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More on Ideal Topological Groups

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Abstract. In this article, we define and study the concept of ideal topological groups. We study its relation to topological groups. We present examples that show that ideal topological groups and topological groups are independent concepts. We give a sufficient condition for a topological group to be an ideal topological group as well as we give a sufficient condition for an ideal topological group to be a topological group. Unlike topological groups, ideal topological groups are not nicely behaved with regard to subgroups. We give an example of a subgroup of an ideal topological group that is not an ideal topological group. We show that every open subgroup of an ideal topological group is also an ideal topological group. Moreover, we investigate $\mathcal{I}$-connectedness of ideal topological groups.

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1. Introduction

The notion of ideal topological spaces was studied in the classic book [11] and also in [13]. In 1990, Jankovic and Hamlett [9] introduced $\mathcal{I}$-open sets in topological spaces and later obtained several properties of ideal topological spaces in [10]. Abd El-Monsef et. al.[1] investigated further properties of $\mathcal{I}$-open sets and introduced $\mathcal{I}$-closed sets, $\mathcal{I}$-continuous mappings and $\mathcal{I}$-open (closed) mappings and studied the relations between them. In 1943, Hewitt [7] introduced the concept of submaximal spaces. Arhangel'skii and Collins [3] studied submaximal spaces and gave characterizations of it. Dontchev [5] defined the $\mathcal{I}$-irresolute mapping and investigated the relationship between $\mathcal{I}$-open classes and preopen classes. In 2020, Jafari and Rajesh [8] initiated the study of ideal topological groups. In this paper, we define the notion of ideal topological groups and present their main properties. Also, we study the relation between topological groups and ideal topological groups. Furthermore, we investigate I-connectedness of ideal topological groups, see [2].

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2. Preliminaries

In this section, we recall some definitions and results which we shall use frequently in the following sections. Note that the fundamental reference for topological spaces is [6]; see also [12] and the main reference for topological groups and their properties is [4].

Definition 1. *[13] A collection $\mathcal{I}$ of subsets of a set X is called an ideal on X if it satisfies the following two conditions:*

(i) If $A \in \mathcal{I}$ and $B \subset A$, then $B \in \mathcal{I}$.

(ii) If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.

Recall that if (X, τ) is a topological space and $\mathcal{I}$ is an ideal on X, then $(X, \tau, \mathcal{I})$, or simply X, is called an ideal topological space.

Definition 2. *[11] Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then $A^\star = \{x \in X : U \cap A \notin \mathcal{I}$ for every open neighborhood U of $x\}$.*

Definition 3. *[10] Let $(X, \tau, \mathcal{I})$ be an ideal topological space. We say that a subset A of X is an $\mathcal{I}$-open if $A \subseteq \operatorname{int}(A^*)$.*

Let $(X, \tau, \mathcal{I})$ be an ideal topological space. A subset A is called $\mathcal{I}$-closed if the set $X - A$ is $\mathcal{I}$-open.

Remark 1. *[1] Arbitrary union of $\mathcal{I}$-open sets is an $\mathcal{I}$-open set. In contrast, the intersection of two $\mathcal{I}$-open sets may not be an $\mathcal{I}$-open set. However, the intersection of an open set with an $\mathcal{I}$-open set is an $\mathcal{I}$-open set.*

Definition 4. *Let X be an ideal topological space and $\mathcal{B}$ be a collection of $\mathcal{I}$-open subsets of X. Then $\mathcal{B}$ is called an $\mathcal{I}$-open base for X if every nonempty $\mathcal{I}$-open set is a union of members of $\mathcal{B}$.*

Definition 5. *Let X be an ideal topological space and $x \in X$. The collection $\mathcal{B}_x$ of $\mathcal{I}$-open neighborhoods of x in X is called an $\mathcal{I}$-open base at x if for any $\mathcal{I}$-open neighborhood U of x, there is $V \in \mathcal{B}_x$ such that $V \subset U$.*

Definition 6. *[1] A mapping $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is called $\mathcal{I}$-continuous if the inverse image of any open set in Y is $\mathcal{I}$-open set in X.*

Definition 7. *[1] A mapping $f : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma)$ is said to be $\mathcal{I}$-continuous at a point x in X if for each open neighborhood V of $f(x)$ in Y, there is an $\mathcal{I}$-open neighborhood U of x in X such that $f(U) \subset V$.*

Theorem 1. *[1] Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a mapping. Then the following are equivalent:*

(i) f is $\mathcal{I}$-continuous.

(ii) f is $\mathcal{I}$-continuous at each point x in X.

