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The Saks-Henstock Lemma and the Change of Variable Formula of the PU integral

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Abstract. Henstock integral is a generalized version of the Riemann integral and in most cases, it is more general than the Lebesgue integral that is not constructed through a measure theoretic standpoint. The PU integral, on one hand, is a Henstock type that utilizes the notion of a partition of unity. In this paper, the Saks-Henstock Lemma and the Change of variable formula for the PU integral will be established.

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1. Introduction

Henstock integral is an integration process that is anchored on how the Riemann integral is constructed. More precisely, in most cases, it is more general than the Lebesgue integral and its construction is free from a measure theoretic standpoint. Thus, it is, relatively, easier than how the Lebesgue integral was constructed. Its definition is through a covering system called δ-fine, where δ is a postive function. A number of its variants were established and one of those is the PU integral, a gauge type of definition that is anchored in the perspective of a covering systems though partitions of unity. In fact, [1] mentioned that the PU integral can be used in the integration of functions on manifolds. In [2, 3], a Henstock-Kurzweil integral type were established and their application to functions on manifolds were presented.

A finite collection of point-interval pair $\{(t_i, I_i)\}_{i=1}^{m}$, where I_i is a compact interval, is of Perron type if $t_i \in I_i$ for all $i \leq m$. For a subset $\boldsymbol{E}$ of $\mathbb{R}^n$, the support of a real-valued

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function f on $\boldsymbol{E}$, written as supp$f$ is the closure of the set of points in the domain whose image under f is nonzero. Moreover, given a gauge δ on $[a,b]$, a finite collection of point interval pairs, of Perron type, $\{(t_i, I_i)\}_{i=1}^m$ is said to be δ-fine Perron partition of $[a,b]$ if I_i is a partition of $[a,b]$ and $I_i \subseteq B(t_i, \delta(t_i))$. If for each $i \leq m$, the condition $t_i \in I_i$ is removed, then we obtain the McShane integral. On another note, a partition of unity [1, 2] on a compact interval $\boldsymbol{E} \subseteq \mathbb{R}^n$ is a finite collection $\{\varphi_i\}_{i=1}^m$ smooth functions with the following conditions:

(i) $\varphi_i \geq 0$ on $\boldsymbol{E}$;

(ii) $\displaystyle\sum_{k=1}^{m} \varphi_k = 1$ a.e. on $\boldsymbol{E}$.

If $\displaystyle\sum_{k=1}^{m} \varphi_k \leq 1$ a.e. on $\boldsymbol{E}$, then $\{\varphi_i\}_{i=1}^m$ is said to be a partial partition of unity. A finite collection of triples $\{(\boldsymbol{\xi}_i, \boldsymbol{I}_i, \varphi)\}_{i=1}^m$ is said to be δ-fine PU-division of $\boldsymbol{E}$ if for each $i \leq m$, supp $\varphi_i \subseteq \boldsymbol{I}_i$ and $\boldsymbol{I}_i \subseteq B(\boldsymbol{\xi}_i, \delta(\boldsymbol{\xi}_i))$. Boonpogkrong, revisited the notion of the PUL integral and it's application in the integrals of a function defined on a manifold. Moreover, Flores and Benitez [4, 5] generalizes the notion in its Stieltjes form and establsihed some convergence theorems. In this paper, the definition of the PU integral will be discussed and the Change of variable formula and the Saks-Henstock Lemma of this integral will be established.

2. Preliminaries

In this section, we introduce the PU integral of a function defined on a Manifold. Throughout the rest of this chapter, if no confusion arises, we denote M as a manifold.

Definition 1. [3, 4] Let X be a Banach space and let $f : [\boldsymbol{a}, \boldsymbol{b}] \to X$. We define the *PU sum* by

$$S(f, D) = \sum_{k=1}^{m} f(\boldsymbol{\xi}_k)\ (\mathcal{R}) \int_{I_k} \varphi_k$$

where D is a δ-fine division of $[\boldsymbol{a}, \boldsymbol{b}]$ and $(\mathcal{R}) \int_{I_k} \varphi_k$ is the Riemann integral of φ_k for all $k \in \{1, 2, \cdots m\}$. For brevity, we write a δ-fine division of $[\boldsymbol{a}, \boldsymbol{b}]$ by $D = \{(\boldsymbol{\xi}, \boldsymbol{I}, \psi)\}$ and a PU sum of f over D by

$$S(f, D) = (D) \sum f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \psi = \sum_{D} f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \psi\ .$$

Definition 2. [4] Let X be a Banach space and $f : [\boldsymbol{a}, \boldsymbol{b}] \to X$ be a Banach-valued funtion. We say that f is *PU integrable* to a vector A over $[\boldsymbol{a}, \boldsymbol{b}]$ if for every $\epsilon > 0$, there exists a gauge δ on $[\boldsymbol{a}, \boldsymbol{b}]$ such that for every δ-fine division D of $[\boldsymbol{a}, \boldsymbol{b}]$, we have

$$\|S(f, D) - A\| < \epsilon.$$

If A is the PU integral of f with over $[\boldsymbol{a}, \boldsymbol{b}]$, then we write

$$A = (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f.$$

Definition 3. [6] Let $g : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$. The *total variation of g over* $[\boldsymbol{a}, \boldsymbol{b}]$ is given by

$$V(g; [\boldsymbol{a}, \boldsymbol{b}]) = \sup \left\{ \sum_{[\boldsymbol{u},\boldsymbol{v}] \in D} \left| \Delta_g([\boldsymbol{u}, \boldsymbol{v}]) \right| : D \text{ is a division of } [\boldsymbol{a}, \boldsymbol{b}] \right\},$$

where

$$\Delta_g([\boldsymbol{u}, \boldsymbol{v}]) = \sum_{\boldsymbol{t} \in \mathcal{V}[\boldsymbol{u},\boldsymbol{v}]} g(\boldsymbol{t}) \prod_{k=1}^{n} (-1)^{\chi_{\{u_k\}}(t_k)}, \qquad \boldsymbol{t} = (t_1, t_2, \cdots, t_n)$$

and $\mathcal{V}[\boldsymbol{u}, \boldsymbol{v}]$ is the set of vertices in $[\boldsymbol{u}, \boldsymbol{v}] = \prod_{k=1}^{n} [u_k, v_k]$. If $V(g, [\boldsymbol{a}, \boldsymbol{b}]) < +\infty$, then g is said to be a *function of bounded variation* on $[\boldsymbol{a}, \boldsymbol{b}]$.

Example 1.

(*i*) For $n = 2$, we have

$$\Delta_g([u_1, v_1]) \times [u_2, v_2] = g(v_1, v_2) - g(u_1, v_2) + g(u_1, u_2) - g(v_1, u_2);$$

(*ii*) For $n = 1$, we have

$$\Delta_g([u_1, v_1]) = g(v_1) - g(u_1).$$

Remark 1. The total variation in n-dimensional Euclidean space is an extension of the usual total variation in Euclidean space.

Definition 4. [7] A topological space is *second countable* if it has a countable basis.

Definition 5. [7] A topological space M is *locally Euclidean of dimension n* if every point $p \in M$ has a neighborhood U_p such that there is a homeomorphism ϕ from U_p onto an open subset O_p of $\mathbb{R}^n$. We call the pair $(U, \phi : U \to \mathbb{R}^n)$ a *chart*, U a *coordinate neighborhood* or a *coordinate open set*, and ϕ a *coordinate map* or *coordinate system* on U. We say that a chart (U, ϕ) is centered at $p \in U$ if $\phi(p) = 0$. A chart (U, ϕ) about p simply means that (U, ϕ) is a chart and $p \in U$.

