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A Note on Nonlinear Mixed (bi-Skew, skew Lie) Triple Derivations on $*$-Algebras

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Abstract. Let $\mathfrak{A}$ be a unital $*$-algebra containing non-trivial projection. We prove that if a map $\Lambda : \mathfrak{A} \to \mathfrak{A}$ such that $\Lambda([[\mathcal{L}, \mathcal{M}]_\bullet, \mathcal{N}]_*) = [[\Lambda(\mathcal{L}), \mathcal{M}]_\bullet, \mathcal{N}]_* + [[\mathcal{L}, \Lambda(\mathcal{M})]_\bullet, \mathcal{N}]_* + [[\mathcal{L}, \mathcal{M}]_\bullet, \Lambda(\mathcal{N})]_*$ for all $\mathcal{L}, \mathcal{M}, \mathcal{N} \in \mathfrak{A}$, then Λ is additive. Moreover, if $\Lambda(\mathfrak{I})$ is self-adjoint, then Λ is a $*$-derivation. Additionally, as an application, we can also apply our results on factor von Neumann algebras, standard operator algebras and prime $*$-algebras.

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Key Words and Phrases: Mixed bi-skew Lie triple derivation, $*$-derivation, $*$- algebra

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1. Introduction

Let $\mathfrak{A}$ be an $*$-algebra over the complex field $\mathbb{C}$. For $\mathcal{L}, \mathcal{M} \in \mathfrak{A}$, we call $[\mathcal{L}, \mathcal{M}]_* = \mathcal{L}\mathcal{M} - \mathcal{M}\mathcal{L}^*$ the skew Lie product and $[\mathcal{L}, \mathcal{M}]_\bullet = \mathcal{L}\mathcal{M}^* - \mathcal{M}\mathcal{L}^*$ denotes the bi-skew Lie product. The skew Lie product, Jordan product, and bi-skew Lie product have become increasingly relevant in various research fields, and numerous authors have shown a keen interest in their exploration. This is evident from the numerous studies by authors (see [1, 2, 4–7, 9, 10, 13]). Recall that an additive map $\Lambda : \mathfrak{A} \to \mathfrak{A}$ is called an additive derivation if $\Lambda(\mathcal{L}\mathcal{M}) = \Lambda(\mathcal{L})\mathcal{M} + \mathcal{L}\Lambda(\mathcal{M})$ for all $\mathcal{L}, \mathcal{M} \in \mathfrak{A}$. If $\Lambda(\mathcal{L}^*) = \Lambda(\mathcal{L})^*$ for all $\mathcal{L} \in \mathfrak{A}$, then Λ is an additive $*$-derivation. Let $\Lambda : \mathfrak{A} \to \mathfrak{A}$ be a map (without the additivity assumption). We say Λ is a nonlinear skew Lie derivation or nonlinear skew Lie triple derivation if

$$\Lambda([\mathcal{L}, \mathcal{M}]_*) = [\Lambda(\mathcal{L}), \mathcal{M}]_* + [\mathcal{L}, \Lambda(\mathcal{M})]_*$$

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or

$$\Lambda([[\mathcal{L},\mathcal{M}]_*,\mathcal{N}]_*) = [[\Lambda(\mathcal{L}),\mathcal{M}]_*,\mathcal{N}]_* + [[\mathcal{L},\Lambda(\mathcal{M})]_*,\mathcal{N}]_* + [[\mathcal{L},\mathcal{M}]_*,\Lambda(\mathcal{N})]_*$$

for all $\mathcal{L},\mathcal{M},\mathcal{N} \in \mathfrak{A}$. Similarly, a map $\Lambda : \mathfrak{A} \to \mathfrak{A}$ is said to be a nonlinear bi-skew Lie derivation or nonlinear bi-skew Lie triple derivation if

$$\Lambda([\mathcal{L},\mathcal{M}]_\bullet) = [\Lambda(\mathcal{L}),\mathcal{M}]_\bullet + [\mathcal{L},\Lambda(\mathcal{M})]_\bullet$$

or

$$\Lambda([[\mathcal{L},\mathcal{M}]_\bullet,\mathcal{N}]_\bullet) = [[\Lambda(\mathcal{L}),\mathcal{M}]_\bullet,\mathcal{N}]_\bullet + [[\mathcal{L},\Lambda(\mathcal{M})]_\bullet,\mathcal{N}]_* + [[\mathcal{L},\mathcal{M}]_\bullet,\Lambda(\mathcal{N})]_\bullet$$

for all $\mathcal{L},\mathcal{M},\mathcal{N} \in \mathfrak{A}$. In 2021, A. Khan [3] established a proof demonstrating that any multiplicative or nonadditive bi-skew Lie triple derivation acting on a factor Von Neumann algebra can be characterized as an additive $*$-derivation.
Numerous authors have recently explored the derivations and isomorphisms corresponding to the novel products created by combining Lie and skew Lie products, skew Lie and skew Jordan product see [8, 11, 12]. As an illustration, Li and Zhang [8] delved into an investigation focused on understanding the arrangement and properties of the nonlinear mixed Jordan triple $*$-derivation within the domain of $*$-algebras. In 2023, Rehman et. al. [12] mixed the concept of Jordan and Jordan $*$-product and gives the complete characterization of nonlinear mixed Jordan $*$-triple derivation on $*$-algebras. Inspired by the above results, in the present paper, we combined skew Lie product and bi-skew Lie product and defined nonlinear mixed bi-skew Lie triple derivations on $*$-algebras. A map Λ: $\mathfrak{A} \to \mathfrak{A}$ is called nonlinear mixed bi-skew Lie triple derivations if

$$\Lambda([[\mathcal{L},\mathcal{M}]_\bullet,\mathcal{N}]_*) = [[\Lambda(\mathcal{L}),\mathcal{M}]_\bullet,\mathcal{N}]_* + [[\mathcal{L},\Lambda(\mathcal{M})]_\bullet,\mathcal{N}]_* + [[\mathcal{L},\mathcal{M}]_\bullet,\Lambda(\mathcal{N})]_*$$

for all $\mathcal{L},\mathcal{M},\mathcal{N} \in \mathfrak{A}$. Our proof establishes that when Λ represents a nonlinear mixed bi-skew Lie triple derivation acting on $*$-algebras, it necessarily possesses additivity. Furthermore, if the image of Λ under the transformation of the identity element ($\Lambda(I)$) is self-adjoint, then Λ can be identified as an $*$-derivation. In simpler terms, the study demonstrates that specific properties, such as additivity and self-adjointness, can be attributed to the nature of nonlinear mixed bi-skew Lie triple derivations on $*$-algebras.