Definition 8. *[1] A mapping $f : (X, \tau) \longrightarrow (Y, \sigma, \mathcal{I})$ is called $\mathcal{I}$-open if for each open set U in X, $f(U)$ is $\mathcal{I}$-open in Y.*

Definition 9. *[5] A mapping $f : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma, \mathcal{J})$ is called $\mathcal{I}$-irresolute if the inverse image of each $\mathcal{J}$-open set in Y is $\mathcal{I}$-open set in X.*

Definition 10. *[8] A bijective mapping $f : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma, \mathcal{J})$ is called $\mathcal{I}$-homeomorphism if f and the inverse mapping f^{-1} are $\mathcal{I}$-continuous.*

Definition 11. *[8] An ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}$-homogeneous if for all $x, y \in X$ there is an $\mathcal{I}$-homeomorphism f of the space X onto itself such that $f(x) = y$.*

Definition 12. *An ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}T_1$-space if given any two distinct points $x, y \in X$, there are two $\mathcal{I}$-open sets U and V containing x and y, respectively, such that $y \notin U$ and $x \notin V$.*

We recall that if $(X, \tau, \mathcal{I})$ is an ideal topological space, then open sets and $\mathcal{I}$-open sets in X are independent [1]. However, if X is an $\mathcal{I}T_1$-space, then we have the following lemma which we shall use in next section. We do not have a reference for this lemma and thus we give a proof of it.

Lemma 1. *Let X be an $\mathcal{I}T_1$ ideal topological space. Then every nonempty open set A of X is $\mathcal{I}$-open set.*

Proof. Fix $x \in X$ such that $x \notin A$. Since X is an $\mathcal{I}T_1$-space, then for each $y \in A$ there are $\mathcal{I}$-open sets U and V_y containing x and y, respectively, such that $y \notin U$ and $x \notin V_y$. Let $W_y = V_y \cap A$. Thus, W_y is $\mathcal{I}$-open and $W_y \subset A$. Let $W = \cup_{y \in A} W_y$. Clearly, W is $\mathcal{I}$-open being the union of $\mathcal{I}$-open sets. But $W = A$. Hence, A is $\mathcal{I}$-open.

Definition 13. *[3] A topological space X is submaximal space if it satisfies one of the following equivalent conditions:*

(i) Every subset of it is locally closed, that is, an intersection of an open subset and a closed subset.

(ii) Every dense subset is open.

(iii) Every preopen subset is open.

Remark 2. *Observe that if $(X, \tau, \mathcal{I})$ is an ideal topological space, then every $\mathcal{I}$-open set is preopen in X [1]. And if X is submaximal, then every $\mathcal{I}$-open set in X is open.*

Theorem 2. *[1] Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ and $g : (Y, \sigma, \mathcal{J}) \to (Z, \mu)$ be two mappings. If f is $\mathcal{I}$-continuous and g is continuous, then $g \circ f$ is $\mathcal{I}$-continuous.*

Theorem 3. *[5] Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ and $g : (Y, \sigma, \mathcal{J}) \to (Z, \mu, \mathcal{K})$ be two mappings. If f is $\mathcal{I}$-irresolute and g is $\mathcal{I}$-continuous, then $g \circ f$ is $\mathcal{I}$-continuous.*

Theorem 4. *[5] Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be an $\mathcal{I}$-continuous mapping. If Y is submaximal space, then f is $\mathcal{I}$-irresolute.*

Theorem 5. *[1] Let $\{X_\alpha : \alpha \in \Delta\}$ be a family of spaces, $X = \prod X_\alpha$ be the product space and $A = \prod_{\alpha=1}^{n} A_\alpha \times \prod_{\alpha \neq \beta} X_\beta$ a non empty subset of X, where n is a positive integer and $A_\alpha \subset X_\alpha$. Then A_α is $\mathcal{I}$-open in X_α for each $1 \leq \alpha \leq n$ if and only if A is $\mathcal{I}$-open in X.*

Theorem 6. *[1] Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be an $\mathcal{I}$-continuous and $U \in \tau$. Then the restriction $f|U$ is an $\mathcal{I}$-continuous.*

3. Ideal topological groups

In this section, we define ideal topological groups and give their basic properties. Also, we study the relation between ideal topological groups and topological groups.

Definition 14. *An ideal topological group G is a group that is also an ideal topological space such that the multiplication mapping $m : G \times G \to G$ and the inverse mapping $inv : G \to G$ both are $\mathcal{I}$-continuous.*

We present some examples of ideal topological groups.

Example 1. *$\mathbb{R}$ under addition with its usual topology is a topological group. If we consider the ideal of finite subsets of $\mathbb{R}$, then it can be shown that any open interval is $\mathcal{I}$-open. Therefore, we deduce that $\mathbb{R}$ is ideal topological group.*

Example 2. *Consider $\mathbb{R}$ under addition with its usual topology and with the ideal of countable subsets of $\mathbb{R}$. Then it is not difficult to show that any open interval is $\mathcal{I}$-open. Therefore, we deduce that $\mathbb{R}$ is ideal topological group.*

Example 3. *Let G be any group with the discrete topology. Then it is known that G is a topological group. If we consider the ideal of nowhere dense subsets in G, then the class of $\mathcal{I}$-open sets is the power set of G. Therefore, G is ideal topological group. In particular, $\mathbb{R}$ under addition with the discrete topology and with the ideal of nowhere dense subsets is ideal topological group.*

It is natural to ask about the relation between topological groups and ideal topological groups. We find that the two concepts are independent, as the following examples show.

