EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 2, Article Number 5638
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

D-Stability Analysis of Structured Matrices Appearing in First and Second Order Economy Models

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Abstract. The D-stability of structured matrices has significant implications in system theory and decision-making systems. The notation of D-stability refers to the characterization of a dynamical model where structured stability is maintained when each eigenvalue of the system matrix remains within a designated region, we consider it in half of right complex plane $\mathbb{C}$ subject to various perturbations. The structured D-stability of time varying dynamical systems implies that the system will remain stable under certain diagonal transformations. This concept is fundamental in system theory and ensures the robust stability and performance even if system experiences structural changes or parameter uncertainties. In this paper, novel results are obtained on the computation of structured stability and structured D-stability of first order and second dynamical models with mathematical forms

$$\frac{d}{dt}(x(t)) = Ax, \quad \frac{d^2}{dt^2}(x(t)) = A\frac{d}{dt}(x(t)) + Bx, \ x \in \mathbb{R}^{n,1},$$

with matrices $A, B \in \mathbb{R}^{n,n}$. New results are developed with the necessary conditions for interconnection among stable, structured D-stable matrices and structured singular values. The numerical experimentation show how structured singular values (μ-values) behaves. The EigTool is used to sketch the behavior of pseudo-spectrum.

2020 Mathematics Subject Classifications: 15A18, 15A16, 15A23
Key Words and Phrases: Dynamic stability, D-stability, structured singular values, pseudo-spectrum

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DOI: https://doi.org/10.29020/nybg.ejpam.v18i2.5638

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1. Introduction

The concept of stable and D-stable matrices play a diverse and a key role to analyze behavior of linear, non-linear systems in economies models. The stable matrices characterize and analyze the behavior of dynamical systems over time. The stability of structured matrices in such models indicates that the deviation from an equilibrium point will diminish over the period of time. This helps to model the economies system. The stable matrices are particularly valuable in macroeconomic models and the dynamics of controller. The structured D-stable operators provides a framework for the analysis, performance and robustness of the stability. The structured D-stability play a versatile and important role once the dynamical system is subject to some external perturbations, for instance, perturbation to policy of adjustment or to vary consumer performance.

We address the problem of discussing structured stability and structured D-stability of first and second order time varying dynamical models $\dot{x} = Ax, \; and \; \ddot{x} = A\dot{x} + Bx, \; x \in \mathbb{R}^{n,1}, \; A, B \in \mathbb{R}^{n,n}$. We make use various tools from linear algebra, matrix analysis, and system theory to derive some novel results. The stability of $\dot{x} = Ax, \; x \in \mathbb{R}^{n,1}$ is considered in the sense of Schur stability, that is, $Re(\lambda_i(DA)) > 0, \; \forall i$. The Jordan canonical form of matrix-product DA is being used for derivation of novel results. Furthermore, new results are established on interconnections between structured stable matrices, structured D-stable matrices, and μ-values.

For matrices $A, B \in \mathbb{R}^{n,n}$, we present novel results on D-stability of second order dynamical system $\ddot{x} = A\dot{x} + Bx, \; x \in \mathbb{R}^{n,1}$. Here $\ddot{x}, \dot{x}$ denotes second, and first order time-derivatives, respectively. These dynamical systems are determined by computing and analyzing spectrum (eigenvalues) of $2n \times 2n$ matrix

$$\begin{pmatrix} A & B \\ I_n & O \end{pmatrix}.$$

The above structured matrix can be obtained by reducing second order system into a first order system [1]. Furthermore, the stability of this matrix is about the computation of strictly positive real part of the eigenvalues. On the other hand, the D-stability demands the determination of the positive (in the strict sense) real part of spectrum of

$$\begin{pmatrix} P_{11} & O \\ 0 & P_{22} \end{pmatrix} \begin{pmatrix} A & B \\ I_n & O \end{pmatrix}.$$

The stability further means that the solution tends towards an equilibrium vector having all zero entries [2]. The spectral properties of above $2n \times 2n$ matrix were studied and analyzed in [3] putting conditions on A, B. Further, it was shown that $2n \times 2n$ matrix have sufficient negative dominant diagonal elements which implies its stability.