Definition 6. [7] A *topological manifold of dimension n* is a Hausdorff, second countable, locally Euclidean space of dimension n.

Example 2. [7] The Euclidean space $\mathbb{R}^n$ is covered by a single chart $(\mathbb{R}^n, 1_{\mathbb{R}^n})$, where $1_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map. It is a prime example of a topological manifold. Every open subset $\boldsymbol{U}$ of $\mathbb{R}^n$ is also a topological manifold, with chart $(\boldsymbol{U}, 1_{\mathbb{R}^n})$.

3. Main Results

Lemma 1. *Let $\boldsymbol{U}$ be an open subset of a compact interval $\boldsymbol{E}^*$ in $\mathbb{R}^r$ and $\psi : \boldsymbol{U} \to \psi(\boldsymbol{U})$ be C^1-diffeomorphism which is monotone. Suppose that $\boldsymbol{E}$ is a compact interval such that $\boldsymbol{E} \subseteq \psi(\boldsymbol{U}) \subseteq \mathbb{R}^n$ and $\varphi : \boldsymbol{E} \to \mathbb{R}$ is continuous and $g : \boldsymbol{E} \to \mathbb{R}$ be a function of bounded variation. Then*

$$(\mathcal{R}) \int_{\boldsymbol{E}} \varphi \; dg = (\mathcal{R}) \int_{\psi^{-1}(\boldsymbol{E})} (\varphi \circ \psi) |\det \psi| \; dg,$$

where $(\mathcal{R}) \int_{\boldsymbol{E}} \varphi \; dg$ is the Riemann-Stieltjes integral of φ with respect to g on $\boldsymbol{E}$ and $|\det \psi|$ is the Euclidean norm of the partial derivatives of ψ.

Proof: Since φ is continuous and g is of bounded variation, then $(\mathcal{R}) \int_{\boldsymbol{E}} \varphi \; dg$ exists in $\mathbb{R}$. Thus, for each $\epsilon > 0$, there exists a constant $\delta_1 > 0$ such that for any δ_1-fine division $D = \{(\boldsymbol{\xi}, \boldsymbol{I})\}$ of $\boldsymbol{E}$, we have

$$\left| \sum_D \varphi(\boldsymbol{\xi}) \Delta_g(\boldsymbol{I}) - (\mathcal{R}) \int_{\boldsymbol{E}} \varphi \; dg \right| < \frac{\epsilon}{2}. \tag{3.1}$$

Note that, $(\varphi \circ \psi)|\det \psi|$ is continuous on $\psi^{-1}(\boldsymbol{E})$. Also, since g is a function of bounded variation on $\psi^{-1}(\boldsymbol{E})$, it follows that

$$(\mathcal{R}) \int_{\psi^{-1}(\boldsymbol{E})} (\varphi \circ \psi) |\det \psi| \; dg$$

exists in $\mathbb{R}$; hence there exists a constant $\delta_0 \leq \delta_1$ such that for any δ_0-fine division $D' = \{(\boldsymbol{t}, \boldsymbol{K})\}$ of $\psi^{-1}(\boldsymbol{E})$, we have

$$\left| \sum_{D'} (\varphi \circ \psi)(\boldsymbol{t}) \Delta_{(g \circ \psi)}(\boldsymbol{K}) - (\mathcal{R}) \int_{\psi^{-1}(\boldsymbol{E})} (\varphi \circ \psi) |\det \psi| \; dg \right| < \frac{\epsilon}{2}. \tag{3.2}$$

Since ψ is a diffeomorphism on $\boldsymbol{U}$, ψ is continuous at every point in $\boldsymbol{U}$. Since $\boldsymbol{E} \subseteq \psi(\boldsymbol{U})$, we have $\psi^{-1}(\boldsymbol{E}) \subseteq \boldsymbol{U}$. Hence, ψ is continuous at every point in $\psi^{-1}(\boldsymbol{E})$. Thus, for each $\boldsymbol{x} \in \psi^{-1}(\boldsymbol{E})$ and $\delta_1 > 0$ there exists $\delta_2(\boldsymbol{x}) > 0$ such that for any $\boldsymbol{y} \in B(\boldsymbol{x}, \delta_2(\boldsymbol{x}))$,

$$|\psi(\boldsymbol{y}) - \psi(\boldsymbol{x})| < \delta_1.$$

For each $\boldsymbol{x} \in \psi^{-1}(\boldsymbol{E})$, let $\delta(\boldsymbol{x}) = \min\{\delta_0(\boldsymbol{x}), \delta_2(\boldsymbol{x})\}$. Now, let $D_0 = \{(\boldsymbol{\eta}, \boldsymbol{J})\}$ be a fix δ-fine division of $\psi^{-1}(\boldsymbol{E})$. Hence, for each $(\boldsymbol{\eta}, \boldsymbol{J})$ in D_0 and $\boldsymbol{x} \in \boldsymbol{J}$ we have

$$|\psi(\boldsymbol{\eta}) - \psi(\boldsymbol{x})| < \delta_1. \tag{3.3}$$

Also, $\psi(\boldsymbol{J})$ is a compact interval in $\mathbb{R}^n$ that partitions $\boldsymbol{E}$. By (3.3), $\{(\psi(\boldsymbol{\eta}), \psi(\boldsymbol{J}))\}$ is δ_1-fine division of $\boldsymbol{E}$. Thus, by (3.1)

$$\frac{\epsilon}{2} > \left| (\mathcal{R}) \int_{\boldsymbol{E}} \varphi \; dg - \sum_{D_0} \varphi(\psi(\boldsymbol{\eta})) \cdot \Delta_g(\psi(\boldsymbol{J})) \right|$$

$$= \left|(\mathcal{R})\int_{\boldsymbol{E}} \varphi\, dg - \sum_{D_0} (\varphi\circ\psi)(\boldsymbol{\eta})\cdot \Delta_{g\circ\psi}(\boldsymbol{J})\right| \tag{3.4}$$

Note that since $\delta \leq \delta_0$, D_0 is also a δ_0-fine division of $\psi^{-1}(\boldsymbol{E})$. Hence, by (3.2)

$$\left|\sum_{D_0}(\varphi\circ\psi)(\boldsymbol{\eta})\Delta_{g\circ\psi}(\boldsymbol{J}) - (\mathcal{R})\int_{\psi^{-1}(\boldsymbol{E})}(\varphi\circ\psi)|\det\psi|\, dg\right| < \frac{\epsilon}{2}. \tag{3.5}$$

Therefore, by (3.4) and (3.5)

$$\begin{aligned}
0 &\leq \left|(\mathcal{R})\int_{\boldsymbol{E}}\varphi\, dg - (\mathcal{R})\int_{\psi^{-1}(\boldsymbol{E})}(\varphi\circ\psi)|\det\psi|\, dg\right| \\
&= \left|(\mathcal{R})\int_{\boldsymbol{E}}\varphi\, dg - \sum_{D_0}(\varphi\circ\psi)(\boldsymbol{t})\Delta_{(g\circ\psi)}(\boldsymbol{J})\right. \\
&\qquad \left. + \sum_{D_0}(\varphi\circ\psi)(\boldsymbol{t})\Delta_{(g\circ\psi)}(\boldsymbol{J}) - (\mathcal{R})\int_{\psi^{-1}(\boldsymbol{E})}(\varphi\circ\psi)|\det\psi|\, dg\right| \\
&\leq \left|(\mathcal{R})\int_{\boldsymbol{E}}\varphi\, dg - \sum_{D_0}(\varphi\circ\psi)(\boldsymbol{t})\Delta_{(g\circ\psi)}(\boldsymbol{J})\right| \\
&\qquad + \left|\sum_{D_0}(\varphi\circ\psi)(\boldsymbol{t})\Delta_{(g\circ\psi)}(\boldsymbol{J}) - (\mathcal{R})\int_{\psi^{-1}(\boldsymbol{E})}(\varphi\circ\psi)|\det\psi|\, dg\right| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have

$$(\mathcal{R})\int_{\boldsymbol{E}}\varphi\, dg = (\mathcal{R})\int_{\psi^{-1}(\boldsymbol{E})}(\varphi\circ\psi)|\det\psi|\, dg. \qquad \square$$