Example 4. *Let G be a group with a topology and with the ideal of of all subsets of G. Then the class of $\mathcal{I}$-open sets contains only the empty set $\emptyset$. Therefore, G cannot be an ideal topological group since the multiplication mapping and the inverse mapping are not $\mathcal{I}$-continuous. In particular, $\mathbb{R}$ under addition with its usual topology is a topological group. If we consider the ideal of all subsets of $\mathbb{R}$, then $\emptyset$ is the only $\mathcal{I}$-open set. Thus, $\mathbb{R}$ is not an ideal topological group since the multiplication mapping and the inverse mapping are not $\mathcal{I}$-continuous.*

Example 5. *Consider* $\mathbb{Z}_3 = \{0, 1, 2\}$*, the group of integers mod 3, with a topology* $\tau = \{\mathbb{Z}_3, \emptyset, \{1, 2\}\}$ *on* $\mathbb{Z}_3$*. Let* $\mathcal{I} = \{\emptyset, 0\}$ *be an ideal on* $\mathbb{Z}_3$*. Then it is clear that the class of* $\mathcal{I}$*-open sets is the power set of* $\mathbb{Z}_3$ *except* $\{0\}$*. It is not difficult to check that the multiplication mapping and the inverse mapping are* $\mathcal{I}$*-continuous. Therefore,* $\mathbb{Z}_3$ *is an ideal topological group. However,* $\mathbb{Z}_3$ *is not a topological group since the multiplication mapping is not continuous at the element* $(0, 1)$*.*

Using Lemma 1, a sufficient condition for a topological group to be an ideal topological group is presented in the following result.

Theorem 7. *Let* G *be an ideal topological space. If* G *is an* $\mathcal{I}T_1$ *topological group, then* G *is an ideal topological group.*

Proof. Suppose that G is an $\mathcal{I}\mathrm{T}_1$ topological group. We shall show that the multiplication mapping $m : G \times G \to G$ and the inverse mapping $inv : G \to G$ both are $\mathcal{I}$-continuous. First we show that m is $\mathcal{I}$-continuous. Let W be an open set in G. Then there exist open sets U and V in G such that $UV \subset W$ since G is a topological group. Using Lemma 1, U and V are $\mathcal{I}$-open in G. Therefore, m is $\mathcal{I}$-continuous. Similarly, we can show that inv is $\mathcal{I}$-continuous. Hence, G is an ideal topological group.

Note that if G is a submaximal space, then every an $\mathcal{I}$-open set of G is open. Therefore, we have the following straightforward result.

Theorem 8. *Let* G *be an ideal topological group. If* G *is submaximal, then* G *is a topological group.*

Proof. Suppose that G is submaximal. We shall show that the multiplication mapping $m : G \times G \to G$ and the inverse mapping $inv : G \to G$ both are continuous. First we show that m is continuous. Let W be an open set in G. Then there exist $\mathcal{I}$-open sets U and V in G such that $UV \subset W$ since G is an ideal topological group. Since G is submaximal, U and V are open in G. Therefore, m is continuous. Similarly, we can show that inv is continuous. Hence, G is a topological group.

Theorem 9. *Let* G *be an ideal topological group and* $g \in G$*. Then each left (right) translation map* $l_g : G \to G$ *(*$r_g : G \to G$*) is an* $\mathcal{I}$*-homeomorphism.*

Proof. We prove that left translation map l_g is an $\mathcal{I}$-homeomorphism. Obviously, l_g is a bijective mapping. Let x be an element in G; let W be an open neighborhood of $l_g(x) = gx$. Since G is an ideal topological group, there are $\mathcal{I}$-open sets U and V containing g and x, respectively, such that $UV \subset W$. This shows that l_g is $\mathcal{I}$-continuous. On the other hand, the inverse mapping of l_g is defined as $(l_g(x))^{-1} = g^{-1}x = l_{g^{-1}}(x)$. This indicates that $(l_g(x))^{-1}$ is $\mathcal{I}$-continuous. We get, l_g is an $\mathcal{I}$-homeomorphism. Similarly, we can show that right translation map r_g is an $\mathcal{I}$-homeomorphism.

Theorem 9 above implies that l_g and r_g are $\mathcal{I}$-open mappings for each $g \in G$. Hence, we have the following immediate result.

Corollary 1. *In any ideal topological group, every open set is $\mathcal{I}$-open.*

Proposition 1. *Let G be an ideal topological group. Let A and B be subsets in G and $g \in G$. Then*

(i) If A is an open set, then Ag and gA both are $\mathcal{I}$-open sets.

