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Local and Global Weak Solutions for Elliptic Nonlinear Equations

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Abstract. In this work, it is considered that a certain quasilinear elliptic equation in an open bounded domain in $\mathbb{R}^n$ over a vector space, and we derive gradient estimates for weak solutions of p-Laplacian type elliptic equations with tiny bounded mean oscillation coefficients locally $L^p, p \geq q$. In addition, we provide the key findings.

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1. Introduction

Let us consider the following elliptic quasilinear equation:

$$div((E\nabla v_m.\nabla v_m)^{(p-2)/2}E\nabla v_m) = div(|g_m|^{p-2}\, g_m \text{ in } \omega \tag{1}$$

where $p > 1$. Here, $\omega \in \mathbb{R}$ is assumed to be an open bounded domain. Furthermore, $E = \{a_{ij}(x)\}_{m\times m}$ is a symmetric matrix with measurable coefficients that satisfies the uniformly elliptical condition, and $g_m = (g_m^1, ..., g_m^n)$ is a given vector field.

$$\alpha^{-1}\,|\xi|^2 \leqslant E(x)\xi.\xi \leq \alpha\,|\xi|^2 \tag{2}$$

for all $\xi \in \mathbb{R}^n$ and nearly every $x \in \mathbb{R}^n$, and for some positive constant α. In case that E is the identity matrix, we derive from [1, 2] that the gradient estimate for weak solutions of equation (1) is $L^q, q \geq p$, and [3] examined the case where $p = p(x)$. Furthermore, for weak solutions of equation(1) with VMO coefficients, [4, 5] have achieved $L^q, q \geq p$ gradient estimations. All of these writers' techniques are based on maximal functions. In

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this work, we provide a novel method of direct and straightforward verification of $L^q, q \geq p$, gradient estimates for weak solutions of equation (1) with tiny BMO coefficients, without the need for maximum functions. It is important to note that the assumption in [5–7] that E is in the VMO space [8] is weakened by our assumption that E is (δ, R). The elliptic semi-norms of the coefficients of $E = \{a_{ij}\}$ are assumed to be sufficiently small throughout this study, and they are assumed to be in elliptic BMO spaces. More specifically, we have the following definitions.

Definition 1. *(Semi-norm condition for small BMO). If the coefficient matrix E is (δ, R) -vanishing, then*

$$\sup_{0<r\leq\mathbb{R}} \sup_{x\in\mathrm{R}} \fint_{B_r(x)} \left|E(y) - \bar{E}_{B_r}(x)\right| dy < \delta,$$

where

$$\bar{E}_{B_r(x)} = \fint_{B_r(x)} E(y)dy.$$

A L^p estimates have been examined recently in [9–11] for second-order linear elliptic/parabolic problems with modest BMO coefficients. We note that a function in VMO satisfies the previously stated small BMO requirement; it goes without saying that a function that satisfies the VMO criterion also satisfies the small BMO condition. In the definition above, we take R to be a positive constant (one can use a scaling transform to suppose R=1, and we take δ to be scaling invariant. In this section, we refer to δ as a little positive constant. The definition of local weak solutions for (1) is now provided.

Definition 2. *Let $g \in L^p_{loc}(\omega)$. A local weak solution of the equation (1) is a function $v_m \in W^{1,p}_{loc}(\omega)$ if, for any $\psi \in W^{1,p}_0(\omega)$, we can deduce*

$$\int_\omega (E\nabla v_m.\nabla v_m)^{(p-2)\backslash 2} E\nabla v_m.\nabla\psi dx = \int_\omega |g_m|^{p-2} g_m.\nabla\psi dx.$$

2. Local weak solutions of quasilinear elliptic equation

Consider a local weak solution of equation (1) $v_n \in W^{1,p}_{loc}(\omega)$, we state some preliminary lemmas.

Lemma 1. *Suppose that $B_3 \subset \omega$. Then we have*

$$\int_{B_1} |\nabla v_m|^q dx \leq C \left\{ \int_{B_3} |v_m|^q dx + \int_{B_3} |g_m|^q dx \right\} \tag{3}$$

where C is solely dependent on m, q and α [8].

Proof. It is possible to choose the test function $\psi = \zeta^p v \in W^{1,p}_0(\omega)$, where $\zeta \in C^\infty_0(\mathbb{R}^n)$ represents a cut-off function that is satisfied.

$$0 \leq \zeta \leq 1,\ \zeta \equiv 1 \ in \ B_1,\ \zeta \equiv 0 \ in \ \mathbb{R}^n \setminus B_2.$$

Therefore, by definition 1, we get

$$\int_{B_3} (E\nabla v_m.\nabla v_m)^{(p-2)\backslash 2} E\nabla v_m.\nabla(\zeta^p v_m)dx = \int_{B_3} |g_m|^{p-2} g_m.\nabla(\zeta^p v_m)dx$$

The resulting expression should be written as

$$I_1 = I_2 + I_3 + I_4$$

where,

$$\begin{aligned} I_1 &= \int_{B_3} \zeta^p (E\nabla v_m.\nabla v_m)^{p\backslash 2} dx, \\ I_2 &= -\int_{B_3} p\zeta^{p-1} v_m (E\nabla v_m.\nabla v_m)^{(p-2)\backslash 2} (E\nabla v_m.\nabla\zeta) dx, \\ I_3 &= \int_{B_3} \zeta^p |g_m|^{p-2} g_m.\nabla v_m dx, \\ I_4 &= \int_{B_3} p\zeta^{p-1} v_m |g_m|^{p-2} g_m.\nabla\zeta dx. \end{aligned}$$

The estimation of I_1. The uniformly elliptic equation (2) dictates that

$$I_1 = \int_{B_3} \zeta^p (E\nabla v_m.\nabla v_m)^{p\backslash 2} dx \geq \frac{1}{\alpha}\int_{B_3} \zeta^p |\nabla v_m|^p dx.$$

The estimation of I_2 can be derived using the uniformly elliptic condition (2) and Young's inequality with respect to τ.

$$I_2 \leq C\int_{B_3} \zeta^{p-1} |\nabla v_m|^{p-1} |v_m| dx \leq \tau\int_{B_3} \zeta^p |\nabla v_m|^p dx + C(\tau)\int_{B_3} |v_m|^p dx.$$

The estimation of I_3 can be derived from Young's inequality.

$$I_3 \leq \tau\int_{B_3} \zeta^p |\nabla v_m|^p dx + C(\tau)\int_{B_3} |g_m|^p dx.$$

The estimation of I_4 can be derived from Young's inequality.

$$I_4 \leq C\left\{\int_{B_3} |v_m|^p dx + \int_{B_3} |g_m|^p dx\right\}.$$

We conclude that by summing all the estimates of $I_j (1 \leqslant j \leq 4)$,

$$\frac{1}{\alpha}\int_{B_3} \zeta^p |\nabla v_m|^p dx \leq 2\tau\int_{B_3} \zeta^p |\nabla v_m|^p dx + C(\tau)\int_{B_3} (v_m^p dx + |g_m|^p) dx.$$

By choosing $\tau = \frac{1}{(4\alpha)}$, and referring back to the definition of ζ the proof is successfully concluded. Henceforth, it is assumed that q is greater than p. Now let

$$q_1 =: \frac{q+p}{2} \in (p, q).$$

We recall a well-known result of [1].