Theorem 1. (Change of Variable Formula). *Let $f : [\boldsymbol{a},\boldsymbol{b}] \to X$ be PU integrable over $[\boldsymbol{a},\boldsymbol{b}]$. Let $\boldsymbol{U}$ be an open subset of a compact interval $\boldsymbol{E}$ in $\mathbb{R}^m$. Let $\Psi : \boldsymbol{U} \to \Psi(\boldsymbol{U})$ be C^1-diffeomorphism and $[\boldsymbol{a},\boldsymbol{b}] \subseteq \Psi(\boldsymbol{U}) \subseteq \mathbb{R}^n$. Then $(f\circ\Psi)\cdot|\det\psi|\cdot\chi_{\Psi^{-1}([\boldsymbol{a},\boldsymbol{b}])}$ is PU integrable over $\boldsymbol{E}$ and*

$$(\mathcal{P})\int_{\boldsymbol{E}}(f\circ\Psi)\cdot|\det\Psi|\cdot\chi_{\Psi^{-1}([\boldsymbol{a},\boldsymbol{b}])} = (\mathcal{P})\int_{\Psi^{-1}([\boldsymbol{a},\boldsymbol{b}])}(f\circ\Psi)\cdot|\det\Psi| = (\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} f.$$

Proof: Fix $\epsilon > 0$. Since f is PU integrable over $[\boldsymbol{a},\boldsymbol{b}]$, we choose a gauge δ_0 on $[\boldsymbol{a},\boldsymbol{b}]$ such that

$$\left\|\sum_{D} f(\boldsymbol{\xi})\int_{\boldsymbol{I}}\varphi - (\mathcal{P})\int_{[\boldsymbol{a},\boldsymbol{b}]} f\right\| < \frac{\epsilon}{2}$$

for every δ_0-fine division $D = \{(\boldsymbol{\xi}, \varphi, \boldsymbol{I})\}$ of $[\boldsymbol{a}, \boldsymbol{b}]$. Since Ψ is a C^1-diffeomorphism and $\boldsymbol{U}$ is an open set in $\mathbb{R}^r$, $\Psi(\boldsymbol{U})$ is open in $\mathbb{R}^n$. So, choose δ_0 in such a way that for each $\boldsymbol{\xi} \in \Psi(\boldsymbol{U})$,

$$B(\boldsymbol{\xi}, \delta_0(\boldsymbol{\xi})) \subseteq \Psi(\boldsymbol{U}).$$

Define $\delta : \boldsymbol{E} \to \mathbb{R}^+$ such that for $\boldsymbol{u} \in \Psi^{-1}([\boldsymbol{a}, \boldsymbol{b}])$, we have

$$B(\boldsymbol{u}, \delta(\boldsymbol{u})) \subseteq \Psi^{-1}\Bigg(B\Bigg(\Psi(\boldsymbol{u}), \frac{\delta_0(\Psi(\boldsymbol{u}))}{2\sqrt{p}}\Bigg)\Bigg), \tag{3.6}$$

where $1 \leq p$ is a fixed positive real number; also, for $\boldsymbol{u} \in \boldsymbol{E} \setminus \Psi^{-1}([\boldsymbol{a}, \boldsymbol{b}])$,

$$B(\boldsymbol{u}, \delta(\boldsymbol{u})) \cap \boldsymbol{E} \subseteq \boldsymbol{E} \setminus \Psi^{-1}([\boldsymbol{a}, \boldsymbol{b}]). \tag{3.7}$$

Let $D = \{(\boldsymbol{\xi}, \boldsymbol{I}, \varphi)\}$ be a δ-fine division of $\boldsymbol{E}$. Suppose $D = D_1 \cup D_2$ where $D_1 = \{(\boldsymbol{\xi}, \boldsymbol{I}, \varphi) \in D : \boldsymbol{\xi} \in \Psi^{-1}([\boldsymbol{a}, \boldsymbol{b}])\}$ and $D_2 = D \setminus D_1$. We may assume that

$$D_1 = \{(\boldsymbol{\xi}_k, \boldsymbol{I}_{\boldsymbol{k}}, \varphi_k)\}_{k=1}^{r} \quad \text{and} \quad D_2 = \{(\boldsymbol{\xi}_k, \boldsymbol{I}_{\boldsymbol{k}}, \varphi_k)\}_{k=r+1}^{s}.$$

Let $k \in \{r+1, r+2, \cdots, s\}$ and let $\boldsymbol{x} \in \text{supp } \varphi_k$. Here, $\varphi_k(\boldsymbol{x}) > 0$. Since D is a δ-fine division of $\boldsymbol{E}$,

$$\text{supp } \varphi_k \subseteq \boldsymbol{I}_k \subseteq B(\boldsymbol{\xi}_k, \delta(\boldsymbol{\xi}_k));$$

Which implies

$$\text{supp } \ \varphi_k = \text{supp } \ \varphi_k \cap \boldsymbol{E} \subseteq B(\boldsymbol{\xi}_k, \delta(\boldsymbol{\xi}_k)) \cap \boldsymbol{E}.$$

And by (3.7),

$$\text{supp } \ \varphi_k \subseteq B(\boldsymbol{\xi}_k, \delta(\boldsymbol{\xi}_k)) \cap \boldsymbol{E} \subseteq \boldsymbol{E} \setminus \Psi^{-1}([\boldsymbol{a}, \boldsymbol{b}]);$$

which means supp $\varphi_k \cap \Psi^{-1}([\boldsymbol{a}, \boldsymbol{b}]) = \varnothing$. Thus, for each $\boldsymbol{x} \in \Psi^{-1}([\boldsymbol{a}, \boldsymbol{b}])$, $\varphi_k(\boldsymbol{x}) = 0$ for all $k = r+1, r+2 \cdots, s$. Now, for each $\boldsymbol{x} \in \Psi^{-1}([\boldsymbol{a}, \boldsymbol{b}])$

$$\sum_{k=r+1}^{s} \varphi_k(\boldsymbol{x}) = 0$$

and so

$$\sum_{k=1}^{s} \varphi_k(\boldsymbol{x}) = \sum_{k=1}^{r} \varphi_k(\boldsymbol{x}) + \sum_{k=r+1}^{s} \varphi_k(\boldsymbol{x}) = \sum_{k=1}^{r} \varphi_k(\boldsymbol{x}).$$

For each $k = 1, 2, \cdots r$, let $\boldsymbol{x}_k = \Psi(\boldsymbol{\xi}_k)$ and $\sigma_k = \varphi_k \circ \Psi^{-1}$.

