\|\}.$$

To state the next corollary, we need the following lemma [18].

Lemma 2. *Let $A, B, C, D \in \mathbb{M}_n(\mathbb{C})$. Then*

$$\left\| \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} \|A\| & \|B\| \\ \|C\| & \|D\| \end{bmatrix} \right\|.$$

Corollary 4. *Let $A, B, C, D \in \mathbb{M}_n(\mathbb{C})$. Then*

$$w\left(\left[\begin{array}{cc} A & B \\ C & D \end{array}\right]\right) \leq \sqrt{\begin{array}{c} \frac{1}{2}\|A^*A + C^*C\| + \frac{1}{2}\|B^*B + D^*D\| \\ +\frac{1}{2}\sqrt{(\|A^*A + C^*C\| - \|B^*B + D^*D\|)^2 + 4\|B^*A + D^*C\|^2} \end{array}}.$$

Proof. By inequality (3), we have

$$\begin{aligned} & w\left(\left[\begin{array}{cc} A & B \\ C & D \end{array}\right]\right) \\ \leq & \sqrt{\left\|\left[\begin{array}{cc} A^*A + C^*C & A^*B + C^*D \\ B^*A + D^*C & B^*B + D^*D \end{array}\right]\right\|} \\ \leq & \sqrt{\left\|\left[\begin{array}{cc} \|A^*A + C^*C\| & \|A^*B + C^*D\| \\ \|B^*A + D^*C\| & \|B^*B + D^*D\| \end{array}\right]\right\|} \\ = & \sqrt{r\left(\left[\begin{array}{cc} \|A^*A + C^*C\| & \|A^*B + C^*D\| \\ \|B^*A + D^*C\| & \|B^*B + D^*D\| \end{array}\right]\right)} \\ & \text{(where } r \text{ denotes the spectral radius of matrices)} \\ = & \sqrt{\begin{array}{c} \frac{1}{2}\|A^*A + C^*C\| + \frac{1}{2}\|B^*B + D^*D\| \\ +\frac{1}{2}\sqrt{(\|A^*A + C^*C\| - \|B^*B + D^*D\|)^2 + 4\|B^*A + D^*C\|^2} \end{array}}, \end{aligned}$$

as required.

Corollary 5. *Let $C, D \in \mathbb{M}_n(\mathbb{C})$. Then*

$$w^2\left(\left[\begin{array}{cc} I & I \\ C & D \end{array}\right]\right) \leq 1 + \max\left\{\|C\|^2, \|D\|^2\right\} + \|I + C^*D\|.$$

Proof. Letting $A = B = I$ in Theorem 1, we have

$$\begin{aligned} w^2\left(\left[\begin{array}{cc} I & I \\ C & D \end{array}\right]\right) &\leq \max\{\|I + C^*C\|, \|I + D^*D\|\} + \|I + C^*D\| \\ &\leq 1 + \max\{\|C^*C\|, \|D^*D\|\} + \|I + C^*D\| \\ &= 1 + \max\left\{\|C\|^2, \|D\|^2\right\} + \|I + C^*D\|. \end{aligned}$$

We need the following lemma [17] to give a lower bound for inequality (5).

Lemma 3. *Let $A, B, C, D \in \mathbb{M}_n(\mathbb{C})$. Then*

$$w\left(\left[\begin{array}{cc} A & B \\ C & D \end{array}\right]\right) \geq \max\left\{w(A), w(D), \frac{w(B+C)}{2}, \frac{w(B-C)}{2}\right\}.$$

Corollary 6. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be positive semidefinite. Then*

$$\frac{1}{2}\|A+B\| \leq w\left(\left[\begin{array}{cc} A & B \\ 0 & 0 \end{array}\right]\right).$$

Proof. Let $U = \frac{1}{\sqrt{2}}\left[\begin{array}{cc} I & I \\ -I & I \end{array}\right]$, where I is the identity matrix, then U is unitary. Consequently, we have

$$\begin{aligned} w\left(\left[\begin{array}{cc} A & B \\ 0 & 0 \end{array}\right]\right) &= w\left(U^*\left[\begin{array}{cc} A & B \\ 0 & 0 \end{array}\right]U\right) \\ &= w\left(\left[\begin{array}{cc} \frac{A-B}{2} & \frac{A+B}{2} \\ \frac{A-B}{2} & \frac{A+B}{2} \end{array}\right]\right) \\ &\geq \max\left\{w\left(\frac{A-B}{2}\right), w\left(\frac{A+B}{2}\right), \frac{w(A)}{2}, \frac{w(B)}{2}\right\} \\ &\qquad\qquad \text{(by Lemma 3)} \\ &= \frac{1}{2}\|A+B\|. \end{aligned}$$