The μ-value [4], a well-known mathematical technique to analyze and synthesize both robust nature and the performance of linear systems from control. The applications of μ-values includes the study and analysis of D-stability problem [5]. An exact determination of μ-values is a very hard problem and for a class of block structured perturbations with real parametric uncertainties it becomes an NP-hard problem [6]. There are various techniques to compute structured singular values in the lower-dimensional problems, we refer [7, 8] and the references therein. Computing exact value of structured singular values (μ-value) for large dimensional problems is hard and this motivates to develop the numerical techniques for its approximation from above and below. In [9, 10], a scheme was developed to numerically approximate the bounds of μ-values subject to mixed real and complex uncertainties. The linear matrix inequalities (LMI) based techniques were developed [11] to approximate μ-values from above and below.

In [12], some new results on the interconnections between the notations of H-stability, $D(\alpha)$-stability, a rank-1 perturbation to D-semistable matrices, and the μ-values were presented and analyzed. A detailed review on mathematical methods to approximate μ-values is presented in [13]. In [14], new results and a detailed analysis was presented on spectrum and pseudo-spectrum of D-stable matrices for economic models.

Proposed technique in this article bridge the links in the sense that how we quantify the interconnection between stable matrices, D-stable matrices with μ-values subject to constraints. According to best of our knowledge not a single technique is developed to give the interconnections of stability and D-stability of structured matrices across first order and second order dynamical systems and μ-values. The proposed methodology determines the novel results to discuss the stable, structured D-stable, and μ-values for structured class of matrices appearing across the economy models.

The mathematical technique presented in this article is applicable to n-dimensional complex valued, and real valued structured and unstructured matrices. The use of various tools from linear algebra, matrix analysis and system theory allow us to develop a mathematical technique rather developing a complex geometrical based approach to discuss the interconnections among structured stability, structured D-stability, and the μ-values. The proposed mathematical methodology guarantees that all the equilibrium states of an economy models shall remains in a stable mood subject to structured or unstructured perturbations. Our technique further allows us to analyze both robust and performance of economy models subject to structured or unstructured parametric uncertainties. The economic models based on linear assumptions can benefit from our presented methodology to study stability and D-stability and its interconnection with structured singular values to ensure that a number of economic indicators will remain within considerable range.

We organized our paper as: Section 2 is on conditions for stability and D-stability, where we have recalled basic notations, definitions, and results concerning our proposed study. The novel results on structured stable matrices, and structured D-stable matrices corresponding to first order dynamical systems are analyzed in section 3. In the section 4, we present the numeric experimentation's and comparison on computation and approximation of lower bounds to μ-values for structured matrices corresponding to dynamical systems under consideration. Finally, the conclusion is presented in the section 5 of this article.

2. Conditions for stability and D-stability

Let $\vec{\hat{x}} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)$ denotes steady state of dynamical system, and $f_i, \forall i$, continuously differentiable functions, then the n-dimensional continuous (in time) dynamical system may be written mathematically as

$$\frac{d}{dt}x_i(t) = f_i\left(x_1(t), x_2(t), \ldots, x_n(t)\right), \forall i = 1:n.$$

Linearized version of above dynamical model is rewritten as follow:

$$\frac{d}{dt}x_{i\beta}(t) = \sum_j \frac{\partial f_i(x)}{\partial x_j} x_{i\beta}(t),$$

where $x_{i\beta}(t) = x_i(t) - \hat{x}_i,\ \forall i$.

Remark 1. *If all the eigenvalues corresponding to coefficient matrix have strictly positive real parts, that is, $Re(\lambda_i(t)) > 0, \forall t$, then linear dynamical system is stable.*

It is well-known that coefficient matrix is Jacobian of system

$$\frac{d}{dt}x_i(t) = f_i(x_1(t), x_2(t), \ldots x_n(t)), \forall i = 1:n,$$

and is presented by

$$J_c = \begin{pmatrix} \frac{\partial f_1(\hat{x})}{\partial x_1} & \frac{\partial f_1(\hat{x})}{\partial x_n} \\ & \\ \frac{\partial f_n(\hat{x})}{\partial x_1} & \frac{\partial f_n(\hat{x})}{\partial x_n} \end{pmatrix}.$$

If $Re(\lambda_i(J_c)) > 0\ \forall i$, then system is stable. The continuous-time linear dynamic model

$$\frac{d}{dt}x(t) = Ax(t),\ x(0) = x_0,$$

with $x(t) \in \mathbb{R}^{n,1}$ be a state variable, the initial condition is $x_0 \in \mathbb{R}^{n,1}$, and $A \in \mathbb{R}^{n,n}$ matrix. An optimal and feasible solution to an optimization problem $A \in \mathbb{R}^{n,n}$ is obtain as

$$A = arg\ min \frac{dX}{dt} - \hat{A}X_{F},$$

where $\|\cdot\|_F$ denotes Frobenius matrix-norm, and the **min** is taken over $\hat{A} \in \mathbb{R}^{n,n}$, and **arg** means the argument. Furthermore, a minimal-norm solution obtained [15] to the dynamical system has the form $A = X^+ \frac{dX}{dt}$, with X^+, the pseudo-inverse to matrix X.