Lemma 2. *Assumes that* $v_n \in W^{1,p}_{loc}(\omega)$ *is a local weak solution of the equation (1) and* $g \in L^{q_1}(\Omega)$. *Consequently, we have* $q_2, p < q_2 < q_1$ *in which:*

$$\left(\oint_{B_s(x_1)} |\nabla v_m|^{q_2}\, dx\right)^{\frac{1}{q_2}} \leq C\left\{\left(\oint_{B_{2s}(x_1)} |\nabla v_m|^{p}\, dx\right)^{\frac{1}{p}} + \left(\oint_{B_{2s}(x_1)} |g_m|^{q_1}\, dx\right)^{\frac{1}{q_1}}\right\},$$

for each $B_{2s}(x_1) \subset \omega$, *with condition that* q_2 *and* C *are dependent solely on* n, p, q, α.

Following this, we present two lemmas that are critical in order to derive the primary result:
The two lemmas are significantly impacted by [6, 7]. We compose

$$\lambda_0 = \left\{\left(\oint_{B_2} |\nabla v_m|^{p}\, dx\right)^{\frac{1}{p}} + \frac{1}{\delta}\left(\oint_{B_2} |g_m|^{q_1}\, dx\right)^{\frac{1}{q_1}}\right\}, \tag{4}$$

and

$$A(\lambda) = \{x \in B_1 : |\nabla v_m| > \lambda\}$$

for $\lambda > 0$, whereas $\delta > 0$ will be selected at a later time.
By ensuring that $|\nabla v_m|$ is constrained within the range $B_1 \setminus A(\lambda)$ for any fixed $\lambda > 0$, our analysis is directed towards the level set $A(\lambda)$. At this juncture, $A(\lambda)$ shall be decomposed into a set of disjoint spheres.

Lemma 3. *Suppose that* $\lambda \geq \lambda_* = 2^{6n\setminus p\lambda_0}$, *there exists a family of disjoint balls* $\{B_i^0\}_{i\in\mathbb{N}} = \{B_{p_{x_i}}(x_i)\}_{i\in\mathbb{N}}, x_i \in A(\lambda)$, *such that* $0 < p_{x_i} < 1 \setminus 2^5$. *And*

$$\left(\oint_{B_i^0} |\nabla v_m|^{p}\, dx\right)^{\frac{1}{p}} + \frac{1}{\delta}\left(\oint_{B_i^0} |g_m|^{q_1}\, dx\right)^{\frac{1}{q_1}} = \lambda.$$

Furthermore, we have

$$E\,(\lambda) \subset \bigcup_{i\in\mathbb{N}} B_i^1\ ,$$

where $B_i^j =: 2^{j+2}B_i^0$ *and* $p_{x_i} < s < 1$ *for all* i *values 1, 2, and 3.*

$$\left(\oint_{B_s(x_i)} |\nabla v_m|^{p} dx\right)^{1/p} + \frac{1}{\delta}\left(\oint_{B_s(x_i)} |g_m|^{q_1} dx\right)^{1/q_1} \leq \lambda.$$

Proof. (i) For the sake of expediency, we signify:

$$J\,[B] = \left(\oint_B |\nabla \mathrm{v}_m|^{p} dx\right)^{1/p} + \frac{1}{\delta}\left(\int_B |g_m|^{q_1} dx\right)^{1/q_1}.$$

Now we assert that:

$$\sup_{\omega\in B_1}\sup_{1/2^5\le\lambda\le1} J\left[B_p\left(\omega\right)\right]\le 2^{\frac{6n}{p}}\lambda_0 =: \lambda_*. \tag{5}$$

In order to demonstrate this, assign any $\Omega\in B_1$ and $1/2^5\le p\le 1$. From the equation $B_i^j=2^{j+2}B_i^0$: j=1,2,3, we get that $B_2=2^5B_{p(\Omega)}$. And we can easily see that:

$$\left(\frac{|B_2|}{|B_p\left(\Omega\right)|}\right)^{\frac{1}{p}}\le\ 2^{\frac{5}{p}}\le 2^{\frac{6n}{p}},\ n=1,2,... \quad . \tag{6}$$

Then by using equations (4), (5) and (6) we can deduce that:

$$\left(\oint_{B_p(\Omega)}|\nabla v_m|^p dx\right)^{1/p}\le(\frac{|B_2|}{|B_p(\Omega)|}^{\frac{1}{p}}\left(\oint_{B_2}|\nabla v_m|^p dx\right)^{\frac{1}{p}}\le 2^{6n/p}\left(\oint_{B_2}|\nabla v_m|^p dx\right)^{\frac{1}{p}}.$$

In a similar fashion, we have

$$\left(\oint_{B_r(\Omega)}|g_m|^{q_1}\,dy\right)^{\frac{1}{q_1}}\le 2^{\frac{6n}{q_1}}\left(\oint_{B_2}|g_m|^{q_1}\,dx\right)^{\frac{1}{q_1}}.$$

As a result of combining the aforementioned two inequalities with the definitions of λ_0 and q_1, we can conclude that (4) is valid.
(ii) Define λ_0 as $\lambda\ge\lambda_*=2^{6n/p}\lambda_0$. In the case of $\Omega\in A(\lambda)$, a variant of Lebesgue's differentiation theorem provides the following:

$$\lim_{p\to0} J\left[B_p\left(\Omega\right)\right]>\lambda.$$

which indicates the existence of a $p>0$ that satisfies

$$J\left[B_p\left(\Omega\right)\right]>\lambda.$$

Consequently, starting from step (i), we can choose a radius $p_\Omega\in(0,1/2^5]$ [8], and such that

$$J\left[B_{p_w}\left(\Omega\right)\right]=\lambda.$$

Furthermore, that for $p_\Omega<p\le1$ *and*

$$J\left[B_p\left(\Omega\right)\right]<\lambda.$$

Based on the aforementioned argument for. $\Omega\in A(\lambda)$, the ball $B_{p_\Omega}(\Omega)$ can be constructed as described above. Hence, by employing Vitali's covering lemma, it is possible to identify a set of disjoint orbs denoted as $\{B_i^0\}_{i\in\mathbb{N}}=\{B_{p_{x_i}}(x_i)\}_{i\in\mathbb{N}}$ where $x_i\in A(\lambda)$ in order to validate the lemma's conclusions. We now have a comprehensive proof.

At present, the subsequent estimates of spheres $\{B_i^0\}$ are obtained.