Corollary 7. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be positive semidefinite. Then*

$$\|A\| + \|B\| \leq w^2\left(\left[\begin{array}{cc} A^{1/2} & B^{1/2} \\ 0 & 0 \end{array}\right]\right) + w^2\left(\left[\begin{array}{cc} B^{1/2} & A^{1/2} \\ 0 & 0 \end{array}\right]\right). \tag{6}$$

Proof. By Lemma 3, we have

$$\begin{aligned} w^2\left(\left[\begin{array}{cc} A^{1/2} & B^{1/2} \\ 0 & 0 \end{array}\right]\right) &\geq \left[\max\left\{w\left(A^{1/2}\right), \frac{w\left(B^{1/2}\right)}{2}\right\}\right]^2 \\ &= \max\left\{\|A\|, \frac{1}{4}\|B\|\right\}. \end{aligned} \tag{7}$$

Similarly, we have

$$w^2\left(\left[\begin{array}{cc} B^{1/2} & A^{1/2} \\ 0 & 0 \end{array}\right]\right) \geq \max\left\{\|B\|, \frac{1}{4}\|A\|\right\}. \tag{8}$$

By adding inequalities (7) and (8) and then using the fact that $\max(a,b) = \frac{a+b+|a-b|}{2}$, we have

$$w^2\left(\left[\begin{array}{cc} A^{1/2} & B^{1/2} \\ 0 & 0 \end{array}\right]\right) + w^2\left(\left[\begin{array}{cc} B^{1/2} & A^{1/2} \\ 0 & 0 \end{array}\right]\right)$$

$$
\begin{aligned}
&\geq \max\left\{\|A\|, \frac{1}{4}\|B\|\right\} + \max\left\{\|B\|, \frac{1}{4}\|A\|\right\} \\
&= \frac{5}{8}(\|A\| + \|B\|) + \frac{1}{8}(|4\|A\| - \|B\|| + |4\|B\| - \|A\||) \qquad (9) \\
&\geq \|A\| + \|B\|.
\end{aligned}
$$

In inequality (5), by replacing A and B by the positive semidefinite matrices $A^{1/2}$ and $B^{1/2}$ respectively, we have

$$
w\left(\begin{bmatrix} A^{1/2} & B^{1/2} \\ 0 & 0 \end{bmatrix}\right) \leq \sqrt{\max\{\|A\|, \|B\|\} + \left\|A^{1/2}B^{1/2}\right\|}. \qquad (10)
$$

Based on inequalities (7), (8), and (10), we have the following corollary.

Corollary 8. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be positive semidefinite. Then*

$$
\begin{aligned}
\max\{\|A\|, \|B\|\} &\leq \max\left\{w^2\left(\begin{bmatrix} A^{1/2} & B^{1/2} \\ 0 & 0 \end{bmatrix}\right), w^2\left(\begin{bmatrix} B^{1/2} & A^{1/2} \\ 0 & 0 \end{bmatrix}\right)\right\} \\
&\leq \max\{\|A\|, \|B\|\} + \left\|A^{1/2}B^{1/2}\right\|.
\end{aligned}
$$

In particular, if $A^{1/2}B^{1/2} = 0$, then

$$
\max\{\|A\|, \|B\|\} = \max\left\{w^2\left(\begin{bmatrix} A^{1/2} & B^{1/2} \\ 0 & 0 \end{bmatrix}\right), w^2\left(\begin{bmatrix} B^{1/2} & A^{1/2} \\ 0 & 0 \end{bmatrix}\right)\right\}.
$$

Proof. The first inequality follows from inequalities (7) and (8).
In inequality (10), by Interchanging the roles of $A^{1/2}$ and $B^{1/2}$, we have

$$
w^2\left(\begin{bmatrix} B^{1/2} & A^{1/2} \\ 0 & 0 \end{bmatrix}\right) \leq \max\{\|A\|, \|B\|\} + \left\|A^{1/2}B^{1/2}\right\|. \qquad (11)
$$

So, we can obtain the second inequality from inequalities (10) and (11).