2. Main Result

Our First Theorem is as follows:

Theorem 2.1. *Let $\mathfrak{A}$ be a unital $*$-algebra with unity $\mathfrak{I}$ containing a non-trivial projection P satisfies*

$$\mathcal{X}\mathfrak{A}\mathcal{P} = 0 \implies \mathcal{X} = 0 \qquad (\blacktriangle)$$

and

$$\mathcal{X}\mathfrak{A}(\mathfrak{I} - \mathcal{P}) = 0 \implies \mathcal{X} = 0. \qquad (\blacktriangledown)$$

Define a map $\Lambda : \mathfrak{A} \to \mathfrak{A}$ *such that*

$$\Lambda([[\mathcal{L}, \mathcal{M}]_\bullet, \mathcal{N}]_*) = [[\Lambda(\mathcal{L}), \mathcal{M}]_\bullet, \mathcal{N}]_* + [[\mathcal{L}, \Lambda(\mathcal{M})]_\bullet, \mathcal{N}]_* + [[\mathcal{L}, \mathcal{M}]_\bullet, \Lambda(\mathcal{N})]_*$$

Then Λ *is an additive.*

Proof. Let $\mathcal{P} = \mathcal{P}_1$ be a non-trivial projection in $\mathfrak{A}$ and $\mathcal{P}_2 = \mathfrak{I} - \mathcal{P}_1$, where $\mathfrak{I}$ is the unity of this algebra. Then by Peirce decomposition of $\mathfrak{A}$, we have $\mathfrak{A} = \mathcal{P}_1\mathfrak{A}\mathcal{P}_1 \oplus \mathcal{P}_1\mathfrak{A}\mathcal{P}_2 \oplus \mathcal{P}_2\mathfrak{A}\mathcal{P}_1 \oplus \mathcal{P}_2\mathfrak{A}\mathcal{P}_2$ and, denote $\mathfrak{A}_{11} = \mathcal{P}_1\mathfrak{A}\mathcal{P}_1, \mathfrak{A}_{12} = \mathcal{P}_1\mathfrak{A}\mathcal{P}_2, \mathfrak{A}_{21} = \mathcal{P}_2\mathfrak{A}\mathcal{P}_1$ and $\mathfrak{A}_{22} = \mathcal{P}_2\mathfrak{A}\mathcal{P}_2$. Note that any $\mathcal{L} \in \mathfrak{A}$ can be written as $\mathcal{L} = \mathcal{L}_{11} + \mathcal{L}_{12} + \mathcal{L}_{21} + \mathcal{L}_{22}$, where $\mathcal{L}_{ij} \in \mathfrak{A}_{ij}$ and $\mathcal{L}_{ij}^* \in \mathfrak{A}_{ji}$ for $i, j = 1, 2$.

Several lemmas are used to prove Theorem 2.1.

Lemma 2.1. $\Lambda(0) = 0$.

Proof. It is trivial that

$$\Lambda(0) = \Lambda([[0, 0]_\bullet, 0]_*) = [[\Lambda(0), 0]_\bullet, 0]_* + [[0, \Lambda(0)]_\bullet, 0]_* + [[0, 0]_\bullet, \Lambda(0)]_* = 0.$$

Lemma 2.2. *For any* $\mathcal{L}_{ij} \in \mathfrak{A}_{ij}, 1 \leq i, j \leq 2$, *we have*

$$\Lambda(\sum_{i,j=1}^{2} \mathcal{L}_{ij}) = \sum_{i,j=1}^{2} \Lambda(\mathcal{L}_{ij}).$$

Proof. Let $M = \Lambda(\mathcal{L}_{11} + \mathcal{L}_{12} + \mathcal{L}_{21} + \mathcal{L}_{22}) - \Lambda(\mathcal{L}_{11}) - \Lambda(\mathcal{L}_{12}) - \Lambda(\mathcal{L}_{21}) - \Lambda(\mathcal{L}_{22})$. In order to prove that $\Lambda(\mathcal{L}_{11} + \mathcal{L}_{12} + \mathcal{L}_{21} + \mathcal{L}_{22}) = \Lambda(\mathcal{L}_{11}) + \Lambda(\mathcal{L}_{12}) + \Lambda(\mathcal{L}_{21}) + \Lambda(\mathcal{L}_{22})$, we show $M = 0$. Since $[[\mathcal{L}_{12}, \mathcal{P}_1]\bullet, \mathcal{P}_1]_* = [[\mathcal{L}_{21}, \mathcal{P}_1]\bullet, \mathcal{P}_1]_* = [[\mathcal{L}_{22}, \mathcal{P}_1]\bullet, \mathcal{P}_1]_* = 0$. It follows from Lemma 2.1 that

$$\begin{aligned}
&\Lambda([[\mathcal{L}_{11} + \mathcal{L}_{12} + \mathcal{L}_{21} + \mathcal{L}_{22}, \mathcal{P}_1]_\bullet, \mathcal{P}_1]_*) \\
&= \Lambda([[\mathcal{L}_{11}, \mathcal{P}_1]_\bullet, \mathcal{P}_1]_*) + \Lambda([[\mathcal{L}_{12}, \mathcal{P}_1]_\bullet, \mathcal{P}_1]_*) \\
&\quad + \Lambda([[\mathcal{L}_{21}, \mathcal{P}_1]_\bullet, \mathcal{P}_1]_*) + \Lambda([[\mathcal{L}_{22}, \mathcal{P}_1]_\bullet, \mathcal{P}_1]_*) \\
&= [[\Lambda(\mathcal{L}_{11}) + \Lambda(\mathcal{L}_{12}) + \Lambda(\mathcal{L}_{21}) + \Lambda(\mathcal{L}_{22}), \mathcal{P}_1]_\bullet, \mathcal{P}_1]_* \\
&\quad + [[\mathcal{L}_{11} + \mathcal{L}_{12} + \mathcal{L}_{21} + \mathcal{L}_{22}, \Lambda(\mathcal{P}_1)]_\bullet, \mathcal{P}_1]_* \\
&\quad + [[\mathcal{L}_{11} + \mathcal{L}_{12} + \mathcal{L}_{21} + \mathcal{L}_{22}, \mathcal{P}_1]_\bullet, \Lambda(\mathcal{P}_1)]_*
\end{aligned}$$