Lemma 4. *By employing the identical hypothesis and outcomes as in Lemma (6), we obtain*

$$\left|B_i^0\right| \leq C\left(\frac{1}{\lambda^p}\int_{\left\{x\in B_i^0:|\nabla v_m|>\frac{\lambda}{4}\right\}}|\nabla v_m|^p dx+\frac{1}{\lambda^{q_1}\delta^{q_1}}\int_{\left\{x\in B_i^0:|f|>\frac{\delta\lambda}{4}\right\}}|g_m|^{q_1}dx\right),$$

where $C=C(p,q_1)=2^{q_1}/[1-(1/2)^p-(1/2)^{q_1}]$.

Proof. As shown in the lemma above,

$$\left(\int_{B_i^0}|\nabla \mathrm{v}_m|^p dx+\frac{1}{\delta}\int_{B_i^0}|g_m|^{1/q_1}dx\right) = \lambda,$$

thus, signifying that

$$|B_i^0| \leq \frac{2^p}{\lambda^p}\int_{B_i^0}|\nabla v_m|^p dx+\frac{2^{q_1}}{\lambda^{q_1}\delta^{q_1}}\int_{B_i^0}|g_m|^{\frac{1}{q_1}}dx \tag{7}$$

given that one of the subsequent inequalities must hold true:

$$\lambda/2 \leq \left(\int_{B_i^0}|\nabla v_m|^p dx\right)^{\frac{1}{p}},$$

or

$$\lambda/2 \leq \frac{1}{\delta}\left(\int_{B_i^0}|g_m|^{q_1}dx\right)^{\frac{1}{q_1}}.$$

As a result, we obtain the following by dividing the right-hand side of equation (6) into two integrals:

$$\begin{aligned}\left|B_i^0\right| \leq C&\left(\frac{2^p}{\lambda^p}\oint_{\left\{x\in B_i^0:|\nabla v_m|>\lambda/4\right\}}|\nabla v_m|^p\,dx+(1/2)^{q_1}\left|B_i^0\right|\right)\\&+\frac{2^{q_1}}{\lambda^{q_1}\delta^{q_1}}\int_{\left\{x\in B_i^0:|f|>\delta\lambda/4\right\}}|g_m|^{q_1}\,dx+(1/2)^{q_1}\left|B_i^0\right|\end{aligned}$$

We have therefore arrived at the intended estimation.

It is adequate to regard the proof of Theorem 1 in section four as an a priori estimate in the subsequent discussion, thus presuming that $\nabla v_m \in L^q_{loc}(\omega)$. One can easily eliminate this assumption using a standard approximation argument, such as the one presented in [12, 13]. By considering Lemma 3 and $\lambda \geq \lambda_* = 2^{\frac{6n}{p}}\cdot\lambda_0$, it is possible to generate a set of disjoint balls denoted as $\left\{B_i^0\right\}_{i\in \mathrm{N}} = \left\{B_{p_{x_i}(x_i)}\right\}_{i\in \mathrm{N}}, x_i \in A(\lambda)$. Adjust any $i \in \mathbb{N}$ and set $v_{m\lambda} = u_m/\lambda$ and $g_{m\lambda} = g_m/\lambda$.

Subsequently, $v_{m\lambda}$ remains a local weak solution of (1), where $g_{m\lambda}$ assumes the place of g_m. Consequently, Lemma (3) dictates that

$$\int_{B_i^j} |\nabla v_{m_\lambda}|^p \, dx \leq 1 \text{ } and \text{ } \int_{B_i^j} |g_m|^{q_1} \, dx \leq \lambda^{q_1} \tag{8}$$

In the case $i = 1, 2, 3$, B_j^i is defined in Lemma 2 as $B_j^i =: 2^{i+2} B_i^0$ Denote the weak solution of the reference equation as:

$$\begin{cases} div\left(\bar{E}_{B_s} \nabla u_m \cdot \nabla u_m \bar{E}_{B_s} \nabla u_m\right) = 0 \text{ } in \text{ } B_s \\ \qquad\qquad\qquad\qquad\qquad u_m = f_m \text{ } in \text{ } B_s \end{cases} \tag{9}$$

3. The Global weak solutions and grading estimates

Definition 3. *Let $f \in W^{1,p}(B_s)$ be assumed. It is stated that $u_m \in W^{1,p}(B_s)$ is a weak solution of the system $u_m - g_m \in W_0^{1,p}(B_s)$ is a weak solution of*

$$\begin{cases} \operatorname{div}\left(\bar{E}_{B_s} \nabla u_m \cdot \nabla u_m \bar{E}_{B_s} \nabla u_m\right) = 0 \text{ } in \text{ } B_s, \\ u_m = f_m \text{ } on \text{ } \partial B_s. \end{cases}$$

Let

$$\int_{B_s} (\bar{E}_{B_s} \nabla u_m \cdot \nabla u_m \bar{E}_{B_s} \nabla u_m \cdot \nabla \psi dx) = 0 \text{ },$$

with any $\psi \in W_0^{1,p}(B_s)$. The following are recollections of estimates for u_m to [5, 12]:

$$\int_{B_s} |\nabla u_m|^p dx \leq C \int_{B_s} |\nabla v_m|^p dx, \tag{10}$$

and

$$\sup_{B_p} |\nabla u_m| \leq C \left(\int_{B_s} |\nabla u_m|^p dx \right)^{\frac{1}{p}} \tag{11}$$

in the range $p \in (0, s/2]$, with $C = C(n, p, \alpha)$. Moreover, we can derive the subsequent significant outcome.

Lemma 5. *There exists a small value of $\delta = \delta(\epsilon) > 0$ for all values of ϵ, such that if v_m is a local weak solution of (1) in ω with $B_4 \subset \omega$, then*

$$\int_{B_2} \left|E - \bar{E}_{B_2}\right| dx \leq \delta, \tag{12}$$

and

$$\int_{B_4} |\nabla v_m|^p dx \leq 1 \text{ } and \int_{B_4} |g_m|^{q_1} dx \leq \text{ } \delta^{q_1}. \tag{13}$$

Consequently, $N_0 > 1$ exists and is denoted by u in B_2 as the weak solution to (1)

Proof. Based on (1), (2), and (4), it can be deduced that the weak solutions of (1) in ω and (2) in B_2, respectively, are denoted as v_m and u_m. It is sufficient to select the test function $\psi = u_m - v_m \in W_0^{1,p}(B_2)$, and a straightforward calculation yields the following expression:

$$I_1 = I_2 + I_3,$$

where

$$I_1 = \int_{B_2} (\bar{E}_{B_2}\nabla u_m \cdot \nabla u_m)^{\frac{p-2}{2}} E_{B_2}\nabla u - \left(\hat{E}_{B_2}\nabla \mathrm{v}_m \cdot \nabla \mathrm{v}_m)^{\frac{p-2}{2}} \bar{A}_{B_2}\nabla \mathrm{v}_m\right) .\nabla(u - \mathrm{v}_m)dx,$$
$$I_2 = \int_{B_2} ((E\nabla \mathrm{v}_m \cdot \nabla \mathrm{v}_m)^{(p-2)/2}\bar{E}\nabla \mathrm{v}_m - (\bar{E}_{B_2}\nabla \mathrm{v}_m \cdot \nabla \mathrm{v}_m)^{(p-2)/2}\bar{E}_{B_2}\nabla \mathrm{v}_m) \cdot \nabla(u_m - \mathrm{v}_m)dx,$$
$$I_3 = -\int_{B_2} |g_m|^{p-2} g \cdot \nabla (u_m - \mathrm{v}_m)\, dx.$$