Note that $\sigma_j : \Psi(\boldsymbol{U}) \to \mathbb{R}$ and

$$B\Bigg(\sigma_k(\Psi(\boldsymbol{y})), \frac{\delta_0(\Psi(\boldsymbol{y}))}{2\sqrt{p}}\Bigg) \subseteq B\Bigg(\sigma_k(\Psi(\boldsymbol{y})), \delta_0(\Psi(\boldsymbol{y}))\Bigg).$$

Notice that $x \in \text{supp}\ \sigma_k \cdot \chi_{[\boldsymbol{a},\boldsymbol{b}]}$ for all $k = 1, 2, \cdots, r$; Then

$$\varphi_k(\Psi^{-1}(x)) = (\varphi_k \circ \Psi^{-1})(x) = \sigma_k(x) > 0,$$

that is, $\Psi^{-1}(\boldsymbol{x}) \in \text{supp}\ \varphi_k$. From (3.6) and since D is a δ-fine division of $\boldsymbol{E}$, we have

$$\Psi^{-1}(\boldsymbol{x}) \in \text{supp}\ \varphi_k \subseteq \boldsymbol{I}_k \subseteq B(\boldsymbol{\xi}_k, \delta(\boldsymbol{\xi_k})) \subseteq \Psi^{-1}\left(B\left(\Psi(\boldsymbol{\xi}_k), \frac{\delta_0(\Psi(\boldsymbol{\xi}_k))}{2\sqrt{p}}\right)\right).$$

Hence,

$$\boldsymbol{x} \in B\left(\Psi(\boldsymbol{\xi}_k), \frac{\delta_0(\Psi(\boldsymbol{\xi}_k))}{2\sqrt{p}}\right).$$

Thus, for each $k = 1, 2, \cdots, r$,

$$\text{supp}\ \sigma_k \cdot \chi_{[\boldsymbol{a},\boldsymbol{b}]} \subseteq (B\left(\Psi(\boldsymbol{\xi}_k), \frac{\delta_0(\Psi(\boldsymbol{\xi}_k))}{2\sqrt{p}}\right)) \cap [\boldsymbol{a}, \boldsymbol{b}].$$

Now, since $B\left(\sigma_k(\Psi(\boldsymbol{y})), \frac{\delta_0(\Psi(\boldsymbol{y}))}{2\sqrt{p}}\right) \subseteq B\left(\sigma_k(\Psi(\boldsymbol{y})), \delta_0(\Psi(\boldsymbol{y}))\right)$ for all $k = 1, 2, \cdots r$, it follows that

$$\begin{aligned}\text{supp}\ \sigma_k \cdot \chi_{[\boldsymbol{a},\boldsymbol{b}]} &\subseteq B\left(\sigma_k(\Psi(\boldsymbol{y})), \frac{\delta_0(\Psi(\boldsymbol{y}))}{2\sqrt{p}}\right) \cap [\boldsymbol{a}, \boldsymbol{b}] \\ &\subseteq B\left(\sigma_k(\Psi(\boldsymbol{y})), \delta_0(\Psi(\boldsymbol{y}))\right) \cap [\boldsymbol{a}, \boldsymbol{b}]\end{aligned}$$

for all $k = 1, 2, \cdots r$. So, we choose a compact interval $\boldsymbol{J}_k$ such that

$$B\left(\sigma_k(\Psi(\boldsymbol{y})), \frac{\delta_0(\Psi(\boldsymbol{y}))}{2\sqrt{p}}\right) \cap [\boldsymbol{a}, \boldsymbol{b}] \subseteq \boldsymbol{J}_k \subseteq B\left(\sigma_k(\Psi(\boldsymbol{y})), \delta_0(\Psi(\boldsymbol{y}))\right) \cap [\boldsymbol{a}, \boldsymbol{b}].$$

for all $k = 1, 2, \cdots, r$. Thus,

$$\begin{aligned}\text{supp}\ \sigma_k \cdot \chi_{[\boldsymbol{a},\boldsymbol{b}]} &\subseteq B\left(\sigma_k(\Psi(\boldsymbol{y})), \frac{\delta_0(\Psi(\boldsymbol{y}))}{2\sqrt{p}}\right) \cap [\boldsymbol{a}, \boldsymbol{b}] \subseteq \boldsymbol{J}_k \\ &\subseteq B\left(\sigma_k(\Psi(\boldsymbol{y})), \delta_0(\Psi(\boldsymbol{y}))\right) \cap [\boldsymbol{a}, \boldsymbol{b}],\end{aligned}$$

that is,

$$\text{supp}\ \sigma_k \cdot \chi_{[\boldsymbol{a},\boldsymbol{b}]} \subseteq \boldsymbol{J}_k \subseteq B\left(\sigma_k(\Psi(\boldsymbol{y})), \delta_0(\Psi(\boldsymbol{y}))\right) \cap [\boldsymbol{a}, \boldsymbol{b}] \tag{3.8}$$

for all $k = 1, 2, \cdots, r$. Note that on $[\boldsymbol{a}, \boldsymbol{b}]$, $\chi_{[\boldsymbol{a},\boldsymbol{b}]} = 1$. Since φ_k and Ψ^{-1} are continuously differentiable functions on $[\boldsymbol{a}, \boldsymbol{b}]$, $\sigma_k \cdot \chi_{[\boldsymbol{a},\boldsymbol{b}]} = \sigma_k = \varphi_k \circ \Psi^{-1}$ is also continuously differentiable function on $[\boldsymbol{a}, \boldsymbol{b}]$. Fix $\boldsymbol{x} \in [\boldsymbol{a}, \boldsymbol{b}]$. Since φ_k is a partition of unity for all $k = 1, 2, \cdots, r$, we have

$$\sum_{k=1}^{r} \sigma_k \cdot \chi_{[\boldsymbol{a},\boldsymbol{b}]}(\boldsymbol{x}) = \sum_{k=1}^{r} \sigma_k(\boldsymbol{x}) = \sum_{k=1}^{r} (\varphi_k(\Psi^{-1}(\boldsymbol{x})) = 1.$$

This means that $\{\sigma_k \cdot \chi^{-1}\}$ is a partition of unity on $[\boldsymbol{a}, \boldsymbol{b}]$. The inclusion (3.8) implies that $D_0 = \{(\boldsymbol{x}_k, \boldsymbol{J}_k, \sigma_k \cdot \chi^{-1})\}_{k=1}^{r}$ is a δ_0-fine division of $[\boldsymbol{a}, \boldsymbol{b}]$.

Next, we will show that for each $k = 1, 2, \cdots, r$, supp $\sigma_k \circ \Psi =$ supp φ_k. Indeed, let $k \in \{1, 2, \cdots, r\}$ and let $x \in$ supp $\sigma_k \circ \Psi$. Then $\sigma_k(\Psi(x)) > 0$ and so $\Psi(x) \in$ supp $\sigma_k =$ supp $\varphi_k \circ \Psi^{-1}$. Hence, $x \in$ supp φ_k and supp $\sigma_k \circ \Psi \subseteq$ supp φ_k. Now, let $k \in \{1, 2, \cdots, r\}$ and $x \in$ supp φ_k. Then $\varphi_k(\Psi^{-1}(\Psi(x))) = \varphi_k(x) > 0$; and so $\Psi(x) \in$ supp $\varphi_k \circ \Psi^{-1} =$ supp σ_k and $\sigma_k(\Psi(x)) = (\sigma_k \circ \Psi)(x) > 0$. Henceforth, $x \in$ supp $\sigma_k \circ \Psi$ and supp $\varphi_k \subseteq$ supp $\sigma_k \circ \Psi$. Thus, supp $\varphi_k =$ supp $\sigma_k \circ \Psi$. By Lemma 1,

$$\int_{\boldsymbol{J}_k} \sigma_k = \int_{\Psi^{-1}(\boldsymbol{J}_k)} (\sigma_k \circ \psi)|\det \Psi| = \int_{\boldsymbol{I}_k} \varphi_k |\det \Psi|.$$