and

$$\begin{aligned}
\Lambda([[\mathcal{L}_{11} + \mathcal{L}_{12} + \mathcal{L}_{21} + \mathcal{L}_{22}, \mathcal{P}_1]_\bullet, \mathcal{P}_1]_*) &= [[\Lambda(\mathcal{L}_{11} + \mathcal{L}_{12} + \mathcal{L}_{21} + \mathcal{L}_{22}), \mathcal{P}_1]_\bullet, \mathcal{P}_1]_* \\
&\quad + [[\mathcal{L}_{11} + \mathcal{L}_{12} + \mathcal{L}_{21} + \mathcal{L}_{22}, \Lambda(\mathcal{P}_1)]_\bullet, \mathcal{P}_1]_* \\
&\quad + [[\mathcal{L}_{11} + \mathcal{L}_{12} + \mathcal{L}_{21} + \mathcal{L}_{22}, \mathcal{P}_1]_\bullet, \Lambda(\mathcal{P}_1)]_*.
\end{aligned}$$

From the above equations, we get $[[M, \mathcal{P}_1]_\bullet, \mathcal{P}_1]_* = 0$. This implies that $M\mathcal{P}_1 - \mathcal{P}_1 M^* \mathcal{P}_1 - \mathcal{P}_1 M^* + \mathcal{P}_1 M \mathcal{P}_1 = 0$. By multiplying $\mathcal{P}_2$ from left, we get $\mathcal{P}_2 M \mathcal{P}_1 = 0$. Similarly, by

applying $\mathcal{P}_2$ instead of $\mathcal{P}_1$, we get $\mathcal{P}_1 M \mathcal{P}_2 = 0$.
Also, for any $\mathcal{X}_{12} \in \mathfrak{A}_{12}$, we have

$$\begin{aligned}\Lambda([[\mathcal{L}_{11}+\mathcal{L}_{12}+\mathcal{L}_{21}+\mathcal{L}_{22},\mathcal{X}_{12}]_\bullet,\mathcal{P}_2]_*) &= [[\Lambda(\mathcal{L}_{11}+\mathcal{L}_{12}+\mathcal{L}_{21}+\mathcal{L}_{22}),\mathcal{X}_{12}]_\bullet,\mathcal{P}_2]_* \\ &\quad +[[\mathcal{L}_{11}+\mathcal{L}_{12}+\mathcal{L}_{21}+\mathcal{L}_{22},\Lambda(\mathcal{X}_{12})]_\bullet,\mathcal{P}_2]_* \\ &\quad +[[\mathcal{L}_{11}+\mathcal{L}_{12}+\mathcal{L}_{21}+\mathcal{L}_{22},\mathcal{X}_{12}]_\bullet,\Lambda(\mathcal{P}_2)]_*.\end{aligned}$$

From Lemma 2.1, we get

$$\begin{aligned}\Lambda([[\mathcal{L}_{11}+\mathcal{L}_{12}+\mathcal{L}_{21}+\mathcal{L}_{22},\mathcal{X}_{12}]_\bullet,\mathcal{P}_2]_*) &= \Lambda([[\mathcal{L}_{11},\mathcal{X}_{12}]_\bullet,\mathcal{P}_2]_*)+\Lambda([[\mathcal{L}_{12},\mathcal{X}_{12}]_\bullet,\mathcal{P}_2]_*) \\ &\quad +\Lambda([[\mathcal{L}_{21},\mathcal{X}_{12}]_\bullet,\mathcal{P}_2]_*)+\Lambda([[\mathcal{L}_{22},\mathcal{X}_{12}]_\bullet,\mathcal{P}_2]_*) \\ &= [[\Lambda(\mathcal{L}_{11}),\mathcal{X}_{12}]_\bullet,\mathcal{P}_2]_*+[[\mathcal{L}_{11},\Lambda(\mathcal{X}_{12})]_\bullet,\mathcal{P}_2]_* \\ &\quad +[[\mathcal{L}_{11},\mathcal{X}_{12}]_\bullet,\Lambda(\mathcal{P}_2)]_*+[[\Lambda(\mathcal{L}_{12}),\mathcal{X}_{12}]_\bullet,\mathcal{P}_2]_* \\ &\quad +[[\mathcal{L}_{12},\Lambda(\mathcal{X}_{12})]_\bullet,\mathcal{P}_2]_*+[[\mathcal{L}_{12},\mathcal{X}_{12}]_\bullet,\Lambda(\mathcal{P}_2)]_* \\ &\quad +[[\Lambda(\mathcal{L}_{21}),\mathcal{X}_{12}]_\bullet,\mathcal{P}_2]_*+[[\mathcal{L}_{21},\Lambda(\mathcal{X}_{12})]_\bullet,\mathcal{P}_2]_* \\ &\quad +[[\mathcal{L}_{21},\mathcal{X}_{12}]_\bullet,\Lambda(\mathcal{P}_2)]_*+[[\Lambda(\mathcal{L}_{22}),\mathcal{X}_{12}]_\bullet,\mathcal{P}_2]_* \\ &\quad +[[\mathcal{L}_{22},\Lambda(\mathcal{X}_{12})]_\bullet,\mathcal{P}_2]_*+[[\mathcal{L}_{22},\mathcal{X}_{12}]_\bullet,\Lambda(\mathcal{P}_2)]_*.\end{aligned}$$

From the above two equations, we get $[[M,\mathcal{X}_{12}]_\bullet,\mathcal{P}_2]_* = 0$. That means $-\mathcal{X}_{12}M^*\mathcal{P}_2 + \mathcal{P}_2 M\mathcal{X}_{12}^* = 0$. By multiplying $\mathcal{P}_1$ from left, we get $\mathcal{P}_2 M\mathcal{X}_{12}^* = 0$. Thus, $\mathcal{P}_2 M\mathcal{P}_2 = 0$ by using (▲) and (▼). In the similar way, we can show that $\mathcal{P}_1 M\mathcal{P}_1 = 0$ by choosing $\mathcal{X}_{21}$ and $\mathcal{P}_1$ instead of $\mathcal{X}_{21}$ and $\mathcal{P}_1$ respectively in above. Hence $M = 0$. It follows that $\Lambda(\sum_{i,j=1}^{2}\mathcal{L}_{ij}) = \sum_{i,j=1}^{2}\Lambda(\mathcal{L}_{ij})$.