(ii) If A is a closed set, then Ag and gA both are $\mathcal{I}$-closed sets.

(iii) If A is an open set, then AB and BA both are $\mathcal{I}$-open sets.

Proof. (i) and (ii) follow using Theorem 9. Since $AB = \cup_{b \in B} Ab$ and the union of $\mathcal{I}$-open sets is $\mathcal{I}$-open, AB is $\mathcal{I}$-open and similarly, BA is $\mathcal{I}$-open. Hence, (iii) is proved.

Theorem 10. *Let G be an ideal topological group. Then the inverse mapping $inv : G \to G$ defined by $inv(x) = x^{-1}$ is an $\mathcal{I}$-homeomorphism.*

Proof. It is clear that *inv* mapping is bijective. Since G is an ideal topological group, *inv* is $\mathcal{I}$-continuous. Since $(inv)^{-1}(x) = x^{-1}$, we have that $(inv)^{-1}$ is $\mathcal{I}$-continuous. Hence, *inv* is an $\mathcal{I}$-homeomorphism.

Corollary 2. *Let G be an ideal topological group. If A is an open subset of G, then A^{-1} is an $\mathcal{I}$-open set.*

Corollary 3. *Let G be an ideal topological group. Let A be an open subset of G. Then there exists a symmetric $\mathcal{I}$-open set U of G such that $U \subset A$.*

Proof. By Corollary 2, A^{-1} is $\mathcal{I}$-open. Let $U = A \cap A^{-1}$. Then U is $\mathcal{I}$-open being the intersection of open set with $\mathcal{I}$-open set. U is symmetric and $U \subset A$.

Theorem 11. *Let G be an ideal topological group, and let A be an $\mathcal{I}$-open set in G. If G is submaximal, then*

(i) Ag and gA both are $\mathcal{I}$-open set for any $g \in G$.

(ii) AB and BA both are $\mathcal{I}$-open set for any $B \subset G$.

(iii) A is $\mathcal{I}$-open if and only if A^{-1} is $\mathcal{I}$-open.

Proof. Since G is submaximal, A is open. Using Proposition 1, (i) and (ii) follow. (iii) follows immediately using Corollary 2.

Theorem 12. *Let G be an ideal topological group. Suppose G is a submaximal space such that $G \times G$ is submaximal. Then the mapping $f : G \times G \to G$ sending (x, y) to xy^{-1} is $\mathcal{I}$-continuous.*

Proof. Consider the mapping $h : G \times G \to G \times G$ such that $h(x, y) = (x, inv(y))$. First we show that h is $\mathcal{I}$-irresolute. Take an $\mathcal{I}$-open set $U \times V$. From the submaximality of $G \times G$, $U \times V$ is open. $h^{-1}(U, V) = (U, inv^{-1}(V))$. Hence, $(U, inv^{-1}(V))$ is $\mathcal{I}$-open since inv is $\mathcal{I}$-continuous. Note that $f(x, y) = m(h(x, y)) = m((x, inv(y)) = xinv(y) = xy^{-1}$, that is, $f = m \circ h$. Since the multiplication mapping m is $\mathcal{I}$-continuous and h is $\mathcal{I}$-irresolute, Theorem 3 implies that f is $\mathcal{I}$-continuous.

Theorem 13. *Let G be an ideal topological group. Let β_e be an open base at the identity element e of G. Then*

(i) For every $U \in \beta_e$, there is V an $\mathcal{I}$-open neighborhood of e such that $V^2 \subset U$.

(ii) For every $U \in \beta_e$, there is V an $\mathcal{I}$-open neighborhood of e such that $V^{-1} \subset U$.

(iii) For every $U \in \beta_e$ and $g \in G$, there is V an $\mathcal{I}$-open neighborhood of e such that $gV \subset U$.

(iv) If G is $\mathcal{I}T_1$-space, then for every $U, V \in \beta_e$, there is W an $\mathcal{I}$-open neighborhood of e such that $W \subset U \cap V$.

Proof. (i) follows from the $\mathcal{I}$-continuity of the multiplication mapping at the identity element e. (ii) follows from the $\mathcal{I}$-continuity of the inverse mapping at the identity element e. (iii) follows from the $\mathcal{I}$-continuity of the left translation mapping in G. For (iv), $U \cap V$ is a neighborhood of e. Therefore, there is $W \in \beta_e$ such that $W \subset U \cap V$. W is an $\mathcal{I}$-open set since G is an $\mathcal{I}T_1$-space.

Theorem 14. *Let G be an ideal topological group. Let β_e be an open base at the identity element e of G. If G is submaximal, then*

(i) For every $U \in \beta_e$, there is V an $\mathcal{I}$-open neighborhood of e such that $V^2 \subset U$.

(ii) For every $U \in \beta_e$, there is V an $\mathcal{I}$-open neighborhood of e such that $V^{-1} \subset U$.