Corollary 9. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be positive semidefinite. Then*

$$
\begin{aligned}
&\max\left\{\frac{\|A\| + \|B\|}{2}, \frac{5}{8}\|A\|, \frac{5}{8}\|B\|\right\} \\
&\leq \frac{1}{2}\left[w^2\left(\begin{bmatrix} A^{1/2} & B^{1/2} \\ 0 & 0 \end{bmatrix}\right) + w^2\left(\begin{bmatrix} B^{1/2} & A^{1/2} \\ 0 & 0 \end{bmatrix}\right)\right] \\
&\leq \max\{\|A\|, \|B\|\} + \left\|A^{1/2}B^{1/2}\right\|.
\end{aligned}
$$

Proof. Using inequality (9), we have

$$w^2\left(\begin{bmatrix} A^{1/2} & B^{1/2} \\ 0 & 0 \end{bmatrix}\right) + w^2\left(\begin{bmatrix} B^{1/2} & A^{1/2} \\ 0 & 0 \end{bmatrix}\right)$$

$$\begin{aligned} &\geq \frac{5}{8}\left(\|A\| + \|B\|\right) + \frac{1}{8}\left(|4\|A\| - \|B\|| + |4\|B\| - \|A\||\right) \\ &= \frac{5}{8}\left(\|A\| + \|B\|\right) + \frac{1}{8}\left(|4\|A\| - \|B\|| + |\|A\| - 4\|B\||\right) \\ &\geq \frac{5}{8}\left(\|A\| + \|B\|\right) + \frac{5}{8}\left|\|A\| - \|B\|\right| \\ &= \frac{5}{4}\max\left\{\|A\|, \|B\|\right\}. \end{aligned} \tag{12}$$

Using inequalities (6) and (12) we get the first inequality. Also, the second inequality can be obtained from inequalities (10) and (11).

We end this paper with the following result.

Theorem 2. *Let $A, B, C, D \in \mathbb{M}_n(\mathbb{C})$. Then*

$$w\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \geq \left\|\begin{bmatrix} \mathrm{Re}(A) & \frac{B+C^*}{2} \\ \frac{C+B^*}{2} & \mathrm{Re}(D) \end{bmatrix}\right\|. \tag{13}$$

Proof. We have

$$\begin{aligned} w\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) &= \max_{\theta\in\mathbb{R}}\left\|Re\left(e^{i\theta}\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)\right\| \\ &\geq \left\|Re\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)\right\| \\ &= \frac{1}{2}\left\|\begin{bmatrix} A & B \\ C & D \end{bmatrix} + \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix}\right\| \\ &= \frac{1}{2}\left\|\begin{bmatrix} A + A^* & B + C^* \\ C + B^* & D + D^* \end{bmatrix}\right\| \\ &= \frac{1}{2}\left\|\begin{bmatrix} 2\mathrm{Re}(A) & B + C^* \\ C + B^* & 2\mathrm{Re}(D) \end{bmatrix}\right\| \\ &= \left\|\begin{bmatrix} \mathrm{Re}(A) & \frac{B+C^*}{2} \\ \frac{C+B^*}{2} & \mathrm{Re}(D) \end{bmatrix}\right\|, \end{aligned}$$

as required.

By inequalities (2) and (13), we have

$$\begin{aligned} \left\|\begin{bmatrix} \mathrm{Re}(A) & \frac{B+C^*}{2} \\ \frac{C+B^*}{2} & \mathrm{Re}(D) \end{bmatrix}\right\| &\leq w\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \\ &\leq \sqrt{\max\left\{\|A^*A + C^*C\|, \|B^*B + D^*D\|\right\} + \|A^*B + C^*D\|}. \end{aligned}$$

3. Conclusion

In this paper, new results related to numerical radii of block matrices were given. Several particular cases were also given. At the end of the paper, an upper bound and a lower bound of the numerical radius of the partitioned matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ were established.

Conflict of interest. The authors declare that they have no conflict of interest.

Data availability. Not applicable.

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