Lemma 2.3. *For each $\mathcal{L}_{12},\mathcal{M}_{12}\in\mathfrak{A}_{12}$ and $\mathcal{L}_{21},\mathcal{M}_{21}\in\mathfrak{A}_{21}$, we have*

- *(i)* $\Lambda(\mathcal{L}_{12}+\mathcal{M}_{12}) = \Lambda(\mathcal{L}_{12})+\Lambda(\mathcal{M}_{12})$.
- *(ii)* $\Lambda(\mathcal{L}_{21}+\mathcal{M}_{21}) = \Lambda(\mathcal{L}_{21})+\Lambda(\mathcal{M}_{21})$.

Proof. (1) Let $T = \Lambda(\mathcal{L}_{12}+\mathcal{M}_{12})-\Lambda(\mathcal{L}_{12})-\Lambda(\mathcal{M}_{12})$. It follows from Lemma 2.1 that

$$\begin{aligned}&\Lambda([[\mathcal{L}_{12}+\mathcal{M}_{12},\mathcal{P}_1]_\bullet,\mathcal{P}_2]_*) \\ &\quad= \Lambda([[\mathcal{L}_{12},\mathcal{P}_1]_\bullet,\mathcal{P}_2]_*)+\Lambda([[\mathcal{M}_{12},\mathcal{P}_1]_\bullet,\mathcal{P}_2]_*) \\ &\quad= [[\Lambda(\mathcal{L}_{12}),\mathcal{P}_1]_\bullet,\mathcal{P}_2]_*+[[\mathcal{L}_{12},\Lambda(\mathcal{P}_1)]_\bullet,\mathcal{P}_2]_*+[[\mathcal{L}_{12},\mathcal{P}_1]_\bullet,\Lambda(\mathcal{P}_2)]_* \\ &\qquad +[[\Lambda(\mathcal{M}_{12}),\mathcal{P}_1]_\bullet,\mathcal{P}_2]_*+[[\mathcal{M}_{12},\Lambda(\mathcal{P}_1)]_\bullet,\mathcal{P}_2]_*+[[\mathcal{M}_{12},\mathcal{P}_1]_\bullet,\Lambda(\mathcal{P}_2)]_*.\end{aligned}$$

Alternatively, we have

$$\begin{aligned}\Lambda([[\mathcal{L}_{12}+\mathcal{M}_{12},\mathcal{P}_1]_\bullet,\mathcal{P}_2]_*) &= [[\Lambda(\mathcal{L}_{12}+\mathcal{M}_{12}),\mathcal{P}_1]_\bullet,\mathcal{P}_2]_*+[[\mathcal{L}_{12}+\mathcal{M}_{12},\Lambda(\mathcal{P}_1)]_\bullet,\mathcal{P}_2]_* \\ &\quad +[[\mathcal{L}_{12}+\mathcal{M}_{12},\mathcal{P}_1]_\bullet,\Lambda(\mathcal{P}_2)]_*.\end{aligned}$$

By comparing the above two expressions, we get $[[T,\mathcal{P}_1]_\bullet,\mathcal{P}_2]_* = 0$. This implies that $\mathcal{P}_2 T\mathcal{P}_1 = 0$. Similarly, $\mathcal{P}_1 T\mathcal{P}_2 = 0$. For any $\mathcal{X}_{12}\in\mathfrak{A}_{12}$, we have

$$\Lambda([[\mathcal{X}_{12},\mathcal{L}_{12}+\mathcal{M}_{12}]_\bullet,\mathcal{P}_2]_*) = [[\Lambda(\mathcal{X}_{12}),\mathcal{L}_{12}+\mathcal{M}_{12}]_\bullet,\mathcal{P}_2]_*+[[\mathcal{X}_{12},\Lambda(\mathcal{L}_{12}+\mathcal{M}_{12})]_\bullet,\mathcal{P}_2]_*$$

$$+[[\mathcal{X}_{12}, \mathcal{L}_{12} + \mathcal{M}_{12}]_\bullet, \Lambda(\mathcal{P}_2)]_*.$$

Since $[[\mathcal{X}_{12}, \mathcal{L}_{12}]_\bullet, \mathcal{P}_2]_* = 0$ and using Lemma 2.1, we have

$$\begin{aligned}\Lambda([[\mathcal{X}_{12}, \mathcal{L}_{12} + \mathcal{M}_{12}]_\bullet, \mathcal{P}_2]_*) &= \Lambda([[\mathcal{X}_{12}, \mathcal{L}_{12}]_\bullet, \mathcal{P}_2]_*) + \Lambda([[\mathcal{X}_{12}, \mathcal{M}_{12}]_\bullet, \mathcal{P}_2]_*) \\ &= [[\Lambda(\mathcal{X}_{12}), \mathcal{L}_{12}]_\bullet, \mathcal{P}_2]_* + [[\mathcal{X}_{12}, \Lambda(\mathcal{L}_{12})]_\bullet, \mathcal{P}_2]_* \\ &\quad +[[\mathcal{X}_{12}, \mathcal{L}_{12}]_\bullet, \Lambda(\mathcal{P}_2)]_* + [[\Lambda(\mathcal{X}_{12}), \mathcal{M}_{12}]_\bullet, \mathcal{P}_2]_* \\ &\quad +[[\mathcal{X}_{12}, \Lambda(\mathcal{M}_{12})]_\bullet, \mathcal{P}_2]_* + [[\mathcal{X}_{12}, \mathcal{M}_{12}]_\bullet, \Lambda(\mathcal{P}_2)]_*.\end{aligned}$$

From the last two expressions, we get $[[\mathcal{X}_{12}, T]_\bullet, \mathcal{P}_2]_* = 0$. That means $\mathcal{X}_{12}T^*\mathcal{P}_2 - \mathcal{P}_2 M\mathcal{X}_{12}^* = 0$. Multiplying left side by $\mathcal{P}_2$ and then using (▲) and (▼), we get $\mathcal{P}_2 T\mathcal{P}_2 = 0$. Similarly, $\mathcal{P}_1 T\mathcal{P}_1 = 0$. Hence, $T = 0$.
(2) By using the similar argument as in (1), we get the required conclusion.