Notice that

$$\begin{aligned}
&\left\| \sum_{k=1}^{s} f(\Psi(\boldsymbol{\xi}_k))|\det \Psi| \chi_{\Psi^{-1}([\boldsymbol{a},\boldsymbol{b}])}(\boldsymbol{\xi}_k) \int_{\boldsymbol{I}_k} \varphi_k - \sum_{k=1}^{p} f(\boldsymbol{x}_k) \int_{\boldsymbol{J}_k} \sigma_k \chi_{[\boldsymbol{a},\boldsymbol{b}]} \right\| \\
&= \left\| \sum_{k=1}^{p} f(\Psi(\boldsymbol{\xi}_k))|\det \Psi| \int_{\boldsymbol{I}_k} \varphi_k \; - \sum_{k=1}^{p} f(\boldsymbol{x}_k) \int_{\boldsymbol{J}_k} \sigma_k \right\| \\
&= \left\| \sum_{k=1}^{p} f(\Psi(\boldsymbol{\xi}_k))|\det \Psi| \int_{\boldsymbol{I}_k} \varphi_k - \sum_{k=1}^{p} f(\Psi(\boldsymbol{\xi}_k))|\det \Psi| \int_{\boldsymbol{I}_k} \varphi_k \right\| \\
&= 0
\end{aligned}$$

implies

$$\sum_{k=1}^{s} f(\Psi(\boldsymbol{\xi}_k))|\det \Psi| \chi_{\Psi^{-1}(E)}(\boldsymbol{\xi}_k) \int_{\boldsymbol{I}_k} \varphi_k \; = \sum_{k=1}^{p} f(\boldsymbol{x}_k) \int_{\boldsymbol{J}_k} \sigma_k \chi_{\boldsymbol{E}}. \tag{3.9}$$

Since D_0 is a δ_0-fine division of $[\boldsymbol{a}, \boldsymbol{b}]$,

$$\left\| \sum_{k=1}^{p} f(\boldsymbol{x}_k) \int_{\boldsymbol{J}_k} \sigma_k \chi_{\boldsymbol{E}} \; - (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f \right\| < \epsilon;$$

that is,

$$\left\| \sum_{k=1}^{s} f(\Psi(\boldsymbol{\xi}_k))|\det \Psi| \chi_{\Psi^{-1}(E)}(\boldsymbol{\xi}_k) \int_{\boldsymbol{I}_k} \varphi_k - (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f \right\| < \epsilon$$

This simply means, $(f \circ \Psi)\chi_{\Psi^{-1}([\boldsymbol{a},\boldsymbol{b}])}$ is PU integrable with over $\boldsymbol{E}$ and

$$(\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f = (\mathcal{P}) \int_{\Psi^{-1}([\boldsymbol{a},\boldsymbol{b}])} (f \circ \Psi) \; d\Psi.$$ □

Lemma 2. *Let* $D_p = \{\boldsymbol{\xi}, \boldsymbol{I}, \varphi\}$ *be a partial* δ*-fine division of* $[\boldsymbol{a}, \boldsymbol{b}]$*,* $\sigma(\boldsymbol{x}) = \sum_{D_1} \varphi(\boldsymbol{x})$*, and* $D_f = \{(\boldsymbol{\eta}, \boldsymbol{J}, \psi)\}$ *be* δ*-fine division of* $[\boldsymbol{a}, \boldsymbol{b}]$*. Then*

$$D_p \cup \{(\boldsymbol{\eta}, \boldsymbol{J}, (1-\sigma)\psi)\}$$

is a δ*-fine division of* $[\boldsymbol{a}, \boldsymbol{b}]$*.*

Proof: Let $\boldsymbol{x} \in [\boldsymbol{a}, \boldsymbol{b}]$. Since D_f is a δ-fine division of $[\boldsymbol{a}, \boldsymbol{b}]$, $\{\psi\}$ is a partition of unity. Thus,

$$\sum_{D_f} \psi(\boldsymbol{x}) = 1.$$

Observe that

$$\begin{aligned}\sum_{D_p} \varphi(\boldsymbol{x}) + \sum_{D_f} (1-\sigma(\boldsymbol{x})) \cdot \psi(\boldsymbol{x}) &= \sigma(\boldsymbol{x}) + \sum_{D_f} \psi(\boldsymbol{x}) - \sigma(\boldsymbol{x}) \sum_{D_f} \psi(\boldsymbol{x}) \\ &= \sigma(\boldsymbol{x}) + 1 - \sigma(\boldsymbol{x}) \cdot 1 = 1.\end{aligned}$$

This means that $\{\varphi\} \cup \{(1-\sigma)\psi\}$ is a partition of unity. Since D_p is a partial δ-fine division of $[\boldsymbol{a}, \boldsymbol{b}]$, it follows that

$$\text{supp } \varphi \subseteq \boldsymbol{I} \subseteq B(\boldsymbol{\xi}, \delta(\boldsymbol{\xi})). \tag{3.10}$$

Now, let $\boldsymbol{x} \in \text{supp } ((1-\sigma) \cdot \psi)$. Then $(1-\sigma(\boldsymbol{x})) \cdot \psi(\boldsymbol{x}) > 0$ and so $\psi(\boldsymbol{x}) > 0$. Hence, $\boldsymbol{x} \in \text{supp } \psi$. Thus, supp $((1-\varphi) \cdot \psi) \subseteq \text{supp } \psi$. So we have

$$\text{supp } \ ((1-\sigma) \cdot \psi) \subseteq \text{supp } \ \psi \subseteq \boldsymbol{J} \subseteq B(\boldsymbol{\eta}, \delta(\boldsymbol{\eta})). \tag{3.11}$$

By the inclusions in 3.10 and 3.11, evidently

$$D \cup \{(\boldsymbol{\eta}, \boldsymbol{J}, (1-\sigma)\psi)\}$$

is a δ-fine division of $[\boldsymbol{a}, \boldsymbol{b}]$. □

Lemma 3. *Let* $f : [\boldsymbol{a}, \boldsymbol{b}] \to X$ *be a PU integrable function over* $[\boldsymbol{a}, \boldsymbol{b}]$*. If* $\boldsymbol{G}$ *be an open and bounded set such that supp* $f \subseteq \boldsymbol{G} \subseteq \mathbb{R}^n$*, then the PU integral*

$$(\mathcal{P}) \int_{[\boldsymbol{a}, \boldsymbol{b}]} f \cdot \phi$$

exists for any continuously differentiable function $\phi : \boldsymbol{G} \to \mathbb{R}$ *with*

$$0 < \phi(\boldsymbol{x}) \leq 1, \text{ for all } \boldsymbol{x} \in \boldsymbol{G}.$$

Proof: Fix $\epsilon > 0$. Then we choose a gauge δ_1 on $[\boldsymbol{a}, \boldsymbol{b}]$ such that for any δ_1-fine division $D = \{(\boldsymbol{\xi}, \varphi, \boldsymbol{I})\}$ of $[\boldsymbol{a}, \boldsymbol{b}]$, we have

$$\left\| \sum_D f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \varphi - (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f \right\| < \epsilon.$$

Assume that $B(\boldsymbol{x}, \delta_1(\boldsymbol{x})) \subseteq G$, for each $\boldsymbol{x} \in \text{supp } f$. Put $M = V(g; [\boldsymbol{a}, \boldsymbol{b}]) \in \mathbb{R}$ and we choose $\delta_2(\boldsymbol{x}) > 0$ such that for any $\boldsymbol{y} \in [\boldsymbol{a}, \boldsymbol{b}]$ with $\boldsymbol{y} \in B(\boldsymbol{x}, \delta_2(\boldsymbol{x}))$, we have

$$\|f(\boldsymbol{x})\| \cdot |\phi(\boldsymbol{y}) - \phi(\boldsymbol{x})| < \frac{\epsilon}{3(M+1)}. \tag{3.12}$$