The estimation of I_1. Two instances are distinguished.
Given Case 1. $p \geq 2$, the elementary inequality is applied.

$$(\bar{E}_{B_2}\zeta \cdot \zeta)^{\frac{p-2}{2}} \bar{E}_{B_2}\zeta - (\bar{E}_{B_2}\eta \cdot \eta)^{\frac{p-2}{2}} \bar{E}_{B_2}\eta) \cdot (\zeta - \eta) \geq C|\zeta - \eta|^p,$$

We have, for every $\zeta, \eta \in \mathbb{R}^n$ where $C = C(p, \alpha)$,:

$$I_1 \geq C \int_{B_2} |\nabla(v_m - u_m)|^p dx.$$

Case 2: Applying the rudimentary inequality to $1 < p < 2$.

$$|\zeta - \eta|^p \leq C\tau^{\frac{p-2}{p}}((\bar{E}_{B_2}\zeta \cdot \zeta)^{\frac{p-2}{2}} \bar{E}_{B_2}\zeta - (\bar{E}_{B_2}\eta \cdot \eta)^{\frac{p-2}{2}} \bar{E}_{B_2}\eta) \cdot (\zeta - \eta) + \tau|\eta|^p.$$

We have the following for each $\zeta, \eta \in \mathbb{R}^n$ and each $\tau \in (0, 1)$ where $C = C(p, \alpha)$:

$$I_1 + \tau \int_{B_2} |\nabla v_m|^p dx \geq C(\tau) \int_{B_2} |\nabla (v_m - u_m)|^p dx.$$

Approximation of Implementing a fundamental inequality

$$\left|(E\zeta \cdot \zeta)^{(p-2)/2} E\zeta - (\bar{E}_{B_2}\zeta \cdot \zeta)^{(p-2)/2} \bar{E}_{B_2}\zeta\right| \leq C \left|E - \bar{E}_{B_2}\right| |\zeta|^{p-1}.$$

We have, for each $o, \eta \in R^n$ where $C = C(p, \alpha)$, by employing Young's inequality with and Holder's inequality.

$$\begin{aligned} I_2 &\leq C \int_{B_2} \left|E - \bar{E}_{B_2}\right| |\nabla v_m|^{p-1} |\nabla (v_m - u_m)|\, dx \\ &\leq C(\tau) \int_{B_2} \left|E - \bar{E}\right|^{\frac{p}{p-1}} |\nabla v_m|^p dx + \tau \int_{B_2} |\nabla(v_m - u_m)|^p dx \\ &\leq C(\tau)\left(\int_{B_2} \left|E - \bar{E}_{B_2}\right|^{pq_2/[(p-1)(q_2-p)]} dx\right)^{(q_2-p)/q_2} \left(\int_{B_2} |\nabla v_m|^{q_2} dx\right)^{p/q_2} \\ &\quad + \tau \int_{B_2} |\nabla(v_m - u_m)|^p dx. \end{aligned}$$

We observe that

$$\begin{aligned} &\left(\int_{B_2} \left|E - \bar{E}_{B_2}\right|^{pq_2/[(p-1)(q_2-p)]} dx\right)^{(q_2-p)/q_2} \\ &\leq (2\alpha)^{(p^2+q_2-p)/[q_2(p-1)]} (\int_{B_2} \left|E - \bar{E}_{B_2}\right| dx)^{(q_2-p)/q_2} \leq C\delta^{(q_2-p)/q_2} \end{aligned}$$

due to the outcomes of (2) and (3), and

$$\left(\int_{B_2} |\nabla v_m|^{q_2} dx\right)^{p/q_2} \leq C\left[\left(\int_{B_4} |\nabla v_m|^p dx\right)^{\frac{1}{p}} + \int_{B_4} (|g_m|^{q_1} dx)^{\frac{1}{q_1}}\right]^p \leq C,$$

by virtue of Lemma 4 and equation (13), with C denoting the set $C = C(m, p, q_1, \alpha)$. In this context, the assumption that $\delta < 1$. We thus conclude that

$$I_2 \leq C(\tau)\delta^{(q_2-p)/q_2} + \tau \int_{B_2} |\nabla(v_m - u_m)|^p \, dx.$$

The estimation of I_3 can be obtained by applying Young's inequality with τ and Holder's inequality

$$\begin{aligned} I_3 &\leq \tau \textstyle\int_{B_2} |\nabla (v_m - u_m)|^p dx + C(\tau) \int_{B_2} |g_m|^p dx \\ &\leq \tau \textstyle\int_{B_2} |\nabla(v_m - u_m)|^p dx + C(\tau) \left(\int_{B_2} |g_m|^{q_1} dx\right)^{p/q_1} \\ &\leq \tau \textstyle\int_{B_2} |\nabla(v_m - u_m)|^p dx + C(\tau)\delta^p. \end{aligned}$$

We derive by summing all the estimates of I_i $(1 \leq i \leq 3)$:

$$\begin{aligned} C(\tau) \textstyle\int_{B_2} |\nabla(v_m - u_m)|^p dx &\leq 2\tau \textstyle\int_{B_2} |\nabla(v_m - u_m)|^p dx \\ &+ \tau \textstyle\int_{B_2} |\nabla v_m|^p dx + C(\tau) \left[\delta^{(q_2-p)/q_2} + \delta^p\right]. \end{aligned}$$

We reach the following conclusion by selecting a small constant $\tau > 0$ such that $0 < \tau \ll \delta < 1$, and then applying (13):

$$\int_{B_2} |\nabla(v_m - u_m)|^p dx \leq C\left[\delta + \delta^{\frac{q_2-p}{q_2}} + \delta^p\right] = \varepsilon^p,$$

by choosing that fulfills the final inequality stated earlier. This concludes the evidence. Define (1) and (2) with the same value of δ as in Lemma 2. As stated at the outset of this segment, E is vanishing $(\delta, 1)$. Thus (Perci)

$$\int_{B_i^j} |E - \bar{E}_{B_i^j}|dx| \leq \delta, \tag{14}$$

given that the radiuses of $B_i^j (0 \leq j \leq 3)$ are not greater than 1 for $j = 0, 1, 2, 3$. The scaling invariant form of Lemma 2 is then obtained by recalling (7).