Lemma 2.4. *For each $\mathcal{L}_{ii}, \mathcal{M}_{ii} \in \mathfrak{A}_{ii}$ such that $1 \leq i \leq 2$, we have*

$$\Lambda(\mathcal{L}_{ii} + \mathcal{M}_{ii}) = \Lambda(\mathcal{L}_{ii}) + \Lambda(\mathcal{M}_{ii}).$$

Proof. Let $T = \Lambda(\mathcal{L}_{ii} + \mathcal{M}_{ii}) - \Lambda(\mathcal{L}_{ii}) - \Lambda(\mathcal{M}_{ii})$. It follows from Lemma 2.1 and $i \neq j$ that

$$\begin{aligned}&\Lambda([[\mathcal{P}_j, \mathcal{L}_{ii} + \mathcal{M}_{ii}]_\bullet, \mathcal{P}_i]_*) \\ &\quad = \Lambda([[\mathcal{P}_j, \mathcal{L}_{ii}]_\bullet, \mathcal{P}_i]_*) + \Lambda([[\mathcal{P}_j, \mathcal{M}_{ii}]_\bullet, \mathcal{P}_i]_*) \\ &\quad = [[\Lambda(\mathcal{P}_j), \mathcal{L}_{ii}]_\bullet, \mathcal{P}_i]_* + [[\mathcal{P}_j, \Lambda(\mathcal{L}_{ii})]_\bullet, \mathcal{P}_i]_* + [[\mathcal{P}_j, \mathcal{L}_{ii}]_\bullet, \Lambda(\mathcal{P}_i)]_* \\ &\qquad +[[\Lambda(\mathcal{P}_j), \mathcal{M}_{ii}]_\bullet, \mathcal{P}_i]_* + [[\mathcal{P}_j, \Lambda(\mathcal{M}_{ii})]_\bullet, \mathcal{P}_i]_* + [[\mathcal{P}_j, \mathcal{M}_{ii}]_\bullet, \Lambda(\mathcal{P}_i)]_*\end{aligned}$$

and

$$\begin{aligned}\Lambda([[\mathcal{P}_j, \mathcal{L}_{ii} + \mathcal{M}_{ii}]_\bullet, \mathcal{P}_i]_*) &= [[\Lambda(\mathcal{P}_j), \mathcal{L}_{ii} + \mathcal{M}_{ii}]_\bullet, \mathcal{P}_i]_* + [[\mathcal{P}_j, \Lambda(\mathcal{L}_{ii} + \mathcal{M}_{ii})]_\bullet, \mathcal{P}_i]_* \\ &\quad +[[\mathcal{P}_j, \mathcal{L}_{ii} + \mathcal{M}_{ii}]_\bullet, \Lambda(\mathcal{P}_i)]_*.\end{aligned}$$

By comparing the last two expressions, we get $[[\mathcal{P}_j, T]_\bullet, \mathcal{P}_i]_* = 0$. This gives $\mathcal{P}_i T\mathcal{P}_j = 0$ with $i \neq j$. Also, for any $\mathcal{X}_{ij} \in \mathfrak{A}_{ij}$, we have

$$\begin{aligned}\Lambda([[\mathcal{X}_{ij}, \mathcal{L}_{ii} + \mathcal{M}_{ii}]_\bullet, \mathcal{P}_i]_*) &= [[\Lambda(\mathcal{X}_{ij}), \mathcal{L}_{ii} + \mathcal{M}_{ii}]_\bullet, \mathcal{P}_i]_* + [[\mathcal{X}_{ij}, \Lambda(\mathcal{L}_{ii} + \mathcal{M}_{ii})]_\bullet, \mathcal{P}_i]_*) \\ &\quad +[[\mathcal{X}_{ij}, \mathcal{L}_{ii} + \mathcal{M}_{ii}]_\bullet, \Lambda(\mathcal{P}_i)]_*.\end{aligned}$$

Under other conditions, $[[\mathcal{X}_{ij}, \mathcal{L}_{ii}]_\bullet, \mathcal{P}_i]_* = 0$ and using Lemma 2.1, we have

$$\begin{aligned}&\Lambda([[\mathcal{X}_{ij}, \mathcal{L}_{ii} + \mathcal{M}_{ii}]_\bullet, \mathcal{P}_i]_*) \\ &\quad = \Lambda([[\mathcal{X}_{ij}, \mathcal{L}_{ii}]_\bullet, \mathcal{P}_i]_*) + \Lambda([[\mathcal{X}_{ij}, \mathcal{M}_{ii}]_\bullet, \mathcal{P}_i]_*) \\ &\quad = [[\Lambda(\mathcal{X}_{ij}), \mathcal{L}_{ii}]_\bullet, \mathcal{P}_i]_* + [[\mathcal{X}_{ij}, \Lambda(\mathcal{L}_{ii})]_\bullet, \mathcal{P}_i]_* + [[\mathcal{X}_{ij}, \mathcal{L}_{ii}]_\bullet, \Lambda(\mathcal{P}_i)]_* \\ &\qquad +[[\Lambda(\mathcal{X}_{ij}), \mathcal{M}_{ii}]_\bullet, \mathcal{P}_i]_* + [[\mathcal{X}_{ij}, \Lambda(\mathcal{M}_{ii})]_\bullet, \mathcal{P}_i]_* + [[\mathcal{X}_{ij}, \mathcal{M}_{ii}]_\bullet, \Lambda(\mathcal{P}_i)]_*.\end{aligned}$$

From the last two expressions, we get $[[\mathcal{X}_{ij}, T]_\bullet, \mathcal{P}_i]_* = 0$. That means $\mathcal{X}_{ij}T^*\mathcal{P}_i - T\mathcal{X}_{ij}^* - \mathcal{P}_i T\mathcal{X}_{ij}^* + \mathcal{X}_{ij}T^* = 0$. Left multiplying by $\mathcal{P}_j$ both sides and using (▲) and (▼), we find $\mathcal{P}_j T\mathcal{P}_j = 0$.