(iii) For every $U \in \beta_e$ and $g \in G$, there is V an $\mathcal{I}$-open neighborhood of e such that $gV \subset U$.

(iv) Assume that $G \times G$ is submaximal. For every $U \in \beta_e$, there is V an $\mathcal{I}$-open neighborhood of e such that $VV^{-1} \subset U$.

(v) For every $U \in \beta_e$ and $g \in G$, there is V an $\mathcal{I}$-open neighborhood of e such that $gVg^{-1} \subset U$.

Proof. (i),(ii) and (iii) are proved in Theorem 13. We show (iv). Using Theorem 12, the mapping $f : G \times G \longrightarrow G$ defined by $f(x, y) = xy^{-1}$ is $\mathcal{I}$-continuous. Therefore, for $U \in \beta_e$, there is $\mathcal{I}$-open neighborhood $A \times B$ of (e, e) in $G \times G$ such that $f(A \times B) \subset U$. But $A \cap B$ is an open neighborhood of e. Thus, there is $V \in \beta_e$ such that $V \subset A \cap B$. Note that V is $\mathcal{I}$-open by Corollary 1. Moreover, $V \times V \subset A \times B$. Then $f(V \times V) \subset f(A \times B) \subset U$.

That is, $VV^{-1} \subset U$. We show (v). Consider the mapping $r_{g^{-1}} \circ l_g : G \longrightarrow G$ defined by $r_{g^{-1}} \circ l_g(a) = gag^{-1}$. Since l_x is $\mathcal{I}$-continuous and $r_{x^{-1}}$ is continuous by the submaximality of G, we have that $r_{x^{-1}} \circ l_x$ is $\mathcal{I}$-continuous by Theorem 2. Therefore, if $U \in \beta_e$, then there is $\mathcal{I}$-open set V containing e such that $r_{g^{-1}} \circ l_g(V) \subset U$. Hence, $gVg^{-1} \subset U$.

Theorem 15. *Every ideal topological group G is an $\mathcal{I}$-homogeneous space.*

Proof. We know that the right translation mapping $r_g : G \longrightarrow G$ given by $r_g(x) = xg$ is $\mathcal{I}$-homeomorphism. Take x and y in G. Let $z = x^{-1}y$. Therefore, $r_z : G \longrightarrow G$ defined by $r_z(x) = xz = x\left(x^{-1}y\right) = y$ is $\mathcal{I}$-homeomorphism. Hence, G is an $\mathcal{I}$-homogeneous space.

Theorem 16. *Let G be an ideal topological group. Suppose that G is submaximal and β_e is an $\mathcal{I}$-open base at the identity element e of G. Then the family $\beta_g = \{Ug : U \in \beta_e\}$ forms an $\mathcal{I}$-open base at the element g of G.*

Proof. Consider the right translation mapping $r_g : G \longrightarrow G$. Let V be an $\mathcal{I}$-open neighborhood of g. By submaximality of G, V is an open neighborhood of g. Since the right translation mapping is $\mathcal{I}$-homeomorphism, there is an $\mathcal{I}$-open neighborhood W of e such that $r_g(W) \subset V$. Since β_e is an $\mathcal{I}$-open base at e, there is $U \in \beta_e$ such that $U \subset W$. But $Ug \subset Wg = r_g(W) \subset V$. Therefore, β_g is an $\mathcal{I}$-open base at the element g.

Theorem 17. *Let G be an ideal topological group. Suppose that G is submaximal and β_e is an $\mathcal{I}$-open base at the identity element e of G. Then the family $\beta_\star = \left\{U^{-1}; U \in \beta_e\right\}$ forms an $\mathcal{I}$-open base at e.*

Proof. Consider the inverse mapping $inv : G \longrightarrow G$. Let V be an $\mathcal{I}$-open neighborhood of e. By submaximality of G, V is an open neighborhood of e. Since inv is an $\mathcal{I}$-homeomorphism, there is an $\mathcal{I}$-open neighborhood W of e such that $inv(W) \subset V$. Since β_e is an $\mathcal{I}$-open base at e, there is $U \in \beta_e$ such that $U \subset W$. But $U^{-1} \subset W^{-1} = inv(W) \subset V$. Therefore, $\beta_\star$ is an $\mathcal{I}$-open base at e.

Theorem 18. *Let G and H be ideal topological groups. Suppose that $f\colon G \to H$ is a homomorphism such that for every $\mathcal{I}$-open set V containing the identity e_H in H, there is an open set U containing the identity element e_G in G with $f(U) \subset V$. Then f is $\mathcal{I}$-continuous.*

Proof. Given $x \in G$. We show that f is $\mathcal{I}$-continuous at x. Suppose that O is an open neighborhood of $f(x) = y$ in H. Since the left translation mapping l_y is $\mathcal{I}$-homeomorphism in H, there is $\mathcal{I}$-open set V containing the identity element e_H of H such that $yV \subset O$. By assumption, there is an open set U containing the identity element e_G in G such that $f(U) \subset V$. Therefore, $f(xU) = f(x)f(U) = yf(U) \subset yV \subset O$. Note that xU is an $\mathcal{I}$-open set containing x by Proposition 1. Hence, f is $\mathcal{I}$-continuous at x.