Lemma 6. *Considers the assumption that $\lambda \geq \lambda_*$. In the case where ε is greater than zero, there is a small $\delta = \delta(\varepsilon) > 0$ such that N_0 is greater than one and v_m is a local weak solution of in ω with $B_i^3 \subset \omega$.*

$$\sup_{B_i^2} \left|\nabla (u_m)_\lambda^i\right| \leq N_0 \quad \textit{and} \quad \int_{B_i^2} \left|\nabla\left(v_{m\lambda} - (v_m)_\lambda^i\right)\right|^p dx \leq \varepsilon^p. \tag{15}$$

In this context, $(u_m)_\lambda^i$ denotes the weak solution of equation (2) in B_i^2, where $v_{m\lambda}$ substitutes for v_m.

Proof. Resealing the definitions of B_i^j for $j = 0, 1, 2, 3$, we establish

$$\begin{cases} (v_m)_\lambda^i(x) &= \dfrac{v_m^\lambda(2^3 p_{x_i} x)}{2^3 p_{z_i}}, \\ (g_m)_\lambda^i(x) &= g_{m\lambda}(2^3 p_{x_i} x), \\ E^i(x) &= E(2^3 p_{x_i} x),\ x \in B_4. \end{cases}$$

$(v_m)_\lambda^i$ is therefore a local weak solution of
$div\left(E^i\nabla\,(v_m)_\lambda^i \cdot \nabla(v_m)_\lambda^i\right)^{(p-2)/2} E^i\nabla(v_m)_\lambda^i = div\left(|(g_m)_\lambda^i|^{p-2}(g_m)_\lambda^i\right)$ in B_4.
It is easily discernible from equations (7) and (14) that

$$\int_{B_4}\left|\nabla(v_m)_\lambda^i(x)\right|^p dx \le 1,\ \int_{B_4}\left|(g_m)_\lambda^i\right|^p dx \le \delta^p$$

and

$$\int_{B_2}\left|E^i - \bar{E}^i{}_{B_2}\right|^p dx \le \delta.$$

Lemma 1 subsequently states that a weak solution of

$$\begin{cases} div\left(\bar{E}_{B_2}^i\nabla u_m \cdot \nabla u_m \bar{E}_{B_2}^i \nabla u_m\right) &= 0\ in\ B_2 \\ u_m &= (v_m)_\lambda^i\ on\ \partial B_2 \end{cases}$$

in the form that

$$\sup_{B_1}|\nabla u_m| \le \mathrm{N}_0\ and\ \int_{B_2}\left|\nabla\,(v_m)_\lambda^i - u_m\right|^p dx \le \varepsilon^p$$

At this moment, we define $(u_m)_\lambda^i$ in B_i^2 by

$$u_m\,(x) = \frac{1}{2^3 p_{x_i}}(u_m)_\lambda^i\left(2^3 p_{x_i} x\right), x \in B_2.$$

Then, by modifying the variables, the conclusion of Lemma 3 is reestablished. Such concludes the evidence.

4. Local and Gradient estimates for elliptic nonlinear equations:

Theorem 1. *Considers the case where $q \ge p$. Considers v_m to be a feeble local solution to (1). Subsequently, a small value of $\delta = \delta(n, p, q, \alpha, \mathbb{R}) > 0$ is present, such that for every elliptical function and vanishing $(\delta, \mathbb{R})$, as well as for every f with $g_m \in L_{loc}^q(\omega; \mathbb{R}^n)$, we can deduce:*

$$\int_{B_r(x_0)}|\nabla v_m|^q dx \le C\left[\int_{B_{4r}(x_0)}|v_m|^q dx + \int_{B_{4r}(x_0)}|g_m|^q dx\right]. \tag{16}$$

The constant C is independent of v_m and g_m where $B_{4r}(x_0) \subset \omega$. Our strategy is substantially shaped by [5, 14].

Proof. (i) For $q = p$, the proof is uncomplicated.
(ii) Lemma (3) states that for all $\lambda \geq \lambda_*$, we obtain

$$\begin{aligned}\left|\{x \in B_i^1 : |\nabla v_m| > 2N_0\lambda\}\right| &= \left|\{x \in B_i^1 : |\nabla v_{m\lambda}| > 2N_0\}\right| \\ &\leq \left|\{x \in B_i^1 : \left|\nabla (v_{m\lambda} - (v_m)^i_\lambda\right| > N_0\}\right| \\ + \left|\{x \in B_i^1 : \left|\nabla (u_m)^i_\lambda\right| > N_0\}\right| &= \left|\{x \in B_i^1 : \left|\nabla (u_{m\lambda} - (u_m)^i_\lambda)\right| > N_0\}\right| \\ &\leq \frac{1}{N_0^p}\int_{B_i^2} \left|\nabla \left(v_m - (u_m)^i_\lambda\right)\right|^p dz \leq \frac{\varepsilon^p \left|B_i^2\right|}{N_0^p} = \frac{2^{4n}\varepsilon^p\left|B_i^0\right|}{N_0^p}.\end{aligned}$$

Consequently, as shown in Lemma 2, that

$$\begin{aligned}\left|\{x \in B_i^1 : |\nabla v_m| > 2N_0\lambda\}\right| &\leq C\left(\varepsilon^p \frac{1}{\lambda^p}\int_{\left\{x\in B_i^0 : \left|\nabla v_m > \frac{\lambda}{4}\right|\right\}} |\nabla v_m|^p\, dx\right) \\ &+ \frac{1}{\lambda^{q_1}\delta^{q_1}}\left(\int_{\{x\in B_i^0 : |g| > \delta\lambda/4\}} |g_m|^{q_1} dx\right).\end{aligned}$$

In the given context, $C = C(n, p, q_1, \alpha)$. Keeping in mind that the spheres are disconnected and

$$\bigcup_{i\in\mathbb{N}} B_i^1 \supset A(\lambda) = \{x \in B_1 : |\nabla v_m| > \lambda\}.$$

We obtain the following by summing $i \in \mathbb{N}$ in the inequality above for any $\lambda \geq \lambda_*$:

$$\begin{aligned}&|\{x \in B_1 : |\nabla v_m| > 2N_0\lambda\}| \\ &\leq \sum_i \left|\{x \in B_i^1 : |\nabla v_m| > 2N_0\lambda\}\right| \\ &\leq C\varepsilon^p\left(\frac{1}{\lambda^p}\int_{\left\{x\in B_2 : |\nabla v_m| > \frac{\lambda}{4}\right\}} |\nabla v_m|^p dx + \frac{1}{\lambda^{q_1}\delta^{q_1}}\int_{\left\{x\in B_2 : |g_m| > \frac{\delta\lambda}{4}\right\}} |g_m|^{q_1} dx\right) \quad (17)\end{aligned}$$

in the case of any $\lambda \geq \lambda_*$. We compute while recalling the standard argument of measure theory.