Lemma 2.5. *Λ is an additive map.*

Proof. For any $\mathcal{L}, \mathcal{M} \in \mathfrak{A}$, we write $\mathcal{L} = \mathcal{L}_{11} + \mathcal{L}_{12} + \mathcal{L}_{21} + \mathcal{L}_{22}$ and $\mathcal{M} = \mathcal{M}_{11} + \mathcal{M}_{12} + \mathcal{M}_{21} + \mathcal{M}_{22}$. By using Lemmas 2.2 - 2.4, we get

$$\begin{aligned}
&\Lambda(\mathcal{L} + \mathcal{M}) \\
&= \Lambda(\mathcal{L}_{11} + \mathcal{L}_{12} + \mathcal{L}_{21} + \mathcal{L}_{22} + \mathcal{M}_{11} + \mathcal{M}_{12} + \mathcal{M}_{21} + \mathcal{M}_{22}) \\
&= \Lambda(\mathcal{L}_{11} + \mathcal{M}_{11}) + \Lambda(\mathcal{L}_{12} + \mathcal{M}_{12}) + \Lambda(\mathcal{L}_{21} + \mathcal{M}_{21}) + \Lambda(\mathcal{L}_{22} + \mathcal{M}_{22}) \\
&= \Lambda(\mathcal{L}_{11}) + \Lambda(\mathcal{M}_{11}) + \Lambda(\mathcal{L}_{12}) + \Lambda(\mathcal{M}_{12}) + \Lambda(\mathcal{L}_{21}) + \Lambda(\mathcal{M}_{21}) + \Lambda(\mathcal{L}_{22}) + \Lambda(\mathcal{M}_{22}) \\
&= \Lambda(\mathcal{L}_{11} + \mathcal{L}_{12} + \mathcal{L}_{21} + \mathcal{L}_{22}) + \Lambda(\mathcal{M}_{11} + \mathcal{M}_{12} + \mathcal{M}_{21} + \mathcal{M}_{22}) \\
&= \Lambda(\mathcal{L}) + \Lambda(\mathcal{M}).
\end{aligned}$$

Hence, Λ is additive. This completes the proof of Theorem 2.1.

Theorem 2.2. *Let $\mathfrak{A}$ be a unital $*$-algebra with unity I containing a non-trivial projection $\mathcal{P}$ satisfies ($\blacktriangle$) and ($\blacktriangledown$). Let the map $\Lambda : \mathfrak{A} \to \mathfrak{A}$ satisfy the condition*

$$\Lambda([[\mathcal{L}, \mathcal{M}]_\bullet, \mathcal{N}]_*) = [[\Lambda(\mathcal{L}), \mathcal{M}]_\bullet, \mathcal{N}]_* + [[\mathcal{L}, \Lambda(\mathcal{M})]_\bullet, \mathcal{N}]_* + [[\mathcal{L}, \mathcal{M}]_\bullet, \Lambda(\mathcal{N})]_*$$

for $\mathcal{L}, \mathcal{M}, \mathcal{N} \in \mathfrak{A}$. If $\Lambda(I)$ is self-adjoint, then Λ is an $*$-derivation.

Proof of Theorem 2.2 We present the proof of the above theorem with several lemmas.

Lemma 2.6. *We show that if $\Lambda(\mathfrak{I})$ is self-adjoint then $\Lambda(i\mathfrak{I}) = \Lambda(\mathfrak{I}) = 0$.*

Proof. we know that

$$\begin{aligned}
\Lambda([[i\mathfrak{I}, \mathfrak{I}]_\bullet, \mathfrak{I}]_*) &= [[\Lambda(i\mathfrak{I}), \mathfrak{I}]_\bullet, \mathfrak{I}]_* + [[i\mathfrak{I}, \Lambda(\mathfrak{I})]_\bullet, \mathfrak{I}]_* + [[i\mathfrak{I}, \mathfrak{I}]_\bullet, \Lambda(\mathfrak{I})]_* \\
&= 2\Lambda(i\mathfrak{I}) - 2\Lambda(i\mathfrak{I})^* + 2i\Lambda(\mathfrak{I})^* + 2i\Lambda(\mathfrak{I}) + 4i\Lambda(\mathfrak{I}).
\end{aligned}$$

Also, from the other side, we have

$$\Lambda([[i\mathfrak{I}, \mathfrak{I}]_\bullet, \mathfrak{I}]_*) = 4\Lambda(i\mathfrak{I}).$$

By using above two equations, we get

$$2\Lambda(i\mathfrak{I}) - 2\Lambda(i\mathfrak{I})^* + 2i\Lambda(\mathfrak{I})^* + 2i\Lambda(\mathfrak{I}) + 4i\Lambda(\mathfrak{I}) - 4\Lambda(i\mathfrak{I}) = 0. \tag{2.1}$$

Alternatively, we have

$$\Lambda([[i\mathfrak{I}, \mathfrak{I}]_\bullet, i\mathfrak{I}]_*) = -4\Lambda(\mathfrak{I}).$$

Also, we have

$$\Lambda([[i\mathfrak{I}, \mathfrak{I}]_\bullet, i\mathfrak{I}]_*) = 2i\Lambda(i\mathfrak{I}) - 2i\Lambda(i\mathfrak{I})^* - 2\Lambda(\mathfrak{I})^* - 2\Lambda(\mathfrak{I}) + 4i\Lambda(i\mathfrak{I}).$$

From the last two expressions, we have

$$4\Lambda(\mathfrak{I}) + 2i\Lambda(i\mathfrak{I}) - 2i\Lambda(i\mathfrak{I})^* - 2\Lambda(\mathfrak{I})^* - 2\Lambda(\mathfrak{I}) + 4i\Lambda(i\mathfrak{I}) = 0 \tag{2.2}$$

Multiplying (2.2) by i, we get

$$4i\Lambda(\mathfrak{I}) - 2\Lambda(i\mathfrak{I}) + 2\Lambda(i\mathfrak{I})^* - 2i\Lambda(\mathfrak{I})^* - 2i\Lambda(\mathfrak{I}) - 4\Lambda(i\mathfrak{I}) = 0 \tag{2.3}$$

Adding (2.1) and (2.3), we get

$$\Lambda(i\mathfrak{I}) = i\Lambda(\mathfrak{I}). \tag{2.4}$$

Using (2.4) in (2.3), we get

$$\Lambda(\mathfrak{I})^* = -\Lambda(\mathfrak{I}). \tag{2.5}$$

Since $\Lambda(\mathfrak{I})$ is self-adjoint, then

$$\Lambda(\mathfrak{I}) = \Lambda(i\mathfrak{I}) = 0.$$

Lemma 2.7. Λ *preserves star, i.e.,* $\Lambda(\mathcal{L}^*) = \Lambda(\mathcal{L})^*$ *for all* $\mathcal{L} \in \mathfrak{A}$.