Theorem 19. *Let G and H be ideal topological groups. Suppose that G and H both are submaximal and $f\colon G \to H$ is a homomorphism. If f is $\mathcal{I}$-continuous at the identity element e_G of G, then f is $\mathcal{I}$-continuous.*

Proof. Given $x \in G$. We show that f is $\mathcal{I}$-continuous at x. Suppose that O is an open neighborhood of $f(x) = y$ in H. Since the left translation map l_y is an $\mathcal{I}$-homeomorphism of H, there is an $\mathcal{I}$-open neighborhood V of the identity element e_H of H such that $yV \subset O$. Since H is submaximal, V is an open neighborhood of e_H. But f is $\mathcal{I}$-continuous at the identity element e_G of G. Therefore, there is an $\mathcal{I}$-open neighborhood U of e_G such that $f(U) \subset V$. Note that the set xU is an $\mathcal{I}$-open neighborhood of x since G is submaximal. Thus, $f(xU) = f(x)f(U) = yf(U) \subset yV \subset O$. Hence, f is $\mathcal{I}$-continuous at x.

Next, we will investigate subgroups in ideal topological groups.

Theorem 20. *Let G be an ideal topological group and H be a subgroup of G. If H contains a nonempty open set, then H is $\mathcal{I}$-open in G.*

Proof. Suppose that U is a nonempty open subset of G such that $U \subset H$. For any $h \in H$, the set $l_h(U) = hU$ is an $\mathcal{I}$-open set in G since the left translation mapping is an $\mathcal{I}$-homeomorphism. Therefore, the set $H = \bigcup_{h\in H}(hU)$ is $\mathcal{I}$-open in G.

Corollary 4. *Let G be an ideal topological group and H be a subgroup of G. If G is submaximal and H contains a nonempty $\mathcal{I}$-open set, then H is $\mathcal{I}$-open in G.*

Unlike topological groups, ideal topological groups are not well behaved with respect to subgroups. The following example demonstrates that a subgroup of an ideal topological group is not necessarily an ideal topological group.

Example 6. *In Example 1, $\mathbb{R}$ is an ideal topological group. We know that $\mathbb{Z}$ is a subgroup of $\mathbb{R}$. Moreover, $\mathbb{Z}$ is closed in $\mathbb{R}$ and $\mathbb{Z}$ has the discrete topology. Therefore, the class of $\mathcal{I}$-open sets in $\mathbb{Z}$ contains only $\emptyset$. This implies that $\mathbb{Z}$ is not ideal topological group since neither the multiplication mapping nor the inverse mapping is $\mathcal{I}$-continuous.*

Theorem 21. *Every open subgroup H of an ideal topological group G is also an ideal topological group.*

Proof. We shall show that the multiplication mapping $m_H : H\times H \to H$ and the inverse mapping $inv_H : H \to H$ both are $\mathcal{I}$-continuous. First we show that m_H is $\mathcal{I}$-continuous. Let W be an open set in H. Then W is open in G. Since G is an ideal topological group, the multiplication mapping $m_G : G \times G \to G$ is $\mathcal{I}$-continuous. But H is open in G and $H \times H$ is open in $G \times G$. Using Theorem 6, the restriction $m_G|_{H\times H} : H \times H \to G$ is $\mathcal{I}$-continuous. Thus, there exists an $\mathcal{I}$-open set $U \times V$ in $H \times H$ and $UV \subset W$. Note that U and V both are $\mathcal{I}$-open in H by Theorem 5. Therefore, m_H is $\mathcal{I}$-continuous. Similarly, we can show that the inverse mapping inv_H is $\mathcal{I}$-continuous. Hence, H is ideal topological group.

Theorem 22. *Every open subgroup H of an ideal topological group G is $\mathcal{I}$-closed in G.*

Proof. Since H is open, Hg is $\mathcal{I}$-open for each $g \in G$ by Proposition 1. Consider the family $\alpha = \{Hg : g \in G\}$ of all right cosets of H in G. Note that α is a disjoint $\mathcal{I}$-open covering of G. Therefore, each element of α is $\mathcal{I}$-closed in G. In particular, $H = He$ is $\mathcal{I}$-closed in G.

Corollary 5. *Let G be an ideal topological group and H be a subgroup of G. If H is $\mathcal{I}$-open and G is submaximal, then H is $\mathcal{I}$-closed in G.*

4. $\mathcal{I}$-connectedness in ideal topological groups

In this section, first we review the definition of $\mathcal{I}$-connectedness in ideal topological spaces. We give examples of $\mathcal{I}$-connected and $\mathcal{I}$-disconnected ideal topological spaces and ideal topological groups. Furthermore, some properties of $\mathcal{I}$-connectedness of ideal topological groups are studied.