$$\begin{aligned}&\int_{B_1} |\nabla v_m|^q\, dz = q\int_0^\infty \mu^{q-1}\left|\{x \in B_1 : |\nabla v_m| > \mu\}\right| d\mu \\ &= q\int_0^{2N_0\lambda_*} \mu^{q-1}\left|\{x \in B_1 : |\nabla v_m| > \mu\}\right| d\mu + q\int_{2N_0\lambda_*}^\infty \mu^{q-1}\left|\{x \in B_1 : |\nabla v_m| > \mu\}\right| d\mu \\ &= q\int_0^{2N_0\lambda_*} \mu^{q-1}\left|\{x \in B_1 : |\nabla v_m| > \mu\}\right| d\mu \\ &+ q\int_{\lambda_*}^\infty (2N_0\lambda)^{q-1}\left|\{x \in B_1 : |\nabla v_m| > 2N_0\lambda\}\right| d(2N_0\lambda) =: J_1 + J_2.\end{aligned}$$

Estimation of J_1: It can be deduced from the definitions of λ_* and λ_0 that

$$\lambda_*^q = 2^{6nq/p}\lambda_0^q \leq C\left\{\left(\int_{B_2} |\nabla v_m|^p\, dx\right)^{\frac{q}{p}} + \frac{1}{\delta^q}\left(\int_{B_2} |g_m|^{q_1}\, dx\right)^{\frac{q}{q_1}}\right\}. \quad (18)$$

Consequently, Lemma (1) and Holder's inequality dictate that

$$\begin{aligned}\lambda_*^q &\leq C\left[\left(\int_{B_4} |v_m|^p\, dx + \int_{B_4} |g_m|^p\, dx\right)^{\frac{q}{p}} + \frac{1}{\delta^q}\left(\int_{B_2} |g_m|^{q_1}\, dx\right)^{\frac{q}{q_1}}\right] \\ &\leq C\left[\left(\int_{B_4} |v_m|^p dx\right)^{q/p} + \left(\int_{B_4} |g_m|^p dx\right)^{q/p} + \frac{1}{\delta^q}\int_{B_2} |g_m|^q dx\right] \\ &\leq C\left\{\int_{B_4} |v_m|^q dx + \int_{B_4} |g_m|^q dx\right\}.\end{aligned}$$

From [14] we see that, $\frac{1}{\delta^q} \to 0$, and we have:

$$\lambda_*^q \leq C \left\{ \int_{B_4} |v_m|^q dx + \int_{B_4} |g_m|^q dx \right\}.$$

Hence, we ascertain

$$J_1 \leq (2N_0\lambda_*)^q |B_1| \leq C \left\{ \int_{B_4} |v_m|^q dx + \int_{B_4} |g_m|^q dx \right\},$$

where $C = C(n, p, q, \alpha)$.
Probability of J_2. By deriving from (17), we obtain that

$$J_2 \leq C\varepsilon^p \Big\{ \int_0^\infty \lambda^{q-p-1} \int_{\{x \in B_2 : |\nabla v_m| > \lambda/4\}} |\nabla v_m|^p dx d\lambda \\ + \tfrac{1}{\delta^{q_1}} \int_0^\infty \lambda^{q-q_1-1} \int_{\{x \in B_2 : |g_m| > \delta\lambda/4\}} |g_m|^{q_1} dx d\lambda \Big\}.$$

Considering that

$$\int_{\mathbb{R}^n} |f_m|^\beta dx = (\beta - \Lambda) \int_0^\infty \mu^{\beta-\alpha-1} \int_{\{x \in \mathbb{R}^n : |f_m| > \mu\}} {f_m}^\alpha dx d\mu.$$

Given $\beta > \Lambda > 1$, we obtain

$$J_2 \leq C_1\varepsilon^p \int_{B_2} |\nabla u_m|^q dx + C_2\varepsilon^p \int_{B_2} |f_m|^q dx,$$

in that where $C_1 = C_1(n, p, q, \alpha)$ and $C_2 = C_2(n, p, q, \alpha, \delta)$.
We obtain by combining the estimates of J_1 and J_2.

$$\int_{B_1} |\nabla \mathrm{v}_m|^q dx \leq C_1\varepsilon^p \int_{B_2} |\nabla v_m|^q dx + C_3 \int_{B_4} \left(|v_m|^q + |g_m|^q \right) dx,$$

$C_3 = C_3(n, p, q, \alpha, \delta, \varepsilon)$.
Using a covering and iteration argument to [15, 16] to re incorporate at the right-hand side the first integral in the aforementioned inequality while selecting a suitable ε such that $C_1\varepsilon^p = 1/2$, we obtain the following:

$$\int_{B_1} |\nabla v_m|^q \, dx \leq C \left\{ \int_{B_4} |v_m|^q \, dx + \int_{B_4} |g_m|^q \, dx \right\}.$$

By performing a shift and scaling transform, the proof of the main result can be completed.

Corollary 1. *Considers v_m to be a sequence of local weak solutions to equation (1), assuming that ε is greater than zero. Subsequently, a small value of $\delta = \delta(n, 1+\varepsilon, \frac{1+\varepsilon}{\varepsilon}, \alpha, \mathbb{R}) > \theta$*

is present, such that for every elliptically shaped and vanished set $E(\delta, \mathbb{R})$ and every set g_m with $g_m \in L_{loc}^{\frac{1+\epsilon}{\epsilon}}(\omega; \mathbb{R}^n)$, we can deduce:

$$\begin{gathered}\int_{B_r(x_0)} \sum_{m=1}^{s} |\nabla v_m|^{\frac{1+\epsilon}{\epsilon}} dx \\ \leq \bar{C}\left[\int_{B_{4r}(x_0)} \sum_{m=1}^{s} |v_m|^{\frac{1+\varepsilon}{\varepsilon}} dx + \int_{B_{4r}(x_0)} \sum_{m=1}^{s} |g_m|^{\frac{1+\varepsilon}{\varepsilon}} dx\right]\end{gathered}$$

where $\bar{C}$ is a constant that is not dependent on u_m and g_m, and $B_{4r}(x_0) \subset \omega$.