Proof. From Lemma 2.6, we have

$$\begin{aligned} \Lambda([[\mathcal{L}, i\mathfrak{I}]_\bullet, i\mathfrak{I}]_*) &= [[\Lambda(\mathcal{L}), i\mathfrak{I}]_\bullet, i\mathfrak{I}]_* = [[-i\Lambda(\mathcal{L}) - i\Lambda(\mathcal{L})^*, i\mathfrak{I}]_* \\ &= 2\Lambda(\mathcal{L}) + 2\Lambda(\mathcal{L})^*. \end{aligned}$$

On the other hand, we have

$$\Lambda([[\mathcal{L}, i\mathfrak{I}]_\bullet, i\mathfrak{I}]_*) = 2\Lambda(\mathcal{L}) + 2\Lambda(\mathcal{L}^*).$$

From the last two equations, we get $\Lambda(\mathcal{L}^*) = \Lambda(\mathcal{L})^*$.

Lemma 2.8. *We prove that* $\Lambda(i\mathcal{L}) = i\Lambda(\mathcal{L})$ *for all* $\mathcal{L} \in \mathfrak{A}$.

Proof. It follows from Lemma 2.6 that

$$\Lambda([[i\mathcal{L}, \mathfrak{I}]_\bullet, \mathfrak{I}]_*) = [\Lambda(i\mathcal{L}), \mathfrak{I}]_\bullet, \mathfrak{I}]_* = 2\Lambda(i\mathcal{L}) - 2\Lambda(i\mathcal{L})^*.$$

Hence

$$\Lambda(2i\mathcal{L} + 2i\mathcal{L}^*) = 2\Lambda(i\mathcal{L}) - 2\Lambda(i\mathcal{L})^*. \tag{2.6}$$

From the other side, we have

$$\Lambda([[\mathcal{L}, i\mathfrak{I}]_\bullet, \mathfrak{I}]_*) = [\Lambda(\mathcal{L}), i\mathfrak{I}]_\bullet, \mathfrak{I}]_* = -2i\Lambda(\mathcal{L}) - 2i\Lambda(\mathcal{L})^*$$

It follows that

$$\Lambda(-2i\mathcal{L} - 2i\mathcal{L}^*) = -2i\Lambda(\mathcal{L}) - 2i\Lambda(\mathcal{L})^*. \tag{2.7}$$

Adding (2.6) and (2.7), we get

$$\Lambda(i(\mathcal{L}+\mathcal{L}^*)) = i\Lambda(\mathcal{L}+\mathcal{L}^*). \tag{2.8}$$

Since (2.8) is true for any self-adjoint then for any member of $\mathcal{L}$, we have

$$\Lambda(i\mathcal{L}) = i\Lambda(\mathcal{L}).$$

Lemma 2.9. *We show that* Λ *is a derivation, i.e,* $\Lambda(\mathcal{L}\mathcal{M}) = \Lambda(\mathcal{L})\mathcal{M} + \mathcal{L}\Lambda(\mathcal{M})$.

Proof. It is easy to check that

$$\Lambda([[\mathcal{L},\mathcal{M}]_\bullet,\mathfrak{I}]_*) = 2\Lambda(\mathcal{L}\mathcal{M}^*) - 2\Lambda(\mathcal{M}\mathcal{L}^*).$$

Also, it follows from Lemma 2.6 that

$$\begin{aligned}\Lambda([[\mathcal{L},\mathcal{M}]_\bullet,\mathfrak{I}]_*) &= [[\Lambda(\mathcal{L}),\mathcal{M}]_\bullet,\mathfrak{I}]_* + [[\mathcal{L},\Lambda(\mathcal{M})]_\bullet,\mathfrak{I}]_* \\ &= 2\Lambda(\mathcal{L})\mathcal{M}^* - 2\mathcal{M}\Lambda(\mathcal{L})^* + 2\mathcal{L}\Lambda(\mathcal{M})^* - 2\Lambda(\mathcal{M})\mathcal{L}^*.\end{aligned}$$

By comparing the last two expressions, we have

$$\Lambda(\mathcal{L}\mathcal{M}^*) - \Lambda(\mathcal{M}\mathcal{L}^*) = \Lambda(\mathcal{L})\mathcal{M}^* - \mathcal{M}\Lambda(\mathcal{L})^* + \mathcal{L}\Lambda(\mathcal{M})^* - \Lambda(\mathcal{M})\mathcal{L}^* \tag{2.9}$$

On the other hand, we have

$$\Lambda([[i\mathcal{L},\mathcal{M}]_\bullet,i\mathfrak{I}]_*) = -\Lambda(\mathcal{L}\mathcal{M}^*) - \Lambda(\mathcal{M}\mathcal{L}^*).$$

By using Lemma 2.6 and Lemma 2.8, we have

$$\begin{aligned}\Lambda([[i\mathcal{L},\mathcal{M}]_\bullet,i\mathfrak{I}]_*) &= [[\Lambda(i\mathcal{L}),\mathcal{M}]_\bullet,i\mathfrak{I}]_* + [[i\mathcal{L},\Lambda(\mathcal{M})]_\bullet,i\mathfrak{I}]_* \\ &= i\Lambda(i\mathcal{L})\mathcal{M}^* - i\mathcal{M}\Lambda(i\mathcal{L})^* - \mathcal{L}\Lambda(\mathcal{M})^* - \Lambda(\mathcal{M})\mathcal{L}^* \\ &= -\Lambda(\mathcal{L})\mathcal{M}^* - \mathcal{M}\Lambda(\mathcal{L})^* - \mathcal{L}\Lambda(\mathcal{M})^* - \Lambda(\mathcal{M})\mathcal{L}^*.\end{aligned}$$