Take $\delta(\boldsymbol{x}) = \min\{\delta_1(\boldsymbol{x}), \delta_2(\boldsymbol{x})\}$, for all $\boldsymbol{x} \in \text{supp } f$. Note that for each $\boldsymbol{x} \in \text{supp } f$, if $\boldsymbol{y} \in B(\boldsymbol{x}, \delta(\boldsymbol{x}))$, the inequality (3.12) still holds. For each $\boldsymbol{x} \in \text{supp } f$, we have

$$\|f(\boldsymbol{x})\| \cdot \sup\{|\phi(\boldsymbol{y}) - \phi(\boldsymbol{x})| : \boldsymbol{y} \in B(\boldsymbol{x}, \delta(\boldsymbol{x}))\} < \frac{\epsilon}{3(M+1)} \tag{3.13}$$

Now, let $D_1 = \{(\boldsymbol{\xi}, \varphi, \boldsymbol{I})\}$ and $D_2 = \{(\boldsymbol{\eta}, \psi, \boldsymbol{J})\}$ be any δ-fine divisions of $[\boldsymbol{a}, \boldsymbol{b}]$. For each $\boldsymbol{x} \in \text{supp } f$, since $0 < \phi(\boldsymbol{x}) \leq 1$, we have

$$\sum_{D_1} \phi(\boldsymbol{x}) \cdot \varphi(\boldsymbol{x}) = \phi(\boldsymbol{x}) \cdot \sum_{D_1} \varphi(\boldsymbol{x}) = \phi(\boldsymbol{x}) \leq 1.$$

So, $\{\phi \cdot \varphi\}$ is a partial partition of unity on supp f. Since $\phi(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \in G$, it is evident that

$$\text{supp } (\phi \cdot \varphi) = \text{supp } (\varphi).$$

Note that D_1 is a δ-fine division of $[\boldsymbol{a}, \boldsymbol{b}]$. Then

$$\text{supp } (\phi \cdot \varphi) = \text{supp } \varphi \subseteq \boldsymbol{I} \subseteq B(\boldsymbol{\xi}, \delta(\boldsymbol{\xi})).$$

Hence, $\{(\boldsymbol{\xi}, \phi \cdot \varphi, \boldsymbol{I})\}$ is a partial δ-fine division of $[\boldsymbol{a}, \boldsymbol{b}]$.
Since $0 < \phi(\boldsymbol{x}) \leq 1$ for all supp f,

$$\sum_{D_2} (1 - \phi(\boldsymbol{x})) \cdot \psi(\boldsymbol{x}) = (1 - \phi(\boldsymbol{x})) \cdot \sum_{D_2} \psi(\boldsymbol{x}) = 1 - \phi(\boldsymbol{x}) \leq 1.$$

So, $\{(1 - \phi) \cdot \psi)\}$ is a partial partition of unity on supp f.

Moreover,

$$\text{supp } (1 - \phi) \cdot \psi = \text{supp } \psi \subseteq \boldsymbol{J} \subseteq B(\boldsymbol{\eta}, \delta(\boldsymbol{\eta})).$$

Hence, $\{(\boldsymbol{\eta}, (1 - \phi) \cdot \psi, \boldsymbol{J})\}$ is a partial δ-fine division of $[\boldsymbol{a}, \boldsymbol{b}]$ which, by Lemma 2, will further imply that $\{(\boldsymbol{\xi}, \phi \cdot \varphi, \boldsymbol{I})\} \cup \{(\boldsymbol{\eta}, (1 - \phi) \cdot \psi, \boldsymbol{J})\}$ is δ-fine division of $[\boldsymbol{a}, \boldsymbol{b}]$. But f is PU integrability over $[\boldsymbol{a}, \boldsymbol{b}]$; so,

$$\left\| \left(\sum_{D_1} f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \phi \cdot \varphi \ + \sum_{D_2} f(\boldsymbol{\eta}) \int_{\boldsymbol{J}} (1 - \phi) \cdot \psi \ \right) - (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f \right\| < \frac{\epsilon}{6}. \tag{3.14}$$

Since f is PU integrable over $[\boldsymbol{a},\boldsymbol{b}]$ and D_2 is a δ-fine division of $[\boldsymbol{a},\boldsymbol{b}]$, we then have

$$\left\| \sum_{D_2} f(\boldsymbol{\eta}) \int_{\boldsymbol{J}} \psi \; - (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f \right\| < \frac{\epsilon}{6}. \tag{3.15}$$

By (3.14) and (3.15), we have

$$\begin{aligned}
&\left\| \sum_{D_1} f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \phi \cdot \varphi \; - \sum_{D_2} f(\boldsymbol{\eta}) \int_{\boldsymbol{J}} \phi \cdot \psi \right\| \\
&\quad = \left\| \left(\sum_{D_1} f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \phi \cdot \varphi \; + \sum_{D_2} f(\boldsymbol{\eta}) \int_{\boldsymbol{J}} (1-\phi) \cdot \psi \right) - (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f \right. \\
&\qquad \left. + (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f - \sum_{D_2} f(\boldsymbol{\eta}) \int_{\boldsymbol{J}} \psi \right\| \\
&\quad \leq \left\| \left(\sum_{D_1} f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \phi \cdot \varphi + \sum_{D_2} f(\boldsymbol{\eta}) \int_{\boldsymbol{J}} (1-\phi) \cdot \psi \right) - (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f \right\| \\
&\qquad + \left\| (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f - \sum_{D_2} f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \psi \right\| < \frac{\epsilon}{6} + \frac{\epsilon}{6} = \frac{\epsilon}{3},
\end{aligned}$$

that is,

$$\left\| \sum_{D_1} f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \phi \cdot \varphi - \sum_{D_2} f(\boldsymbol{\eta}) \int_{\boldsymbol{J}} \phi \cdot \psi \right\| < \frac{\epsilon}{3} \tag{3.16}$$

Further, by (3.13)

$$\begin{aligned}
&\left\| \sum_{D_1} f(\boldsymbol{\xi}) \phi(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \varphi - \sum_{D_1} f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \phi \cdot \varphi \right\| \\
&\quad = \left\| \sum_{D_1} f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \phi(\boldsymbol{\xi}) \cdot \varphi - \sum_{D_1} f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \phi \cdot \varphi \right\| \\
&\quad = \left\| \sum_{D_1} f(\boldsymbol{\xi}) \left[\int_{\boldsymbol{I}} \phi(\boldsymbol{\xi}) \cdot \varphi - \int_{\boldsymbol{I}} \phi \cdot \varphi \right] \right\| \\
&\quad = \left\| \sum_{D_1} f(\boldsymbol{\xi}) \left[\int_{\boldsymbol{I}} \left(\phi(\boldsymbol{\xi}) \cdot \varphi - \phi \cdot \varphi \right) \right] \right\| \\
&\quad = \left\| \sum_{D_1} f(\boldsymbol{\xi}) \left[\int_{\boldsymbol{I}} \left(\phi(\boldsymbol{\xi}) - \phi \right) \cdot \varphi \right] \right\| \\
&\quad \leq \left\| \sum_{D_1} f(\boldsymbol{\xi}) \cdot \int_{\boldsymbol{I}} \sup \left\{ |\phi(\boldsymbol{\xi}) - \phi(\boldsymbol{x})| : \boldsymbol{x} \in B(\boldsymbol{\xi}, \delta(\boldsymbol{\xi})) \right\} \cdot \varphi \right\| \\
&\quad \leq \left\| \sum_{D_1} f(\boldsymbol{\xi}) \cdot \sup \left\{ |\phi(\boldsymbol{\xi}) - \phi(\boldsymbol{x})| : \boldsymbol{x} \in B(\boldsymbol{\xi}, \delta(\boldsymbol{\xi})) \right\} \cdot \int_{\boldsymbol{I}} \varphi \right\|
\end{aligned}$$

$$\leq \sum_{D_1} \|f(\boldsymbol{\xi})\| \cdot \sup\Big\{|\phi(\boldsymbol{\xi}) - \phi(\boldsymbol{x})| : \boldsymbol{x} \in B(\boldsymbol{\xi}, \delta(\boldsymbol{\xi}))\Big\} \cdot \int_{\boldsymbol{I}} \varphi$$

$$\leq \frac{\epsilon}{3(M+1)} \cdot \sum_{D_1} \int_{\boldsymbol{I}} \varphi \leq \frac{\epsilon}{3(M+1)} \cdot (M+1) = \frac{\epsilon}{3},$$

that is,

$$\Big\| \sum_{D_1} f(\boldsymbol{\xi})\phi(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \varphi - \sum_{D_1} f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \phi \cdot \varphi \Big\| < \frac{\epsilon}{3} \tag{3.17}$$