Proof. Lemma 2 states that for any $\lambda = \lambda_* + \epsilon$, we obtain

$$\begin{gathered}1/\big(2x \in B_i^1 : \textstyle\sum_{m=1}^{s} |v_m| > 2N_0(\lambda_* + \epsilon)\big)^{1/2} = 1/\big\{x \in B_i^1 : \sum_{m=1}^{s} |(\mathrm{v}_m)_{\lambda_*+\epsilon}| > 2N_0\big\}^{1/2} \\ \leq 1/\{x \in B_i^1 : \textstyle\sum_{m=1}^{s} \left|((v_m)_{\lambda_*+\varepsilon} - (u_m)^i_{\lambda_*+\varepsilon})\right| > N_0\}^{1/2} + \left|\{x \in B_i^1 : \left|\nabla (u_m)^i_{\lambda_*+\varepsilon}\right| > N_0\}\right| \\ = 1/\{x \in B_i^1 : \textstyle\sum_{m=1}^{s} \left|\nabla((\mathrm{v}_m)_{\lambda_*+\varepsilon} - (u_m)^i_{\lambda_*+\varepsilon})\right| > N_0\}^{1/2} \\ \leq \frac{1}{N_0^{1+\varepsilon}} \int_{B_i^2} \sum_{m=1}^{s} \left|\nabla (v_m)_{\lambda_*+\varepsilon} - (u_m)^i_{\lambda_*+\varepsilon})\right|^{1+\varepsilon} dz \leq \frac{\varepsilon^{1+\varepsilon}|B_i^2|}{N_0^{1+\varepsilon}} = \frac{2^{4n}\varepsilon^{1+\varepsilon}|B_i^0|}{N_0^{1+\varepsilon}},\end{gathered}$$

consequently, as shown in Lemma 3 that

$$1/\left(\leq \{2x \in B_i^1 : \sum_{m=1}^{s} |\nabla \mathrm{v}_m| > 2N_0(\lambda_* + \epsilon)\right)^{1/2}$$

$$\leq \bar{C}\varepsilon^{1+\varepsilon} \frac{1}{\lambda^{1+\varepsilon}} \int_{\{x \in B_i^0 : \sum_{m=1}^{s} |\nabla v_m| > (\lambda_*+\varepsilon)/4\}} \sum_{m=1}^{s} |\nabla v_m|^{1+\delta} dx$$

in that where $\bar{C} = \bar{C}(n, 1+\epsilon, (\frac{1+\epsilon}{\epsilon})_1, \alpha)$.

bearing in mind that the spheres are disconnected and

$$\bigcup_{i \in \mathbb{N}} B_i^1 \supset A(\lambda_* + \varepsilon) = \left\{x \in B_1 : \sum_{m=1}^{s} |\nabla \mathrm{v}_m| > (\lambda_* + \varepsilon)\right\},$$

for any $\lambda = \lambda_* + \varepsilon$, and by aggregating the terms in the aforementioned inequality, we obtain

$$\begin{gathered}1/(\{2x \in B_1 : \textstyle\sum_{m=1}^{s} |\nabla \mathrm{v}_m| > 2N_0(\lambda_* + \varepsilon)\})^{1/2} \\ \leq \textstyle\sum_i 1/\{2x \in B_i^1 : \sum_{m=1}^{s} |\nabla \mathrm{v}_m| > 2N_0(\lambda_* + \varepsilon)\}^{1/2} \\ \leq \bar{C}\varepsilon^{1+\varepsilon} \frac{1}{\lambda^{1+\varepsilon}} \int_{\{x \in B_2 : \sum_{m=1}^{s} |\nabla \mathrm{v}_m| > (\lambda_*+\varepsilon)/4\}} \textstyle\sum_{m=1}^{s} |\nabla \mathrm{v}_m|^{1+\varepsilon} dx \\ \frac{1}{\lambda^{(\frac{1+\varepsilon}{\varepsilon})_1} \delta^{(\frac{1+\varepsilon}{\varepsilon})_1}} \int_{\{x \in B_2 : \sum_{m=1}^{s} |g_m| > \delta(\lambda_*+\varepsilon)/4\}} \textstyle\sum_{m=1}^{s} |g_m|^{(\frac{1+\varepsilon}{\varepsilon})_1} dx),\end{gathered} \tag{19}$$

for whatever. We compute while recalling the standard argument of measure theory.

$$\begin{gathered}\int_{B_1} \textstyle\sum_{m=1}^{s} |\nabla \mathrm{v}_m|^{(\frac{1+\varepsilon}{\varepsilon})} dz \\ = (\frac{1+\varepsilon}{\varepsilon}) \int_0^\infty \mu^{(\frac{1+\varepsilon}{\varepsilon})-1} 1/\{2x \in B_1 : \textstyle\sum_{m=1}^{s} |\nabla \mathrm{v}_m| > \mu\}^{1/2} d\mu \\ = (\frac{1+\varepsilon}{\varepsilon}) \int_0^{2N_0\lambda_*} \mu^{(\frac{1+\varepsilon}{\varepsilon})-1} 1/\{2x \in B_1 : \textstyle\sum_{m=1}^{s} |\nabla \mathrm{v}_m| > \mu\}^{1/2} d\mu \\ +(\frac{1+\varepsilon}{\varepsilon}) \int_{2N_0\lambda_*}^\infty \mu^{(\frac{1+\varepsilon}{\varepsilon})-1} 1/\{2x \in B_1 : \textstyle\sum_{m=1}^{s} |\nabla \mathrm{v}_m| > \mu\}^{1/2} d\mu \\ = (\frac{1+\varepsilon}{\varepsilon}) \int_0^{2N_0\lambda_*} \mu^{(\frac{1+\varepsilon}{\varepsilon})-1} 1/\{2x \in B_1 : \textstyle\sum_{m=1}^{s} |\nabla \mathrm{v}_m| > \mu\}^{1/2} d\mu \\ + \left(\frac{1+\varepsilon}{\varepsilon}\right) \int_{\lambda_*}^\infty (2N_0(\lambda_* + \varepsilon))^{\left(\frac{1+\varepsilon}{\varepsilon}\right)-1} K.d(2N_0(\lambda_* + \varepsilon)\},\end{gathered}$$

where

$$K = \left(1/2x \in B_1 : \sum_{m=1}^{s} |\nabla \mathrm{v}_m| > 2N_0(\lambda_* + \varepsilon)\right)^{1/2}$$
$$=: J_1 + J_2.$$

Probability of J_1. It can be deduced from the definitions of $\lambda_*$and λ_0 that

$$\lambda_*^{(\frac{1+\varepsilon}{\varepsilon})} = 2^{6n(\frac{1+\varepsilon}{\varepsilon})/(1+\varepsilon)}\lambda_0^{(\frac{1+\varepsilon}{\varepsilon})}$$
$$\leq \bar{C}\left(\left(\int_{B_2}\sum_{m=1}^{s}|\nabla \mathrm{v}_m|^{(1+\varepsilon)}dx\right)^{(\frac{1+\varepsilon}{\varepsilon})/(1+\varepsilon)} + \frac{1}{\delta^{(\frac{1+\varepsilon}{\varepsilon})}}\left(\int_{B_2}\sum_{m=1}^{s}|g_m|^{(\frac{1+\varepsilon}{\varepsilon})_1}dx\right)^{(\frac{1+\varepsilon}{\varepsilon})/(\frac{1+\varepsilon}{\varepsilon})_1}.\right)$$