Definition 15. *Let X be an ideal topological space. An $\mathcal{I}$-separation of X is a pair A, B of disjoint nonempty $\mathcal{I}$-open subsets of X whose union is X.*

Definition 16. *An ideal topological space X is said to be an $\mathcal{I}$-connected if there does not exist an $\mathcal{I}$-separation of X.*

Example 7. *Any group G with the discrete topology and with the ideal of nowhere dense subsets in G is an ideal topological space in which the class of $\mathcal{I}$-open sets is the power set of G. Thus, G is an $\mathcal{I}$-disconnected space. In particular, $\mathbb{R}$ with the discrete topology and with the ideal of nowhere dense subsets in $\mathbb{R}$ is an $\mathcal{I}$-disconnected ideal topological space.*

Example 8. *Any group G with a topology and with the ideal of all subsets in G is an ideal topological space in which the class of $\mathcal{I}$-open sets contains only the empty set $\emptyset$. Thus, G is an $\mathcal{I}$-connected space. In particular, $\mathbb{R}$ with its usual topology and with the ideal of all subsets in $\mathbb{R}$ is an $\mathcal{I}$-connected ideal topological space.*

The following is an example of $\mathcal{I}$-connected ideal topological group.

Example 9. *Consider $\mathbb{Z}_2 = \{0, 1\}$, the group of integers mod 2, with the trivial topology $\tau : \{\emptyset, \mathbb{Z}_2\}$. Consider the ideal $\mathcal{I} = \{\emptyset, \{1\}\}$. Then it is not difficult to show that the $\mathcal{I}$-open sets are $\emptyset$, $\mathbb{Z}_2$, and $\{0\}$ and $\mathbb{Z}_2$ is an ideal topological group. It is obvious that $\mathbb{Z}_2$ is $\mathcal{I}$-connected.*

The following is an example of $\mathcal{I}$-disconnected ideal topological group.

Example 10. *In Example 1, $\mathbb{R}$ is an ideal topological group. It can be shown that the set of rationals and the set of irrationals are $\mathcal{I}$-open sets. Therefore, $\mathbb{R}$ is $\mathcal{I}$-disconnected.*

Definition 17. *Let X be an ideal topological space; let $S \subset X$. For $x \in S$, the set $S_x = \bigcup_{x \in C \subset S} C$, where C is $\mathcal{I}$-connected in S, is called the $\mathcal{I}$-component of S belonging to x.*

Definition 18. *Let G be an ideal topological group with identity element e of G. The $\mathcal{I}$-component of G is the union of all $\mathcal{I}$-connected subsets of G containing e.*

Theorem 23. *Let X and Y be ideal topological spaces. Let $f : X \to Y$ be an $\mathcal{I}$-continuous surjective mapping. Assume that Y is submaximal space. If X is $\mathcal{I}$-connected, then Y is $\mathcal{I}$-connected.*

Proof. Assume Y is $\mathcal{I}$-disconnected. Then, there are two $\mathcal{I}$-open sets U and V of Y such that $U \cap V = \emptyset$, and $Y = U \cup V$. We can deduce that, $f^{-1}(U) \cap f^{-1}(V) = \emptyset$ and $X = f^{-1}(U) \cup f^{-1}(V)$. But $f^{-1}(U)$ and $f^{-1}(V)$ are $\mathcal{I}$-open sets by Theorem 4 which is a contradiction since X is $\mathcal{I}$-connected. Hence, Y is $\mathcal{I}$-connected.

Theorem 24. *Let G be a submaximal spcae and ideal topological group. The $\mathcal{I}$-component H is an invariant subgroup of G.*

Proof. Recall that a subset H of a group G is called invariant subgroup of G, if $aHa^{-1} = H$ for all $a \in G$. Since the left translation $l_a : G \longrightarrow G$ is an $\mathcal{I}$-homeomorphism. So, we have aH is $\mathcal{I}$-connected, and hence, aHa^{-1} is $\mathcal{I}$-connected, from Theorem 23. Since $e \in aHa^{-1}$ and H is the Union of all $\mathcal{I}$-connected subset of G containing e, it follows that, $aHa^{-1} \subset H$. Replacing a by a^{-1} in this inclusion, we obtain that $a^{-1}Ha \subset H$ or, equivalently, $H \subset aHa^{-1}$.Thus, $aHa^{-1} = H$. Hence, H is an invariant subgroup of G.

Theorem 25. *Let G be an ideal topological group. Suppose U is an open set in G. Then the set $L = \bigcup_{n=1}^{\infty} U^n$ is an $\mathcal{I}$-open set.*

Proof. Since U is open set in an ideal topological group, then, by Proposition 1, $UU = U^2$ is $\mathcal{I}$-open set, $U^2U = U^3$ is $\mathcal{I}$-open set and similarly $U^4, U^5, \ldots$ all are $\mathcal{I}$-open sets in G. Therefore, the set $L = \bigcup_{n=1}^{\infty} U^n$ is $\mathcal{I}$-open, since the union of $\mathcal{I}$-open sets is an $\mathcal{I}$-open set.