In similar fashion,

$$\Big\| \sum_{D_2} f(\boldsymbol{\eta})\phi(\boldsymbol{\eta}) \int_{\boldsymbol{J}} \psi - \sum_{D_2} f(\boldsymbol{\eta}) \int_{\boldsymbol{J}} \phi \cdot \psi \Big\|$$

$$= \Big\| \sum_{D_2} f(\boldsymbol{\eta}) \int_{\boldsymbol{J}} \phi(\boldsymbol{\eta}) \cdot \psi - \sum_{D_2} f(\boldsymbol{\eta}) \int_{\boldsymbol{J}} \phi \cdot \psi \Big\|$$

$$= \Big\| \sum_{D_2} f(\boldsymbol{\eta}) \Big[\int_{\boldsymbol{J}} \phi(\boldsymbol{\eta}) \cdot \psi - \int_{\boldsymbol{J}} \phi \cdot \psi \Big] \Big\|$$

$$= \Big\| \sum_{D_2} f(\boldsymbol{\eta}) \Big[\int_{\boldsymbol{J}} \big(\phi(\boldsymbol{\eta}) \cdot \psi - \phi \cdot \psi\big) \Big] \Big\|$$

$$= \Big\| \sum_{D_2} f(\boldsymbol{\eta}) \Big[\int_{\boldsymbol{J}} \big(\phi(\boldsymbol{\eta}) - \phi\big) \cdot \psi \Big] \Big\|$$

$$\leq \Big\| \sum_{D_2} f(\boldsymbol{\eta}) \cdot \int_{\boldsymbol{J}} \sup\Big\{|\phi(\boldsymbol{\eta}) - \phi(\boldsymbol{x})| : \boldsymbol{x} \in B(\boldsymbol{\eta}, \delta(\boldsymbol{\eta}))\Big\} \cdot \psi \Big\|$$

$$\leq \Big\| \sum_{D_2} f(\boldsymbol{\eta}) \cdot \sup\Big\{|\phi(\boldsymbol{\eta}) - \phi(\boldsymbol{x})| : \boldsymbol{x} \in B(\boldsymbol{\eta}, \delta(\boldsymbol{\eta}))\Big\} \cdot \int_{\boldsymbol{J}} \psi \Big\|$$

$$\leq \sum_{D_2} \|f(\boldsymbol{\eta})\| \cdot \sup\Big\{|\phi(\boldsymbol{\eta}) - \phi(\boldsymbol{x})| : \boldsymbol{x} \in B(\boldsymbol{\eta}, \delta(\boldsymbol{\eta}))\Big\} \cdot \int_{\boldsymbol{J}} \psi$$

$$\leq \frac{\epsilon}{3(M+1)} \cdot \sum_{D_2} \int_{\boldsymbol{J}} \psi \leq \frac{\epsilon}{3(M+1)} \cdot (M+1) = \frac{\epsilon}{3},$$

in other words,

$$\Big\| \sum_{D_2} f(\boldsymbol{\eta})\phi(\boldsymbol{\eta}) \int_{\boldsymbol{J}} \psi - \sum_{D_2} f(\boldsymbol{\eta}) \int_{\boldsymbol{J}} \phi \cdot \psi \Big\| < \frac{\epsilon}{3} \tag{3.18}$$

Therefore, by (3.16), (3.17), and (3.18)

$$\Big\| \sum_{D_1} f(\boldsymbol{\xi})\phi(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \varphi - \sum_{D_2} f(\boldsymbol{\eta})\phi(\boldsymbol{\eta}) \int_{\boldsymbol{I}} \psi \Big\|$$

$$
\begin{aligned}
&= \left\| \sum_{D_1} f(\boldsymbol{\xi})\phi(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \varphi - \sum_{D_1} f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \phi \cdot \varphi + \sum_{D_1} f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \phi \cdot \varphi \right. \\
&\qquad \left. - \sum_{D_2} f(\boldsymbol{\eta}) \int_{\boldsymbol{J}} \phi \cdot \psi + \sum_{D_2} f(\boldsymbol{\eta}) \int_{\boldsymbol{J}} \phi \cdot \psi - \sum_{D_2} f(\boldsymbol{\eta})\phi(\boldsymbol{\eta}) \int_{\boldsymbol{J}} \psi \right\| \\
&\leq \left\| \sum_{D_1} f(\boldsymbol{\xi})\phi(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \varphi - \sum_{D_1} f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \phi \cdot \varphi \right\| \\
&\qquad + \left\| \sum_{D_1} f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \phi \cdot \varphi - \sum_{D_2} f(\boldsymbol{\eta}) \int_{\boldsymbol{J}} \phi \cdot \psi \right\| \\
&\qquad\qquad + \left\| \sum_{D_2} f(\boldsymbol{\eta}) \int_{\boldsymbol{J}} \phi \cdot \psi - \sum_{D_2} f(\boldsymbol{\eta})\phi(\boldsymbol{\eta}) \int_{\boldsymbol{J}} \psi \right\| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}
$$

And the result follows. □

We now show a version of the Saks-Henstock Lemma for the PU integral.

Theorem 2. (Saks-Henstock Lemma) *If $f : [\boldsymbol{a}, \boldsymbol{b}] \to X$ is PU integrable over $[\boldsymbol{a}, \boldsymbol{b}]$, then for every $\epsilon > 0$, there exists a gauge δ on $[\boldsymbol{a}, \boldsymbol{b}]$ such that for any δ-fine partial division $P = \{(\boldsymbol{\xi}, \varphi, \boldsymbol{I})\}$ of $[\boldsymbol{a}, \boldsymbol{b}]$, we have*

$$
\left\| \sum_{P} \left(f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \varphi - (\mathcal{P}) \int_{\boldsymbol{I}} f\varphi \right) \right\| < \epsilon.
$$

Proof: Fix $\epsilon > 0$. Then choose a gauge δ on $[\boldsymbol{a}, \boldsymbol{b}]$ such that whenever $D = \{(\boldsymbol{\xi}, \sigma, \boldsymbol{I})\}$ is a δ-fine division of $[\boldsymbol{a}, \boldsymbol{b}]$, we have

$$
\left\| \sum_{D} f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \sigma - (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f \right\| < \epsilon.
$$

Let $P = \{(\boldsymbol{\xi}, \varphi, \boldsymbol{I})\}$ be a δ-fine partial division of $[\boldsymbol{a}, \boldsymbol{b}]$ and put $\phi = \sum_{P} \varphi$. We choose a $\delta_1(\boldsymbol{x}) \leq \delta(\boldsymbol{x})$ such that for any $\boldsymbol{y} \in [\boldsymbol{a}, \boldsymbol{b}] \cap B(\boldsymbol{x}, \delta_1(\boldsymbol{x}))$, we have

$$
\|f(\boldsymbol{x})\| |\phi(\boldsymbol{y}) - \phi(\boldsymbol{x})| < \frac{\epsilon}{3}.
$$