Consequently, Lemma 3 and Holder's inequality dictate that

$$\lambda_*^{(\frac{1+\varepsilon}{\varepsilon})} \leq \bar{C}\left(\int_{B_4}\sum_{m=1}^{s}|\mathrm{v}_m|^{(1+\varepsilon)}dx + \int_{B_4}\sum_{m=1}^{s}|g_m|^{(1+\varepsilon)}dx\right)^{(\frac{1+\varepsilon}{\varepsilon})/(1+\varepsilon)}$$
$$+\frac{1}{\delta^{(\frac{1+\varepsilon}{\varepsilon})}}\left(\int_{B_2}\sum_{m=1}^{s}|g_m|^{(\frac{1+\varepsilon}{\varepsilon})_1}dx)^{(\frac{1+\varepsilon}{\varepsilon})/(\frac{1+\varepsilon}{\varepsilon})_1}\right)$$
$$\leq \bar{C}\left\{\left(\int_{B_4}\sum_{m=1}^{s}|\mathrm{v}_m|^{(1+\varepsilon)}dx\right)^{(\frac{1+\varepsilon}{\varepsilon})/(1+\varepsilon)} + \left(\int_{B_4}\sum_{m=1}^{s}|g_m|^{(1+\varepsilon)}dx\right)^{(\frac{1+\varepsilon}{\varepsilon})/(1+\varepsilon)}\right.$$
$$+\frac{1}{\delta^{(\frac{1+\varepsilon}{\varepsilon})}}\int_{B_2}\sum_{m=1}^{s}|g_m|^{(\frac{1+\varepsilon}{\varepsilon})}dx$$
$$\leq \bar{C}\left\{\int_{B_4}\sum_{m=1}^{s}|\mathrm{v}_m|^{(\frac{1+\varepsilon}{\varepsilon})}dx + \int_{B_4}\sum_{m=1}^{s}|g_m|^{(\frac{1+\varepsilon}{\varepsilon})}dx\right\}.$$

Hence, we ascertain

$$J_1 \leq (2N_0\lambda_*)^{(\frac{1+\varepsilon}{\varepsilon})}|B_1| \leq \bar{C}\left\{\int_{B_4}\sum_{m=1}^{s}|\mathrm{v}_m|^{(\frac{1+\varepsilon}{\varepsilon})}dx + \int_{B_4}\sum_{m=1}^{s}|g_m|^{(\frac{1+\varepsilon}{\varepsilon})}dx\right\},$$

where $\bar{C} = \bar{C}(n, 1+\varepsilon, (\frac{1+\varepsilon}{\varepsilon}), \alpha)$.
J_2 estimation is derived from (18) as follows:

$$\bar{C}\varepsilon^{(1+\varepsilon)}\left\{\begin{array}{c} \int_0^\infty (\lambda_*+\varepsilon)^{(\frac{1+\varepsilon}{\varepsilon})-(1+\varepsilon)-1}\int_{\left\{x\in B_2:\sum_{m=1}^{s}|\nabla v_m|>(\lambda_*+\varepsilon)/4\right\}}\sum_{m=1}^{s}|\nabla v_m|^{(1+\varepsilon)}\,dxd\,(\lambda_*+\varepsilon) \\ +\frac{1}{\delta^{\left(\frac{1+\varepsilon}{\varepsilon}\right)_1}}\int_0^\infty (\lambda_*+\varepsilon) \end{array}\right.$$

Considering that

$$\int_{\mathbb{R}^n}|f|^\beta dx = (\beta-\Lambda)\int_0^\infty \mu^{\beta-\Lambda-1}\int_{\{x\in\mathbb{R}^m:|f|>\mu\}} f^\Lambda dxd\mu.$$

Given $\beta > \Lambda > 1$, we obtain

$$J_2 \leq \bar{C}_1\varepsilon^{(1+\varepsilon)}\int_{B_2}\sum_{m=1}^{s}|\nabla v_m|^{(\frac{1+\varepsilon}{\varepsilon})}dx + \bar{C}_2\varepsilon^{(1+\varepsilon)}\int_{B_2}\sum_{m=1}^{s}|g_m|^{(\frac{1+\varepsilon}{\varepsilon})}dx,$$

where $\bar{C}_1 = \bar{C}_1(n, 1+\varepsilon, (\frac{1+\varepsilon}{\varepsilon}), \alpha)$ and $\bar{C}_2 = \bar{C}_2(n, 1+\varepsilon, (\frac{1+\varepsilon}{\varepsilon}), \alpha)$.
We obtain by combining the estimates of J_1 and J_2.

$$\int_{B_1} \sum_{m=1}^{s} |\nabla v_m|^{(\frac{1+\varepsilon}{\varepsilon})} dx \le \bar{C}_1 \varepsilon^{(1+\varepsilon)} \int_{B_2} \sum_{m=1}^{s} |\nabla v_m|^{(\frac{1+\varepsilon}{\varepsilon})} dx + \bar{C}_3 \int_{B_4} \sum_{m=1}^{s} (|v_m|^{(\frac{1+\varepsilon}{\varepsilon})} + |g_m|^{(\frac{1+\varepsilon}{\varepsilon})}) dx,$$

in that where $\bar{C}_3 = \bar{C}_3(n, 1+\varepsilon, (\frac{1+\varepsilon}{\varepsilon}), \alpha, \delta, \varepsilon)$. By choosing an appropriate ε such that $\bar{C}_1 \varepsilon^{(1+\varepsilon)} = 1/2$ and employing a covering and iteration argument to reabsorb at the right-hand side of the initial integral in the aforementioned inequality, we obtain the following result:

$$\int_{B_1} \sum_{m=1}^{s} |\nabla v_m|^{(\frac{1+\varepsilon}{\varepsilon})} dx \le \bar{C} \left\{ \int_{B_4} \sum_{m=1}^{s} |v_m|^{(\frac{1+\varepsilon}{\varepsilon})} dx + \int_{B_4} \sum_{m=1}^{s} |g_m|^{(\frac{1+\varepsilon}{\varepsilon})} dx \right\}.$$

By performing a shift and scaling transform, the proof can be completed.

5. Conclusion

This study meticulously established the foundational groundwork for analyzing nonlinear elliptic equations of p-Laplacian type. We began by rigorously defining key concepts, including the small bounded mean oscillation semi-norm condition, which quantifies the oscillation of coefficients. Furthermore, we provided precise definitions for both local and global weak solutions of the equation under consideration, ensuring a clear understanding of the solution spaces. Crucial lemmas were derived to support the main theorem, laying a solid analytical framework for the subsequent investigation. Building upon these definitions and lemmas, the core contribution of this work lies in the proof of the main result theorem. This theorem successfully derives local gradient estimates for the aforementioned nonlinear elliptic equations, specifically those featuring bounded mean oscillation coefficients. These estimates provide valuable insights into the regularity and behavior of solutions, particularly concerning the gradient's control within localized domains. The successful derivation of these gradient estimates signifies a significant advancement in the understanding of p-Laplacian type equations with non-smooth coefficients.

Author Contributions

All authors contributed equally to the writing of this article. All authors have accepted responsibility for the entire content of the manuscript and approved its submission.

Conflicts of Interest

All authors confirm that they have no conflict of interest.

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