By comparing the last two expressions, we have

$$\Lambda(\mathcal{L}\mathcal{M}^*) + \Lambda(\mathcal{M}\mathcal{L}^*) = \Lambda(\mathcal{L})\mathcal{M}^* + \mathcal{M}\Lambda(\mathcal{L})^* + \mathcal{L}\Lambda(\mathcal{M})^* + \Lambda(\mathcal{M})\mathcal{L}^* \tag{2.10}$$

Adding (2.9) and (2.10), we get

$$\Lambda(\mathcal{L}\mathcal{M}^*) = \Lambda(\mathcal{L})\mathcal{M}^* + \mathcal{L}\Lambda(\mathcal{M}^*). \tag{2.11}$$

Replacing $\mathcal{M}^*$ by $\mathcal{M}$, we get

$$\Lambda(\mathcal{L}\mathcal{M}) = \Lambda(\mathcal{L})\mathcal{M} + \mathcal{L}\Lambda(\mathcal{M}).$$

Hence, Λ is a derivation. This completes the proof of Theorem 2.2.

Now, we provide an example to demonstrate the necessity of the conditions (▲) and (▼) in Theorem 2.1.

Example 2.1. *Consider* $\mathcal{A} = \{\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}\}$ *, the algebra of all lower triangular matrix of order* 2 *over the field of complex numbers* $\mathbb{C}$ *and* $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *be unity of* $\mathfrak{A}$*. The map* $* : \mathfrak{A} \to \mathfrak{A}$ *given by* $*(\mathcal{L}) = \mathcal{L}^\theta$*, where* $\mathcal{L}^\theta$ *denotes the conjugate transpose of matrix* A*, is an involution. Hence,* $\mathfrak{A}$ *is a unital* $*$*-algebra with unity* I*. Now, define a map* $\Pi : \mathfrak{A} \to \mathfrak{A}$ *such that* $\Pi\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -ic & 0 \end{pmatrix}$*. Note that* Π *is a derivation on* $\mathfrak{A}$*. So, it also satisfies*

$$\Lambda([[\mathcal{L},\mathcal{M}]_\bullet,\mathcal{N}]_*) = [[\Lambda(\mathcal{L}),\mathcal{M}]_\bullet,\mathcal{N}]_* + [[\mathcal{L},\Lambda(\mathcal{M})]_\bullet,\mathcal{N}]_* + [[\mathcal{L},\mathcal{M}]_\bullet,\Lambda(\mathcal{N})]_*$$

for all $\mathcal{L},\mathcal{M},\mathcal{N} \in \mathfrak{A}$*. Let* $P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ *is a non-trivial projection, so* $P^2 = P$ *and* $P^* = P$*. For* $W = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \neq 0 \in \mathfrak{A}$ *and hence* $W\mathfrak{A}P = (0)$ *but* $0 \neq W \in \mathfrak{A}$*. However,* Π *is not an additive* $*$*-derivation because* $\Pi(\mathcal{L}^*) \neq (\Pi(\mathcal{L}))^*$ *for some* $\mathcal{L} \in \mathfrak{A}$*.*

3. Corollaries

As a direct consequence of Theorem 2.1, we have the following corollaries:

Corollary 3.1. *Let* $\mathfrak{A}$ *be a standard operator algebra on an infinite dimensional complex Hilbert space* $\mathcal{H}$ *containing identity operator* $\mathfrak{I}$*. Suppose that* $\mathfrak{A}$ *is closed under adjoint operation. Define* $\Lambda : \mathfrak{A} \to \mathfrak{A}$ *such that*

$$\Lambda([[\mathcal{L},\mathcal{M}]_\bullet,\mathcal{N}]_*) = [[\Lambda(\mathcal{L}),\mathcal{M}]_\bullet,\mathcal{N}]_* + [[\mathcal{L},\Lambda(\mathcal{M})]_\bullet,\mathcal{N}]_* + [[\mathcal{L},\mathcal{M}]_\bullet,\Lambda(\mathcal{N})]_*$$

for all $\mathcal{L},\mathcal{M},\mathcal{N} \in \mathfrak{A}$*, then* Λ *is an additive. If* $\Lambda(\mathfrak{I})$ *is self-adjoint, then* Λ *is an* $*$*-derivation.*

Corollary 3.2. *Let* $\mathcal{M}$ *ba a factor von Neumann algebra with dim*$\mathcal{M} \geq 2$*. Define* $\Lambda : \mathcal{M} \to \mathcal{M}$ *such that*

$$\Lambda([[\mathcal{L},\mathcal{M}]_\bullet,\mathcal{N}]_*) = [[\Lambda(\mathcal{L}),\mathcal{M}]_\bullet,\mathcal{N}]_* + [[\mathcal{L},\Lambda(\mathcal{M})]_\bullet,\mathcal{N}]_* + [[\mathcal{L},\mathcal{M}]_\bullet,\Lambda(\mathcal{N})]_*$$

for all $\mathcal{L},\mathcal{M},\mathcal{N} \in \mathfrak{A}$*, then* Λ *is an additive. If* $\Lambda(\mathfrak{I})$ *is self-adjoint, then* Λ *is an* $*$*-derivation.*

Corollary 3.3. *Let* $\mathfrak{A}$ *be a prime* $*$*-algebra with unit* $\mathfrak{I}$ *containing non-trivial projection* P*. A map* $\Lambda : \mathfrak{A} \to \mathfrak{A}$ *satisfies*

$$\Lambda([[\mathcal{L},\mathcal{M}]_\bullet,\mathcal{N}]_*) = [[\Lambda(\mathcal{L}),\mathcal{M}]_\bullet,\mathcal{N}]_* + [[\mathcal{L},\Lambda(\mathcal{M})]_\bullet,\mathcal{N}]_* + [[\mathcal{L},\mathcal{M}]_\bullet,\Lambda(\mathcal{N})]_*$$

for all $\mathcal{L},\mathcal{M},\mathcal{N} \in \mathfrak{A}$*,then* Λ *is an additive. If* $\Lambda(\mathfrak{I})$ *is self-adjoint, then* Λ *is an* $*$*-derivation.*

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