So, for each $\boldsymbol{x} \in \operatorname{supp} f$

$$
\|f(\boldsymbol{x})\| \cdot \sup \left\{ |\phi(\boldsymbol{x}) - \phi(\boldsymbol{y})| : \boldsymbol{y} \in B(\boldsymbol{x}, \delta_1(\boldsymbol{x})) \right\} < \frac{\epsilon}{3}. \tag{3.19}
$$

Now, by Lemma 3, $f \cdot (1 - \phi)$ is PU integrable over $[\boldsymbol{a}, \boldsymbol{b}]$. Thus, there is a gauge $\delta_2 \leq \delta_1$ on $[\boldsymbol{a}, \boldsymbol{b}]$ such that for every δ_2-fine division $D = \{(\boldsymbol{\xi}', \psi, \boldsymbol{J})\}$ of $[\boldsymbol{a}, \boldsymbol{b}]$, we have

$$
\left\| \sum_{D} f(\boldsymbol{\xi}') \cdot (1 - \phi(\boldsymbol{\xi}')) \int_{\boldsymbol{J}} \psi - (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f \cdot (1 - \phi) \right\|
$$

$$= \left\| \sum_D f(\boldsymbol{\xi}') \cdot (1-\phi(\boldsymbol{\xi}')) \int_{\boldsymbol{J}} \psi - (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f + (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f \cdot \left(\sum_P \varphi \right) \right\|$$
$$= \left\| \sum_D f(\boldsymbol{\xi}') \cdot (1-\phi(\boldsymbol{\xi}')) \int_{\boldsymbol{J}} \psi \; - (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f + \sum_P \int_{\boldsymbol{I}} f \cdot \varphi \right\| < \frac{\epsilon}{3},$$

that is,

$$\left\| \sum_D f(\boldsymbol{\xi}') \cdot (1-\phi(\boldsymbol{\xi}')) \int_{\boldsymbol{J}} \psi \; + \sum_P \int_{\boldsymbol{I}} f \cdot \varphi - (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f \right\| < \frac{\epsilon}{3}. \tag{3.20}$$

Now, since $P \cup \{(\boldsymbol{\xi}, \boldsymbol{J}, (1-\phi)\psi) : (\boldsymbol{\xi}, \boldsymbol{J}, \psi) \in D\}$ is also a δ-fine division of $[\boldsymbol{a}, \boldsymbol{b}]$ and by integrability of f, we have

$$\left\| \sum_P f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \varphi + \sum_D f(\boldsymbol{\xi}') \int_{\boldsymbol{J}} (1-\phi)\psi - (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f \right\| < \frac{\epsilon}{3}. \tag{3.21}$$

Observe that by (3.19),

$$\begin{aligned}
&\left\| \sum_D f(\boldsymbol{\xi}')(1-\phi(\boldsymbol{\xi}')) \int_{\boldsymbol{J}} \psi - \sum_D f(\boldsymbol{\xi}') \int_{\boldsymbol{J}} (1-\phi)\psi \right\| \\
&\quad = \left\| \sum_D \left\{ f(\boldsymbol{\xi}') \cdot \int_{\boldsymbol{J}} [\phi - \phi(\boldsymbol{\xi}')]\psi \right\} \right\| \\
&\quad \leq \left\| \sum_D \left\{ f(\boldsymbol{\xi}') \cdot \int_{\boldsymbol{J}} \sup \left\{ |\phi(\boldsymbol{x}) - \phi(\boldsymbol{\xi}')| : \boldsymbol{x} \in B(\boldsymbol{\xi}', \delta_1(\boldsymbol{\xi}')) \right\} \cdot \psi \right\} \right\| \\
&\quad = \left\| \sum_D \left\{ f(\boldsymbol{\xi}') \cdot \sup \left\{ |\phi(\boldsymbol{x}) - \phi(\boldsymbol{\xi}')| : \boldsymbol{x} \in B(\boldsymbol{\xi}', \delta_1(\boldsymbol{\xi}')) \right\} \cdot \int_{\boldsymbol{J}} \psi \right\} \right\| \\
&\quad \leq \sum_D \left\{ \left\| f(\boldsymbol{\xi}') \right\| \cdot \sup \left\{ |\phi(\boldsymbol{x}) - \phi(\boldsymbol{\xi}')| : \boldsymbol{x} \in B(\boldsymbol{\xi}', \delta_1(\boldsymbol{\xi}')) \right\} \cdot \int_{\boldsymbol{J}} \psi \right\} \\
&\quad < \frac{\epsilon}{3};
\end{aligned}$$

that is,

$$\left\| \sum_D f(\boldsymbol{\xi}')(1-\phi(\boldsymbol{\xi}')) \int_{\boldsymbol{J}} \psi - \sum_D f(\boldsymbol{\xi}') \int_{\boldsymbol{J}} (1-\phi)\psi \right\| < \frac{\epsilon}{3}. \tag{3.22}$$

Therefore, by (3.20), (3.21), and (3.22)

$$\begin{aligned}
&\left\| \sum_P \left(f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \varphi - \int_{\boldsymbol{I}} f\varphi \right) \right\| \\
&\quad = \left\| \sum_P f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \varphi \; - (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f + (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f \; - \sum_D f(\boldsymbol{\xi}')(1-\phi(\boldsymbol{\xi}')) \int_{\boldsymbol{J}} \psi \right. \\
&\qquad \left. + \sum_D f(\boldsymbol{\xi}')(1-\phi(\boldsymbol{\xi}')) \int_{\boldsymbol{J}} \psi \; - \sum_D f(\boldsymbol{\xi}') \int_{\boldsymbol{J}} (1-\phi)\psi \right.
\end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_D f(\boldsymbol{\xi}') \int_{\boldsymbol{J}} (1-\phi)\psi \ - \sum_P \int_{\boldsymbol{I}} f\varphi \Bigg\| \\
&\le \Bigg\| \sum_P f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \varphi + \sum_D f(\boldsymbol{\xi}') \int_{\boldsymbol{J}} (1-\phi)\psi - (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f \ \Bigg\| \\
&\quad + \Bigg\| (\mathcal{P}) \int_{[\boldsymbol{a},\boldsymbol{b}]} f - \sum_P \int_{\boldsymbol{I}} f\varphi - \sum_D f(\boldsymbol{\xi}')(1-\phi(\boldsymbol{\xi}')) \int_{\boldsymbol{I}} \psi \Bigg\| \\
&\quad + \Bigg\| \sum_D f(\boldsymbol{\xi}')(1-\phi(\boldsymbol{\xi}')) \int_{\boldsymbol{J}} \psi - \sum_D f(\boldsymbol{\xi}') \int_{\boldsymbol{J}} (1-\phi)\psi \Bigg\| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}
$$

And the result follows. □

Corollary 1. *In view of the conditions of Theorem 2, if in addition that X is a finite dimensional Banach space, then for every $\epsilon > 0$, there exists a gauge δ on $[\boldsymbol{a},\boldsymbol{b}]$ such that for any δ-fine partial division $P = \{(\boldsymbol{\xi}, \varphi, \boldsymbol{I})\}$ of $[\boldsymbol{a},\boldsymbol{b}]$, we have*

$$
\sum_P \left\| f(\boldsymbol{\xi}) \int_{\boldsymbol{I}} \varphi - (\mathcal{P}) \int_{\boldsymbol{I}} f\varphi \right\| < \epsilon.
$$

Proof: Note that each norms defined on finite dimensional normed spaces are equivalent. The proof is complete by considering each components of X, endowed with, perhaps, the maximum norm. □

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