Theorem 26. *Let G be an ideal topological group. Suppose U is any symmetric open neighborhood of identity element e. Then the set $L = \bigcup_{n=1}^{\infty} U^n$ is an $\mathcal{I}$-open subgroup of G.*

Proof. We need to prove that L is a subgroup of G. Take $x, y \in L$, and if $x = u^l, y = u^t$, hence we get $x \cdot y = u^l \cdot u^t = u^{l+t}$, and $x^{-1} = \left(u^l\right)^{-1} = \left(u^{-1}\right)^l = u^l$. We have that $x \cdot y$ and x^{-1} both in L. Hence L is a subgroup of G. Therefore, we get $L = \bigcup_{n=1}^{\infty} U^n$ is an $\mathcal{I}$-open subgroup of G.

Corollary 6. *Let G be a submaximal space and ideal topological group. Suppose U is any symmetric $\mathcal{I}$-open neighborhood of e. Then the set $L = \bigcup_{n=1}^{\infty} U^n$ is an $\mathcal{I}$-open and $\mathcal{I}$-closed subgroup of G.*

Proof. The set U is open since G is submaximal. By Theorem 26, we have that L is an $\mathcal{I}$-open subgroup of G. Using Corollary 5, L is $\mathcal{I}$-closed.

Lemma 2. *Let G be a submaximal space and ideal topological group. Suppose U is a neighborhood of the identity element e, and S_e is an $\mathcal{I}$-component of S belonging to identity element e. Then $S_e \subset \bigcup_{n=1}^{\infty} U^n$. In particular, if G is $\mathcal{I}$-connected, then we have $G = \bigcup_{n=1}^{\infty} U^n$.*

Proof. Let U be a neighborhood of the identity element e of G. From corollary 3, there is a symmetric $\mathcal{I}$-open neighborhood V of the identity element e such that $V \subset U$. Clearly, we have $H = \bigcup_{n=1}^{\infty} V^n$ is an $\mathcal{I}$-open and $\mathcal{I}$-closed subgroup of G, from corollary 6. Since S_e is $\mathcal{I}$-connected of S belonging to identity element e, we have $S_e \subset \bigcup_{n=1}^{\infty} V^n \subset \bigcup_{n=1}^{\infty} U^n$. To conclude, if G is $\mathcal{I}$-connected, we have that $G = \bigcup_{n=1}^{\infty} U^n$.

Theorem 27. *Let G be an ideal topological group, and H be an open subgroup of G. Then*

(i) H is an ideal topological group.

(ii) H is an $\mathcal{I}$-open and $\mathcal{I}$-closed in G.

(iii) G is not $\mathcal{I}$-connected.

Proof. (i) and (ii) are proved. Since H and $G - H$ are disjoint $\mathcal{I}$-open sets, (iii) holds.

Theorem 28. *Let G be an ideal topological group. Suppose that G is an $\mathcal{I}$-connected space. Then there is no open proper subgroup of the group G.*

Proof. Suppose that G is an $\mathcal{I}$-connected space and there is an open proper subgroup H of the group G. We prove that $H = \mathcal{I}\mathrm{Cl}(H)$. It is known that $H \subset \mathcal{I}\mathrm{Cl}(H)$ for arbitrary subset H of G. Take $a \in \mathcal{I}\mathrm{Cl}(H)$. The set aH is an $\mathcal{I}$-open set, from $\mathcal{I}$-homeomorphism of the left translation map. Then, we have $aH \cap H \neq \phi$. Suppose that b $\in aH$ and $b \in H$, then there is $h \in H$ such that $b = ah \in H$. We get $a \in Hh^{-1} \subset HH^{-1} = H$. Therefore, we have $H = \mathcal{I}\mathrm{Cl}(H)$. This indicates that the subgroup H is $\mathcal{I}$-open and $\mathcal{I}$-closed set. By Theorem 27, the ideal topological group G is not $\mathcal{I}$-connected which is a contradiction.

5. Conclusions

In this paper, we defined the notion of ideal topological groups. We studied its main fundamental properties. We presented examples that show that ideal topological groups and topological groups are independent concepts. We gave a sufficient condition for a topological group to be an ideal topological group as well as we gave a sufficient condition for an ideal topological group to be a topological group. In contrast to the case of topological groups, not every subgroup of an ideal topological group is an ideal topological group. We showed that every open subgroup of an ideal topological group is an ideal topological group. Moreover, we investigated $\mathcal{I}$-connectedness of ideal topological groups.

In future studies, the operation of taking quotient of ideal topological groups will be the subject of our study. In addition, we will study separation axioms and ideal topological groups action on ideal topological spaces.

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