Remark 2. *The dynamical system*

$$\frac{d}{dt}x(t) = Ax(t), x(0) = x_0$$

may not be a stable dynamical system means that the solution matrix has the form

$$A = arg\ min \frac{dX}{dt} - \hat{A}X_{F},$$

where ***min*** *is taken over* $\hat{A} \in \mathbb{R}^{n,n}$ *and* ***arg*** *means the argument.*

For the structured stability of dynamical model (linear), the eigenvalues of $A \in \mathbb{R}^{n,n}$ must lie in right hand side of the complex plane $\mathbb{C}$.

The n-dimensional real and complex valued matrices are dented with $A \in \mathbb{R}^{n,n}$, and $A \in \mathbb{C}^{n,n}$ respectively. The given matrix $A \in \mathbb{C}^{n,n}$ is called a structured D-stable matrix if matrix product DA appears to be a stable matrix for all $D \in \hat{D}$, where $\hat{D}$ denotes set of positive diagonal structured matrices.

Remark 3. *The matrix products* DA, *and* AD *are similar which implies that* $\lambda_i(DA)\ \forall i$ *are the eigenvalues of* AD, *because* $DA = D(AD)D^{-1}$.

Some simple observations [16] to D-stability of $A \in \mathbb{C}^{n,n}$ are:

1. It is a condition which imply stabilization, and is preserved under matrix multiplication of positive structured diagonal matrices.

2. For a D-stable matrix $A \in \mathbb{C}^{n,n}$ such that $det(A) \neq 0$, then

(a) The inversion A^{-1} is D-stable matrix.

(b) For P (permutation matrix), the matrix $P^T AP$ be a D-stable matrix.

(c) For $D, E \in \hat{D}$, matrix-product DAE is a D-stable structured matrix.

(d) For A, A^* is a structured D-stable matrix where $*$ is complex conjugate transpose of a matrix.

3. For a D-stable matrix $A \in \mathbb{R}^{n,n}$, the $m \times m$ principal sub-matrix is in Euclidean closure of $m \times m$ D-stable structured matrices [17].

Remark 4. *For* $A \in \mathbb{R}^{n,n}$, *structured* D*-stable, a necessary condition is that each of the principal minor of* A *is such that their determinants are non-negative. For complete discussion, we refer [18, 19] and the references therein.*

4. Let $A \in \mathbb{C}^{n,n}$, and $\exists\ \tilde{A} \in \hat{D}$ such that for each $Re(\lambda_i(\tilde{A}A)) > 0$. Then, A is a structured D-stable matrix if and only if $det(A \pm D) \neq 0\ \forall D \in \hat{D}$.

A sufficient condition for $A \in \mathbb{C}^{n,n}$ to be a D-stable matrix is that $\lambda_i(DA + A^*D) > 0\ \forall i$, where $D \in \hat{D}$. For more details, we refer [20–22] and references therein. The another sufficient condition for $A \in \mathbb{C}^{n,n}$ to be structured D-stable matrix is that given matrix A is an M-matrix. For $A \in \mathbb{R}^{n,n}$ to be a class of an M-matrix, all of its off-diagonal elements must be less than or equal to zero while all of the principal minors to be strictly positive. The structured M-matrices are a class of structured stable-matrices [23].

For a positive structured diagonal matrix $D \in \hat{D}$ such that $AD = B = (b_{ij})$ satisfies the strict inequality condition.

$$Re(b_{ii}) > \sum_{j=1, j\neq i}^{n} |b_{ij}|,\ \forall i = 1 : n.$$

Structured matrix $A \in \mathbb{C}^{n,n}$ satisfying above strict inequality condition is known as the quasi-dominant diagonal matrix [24]. The given matrix $A \in \mathbb{R}^{n,n}$ is structured D-stable if $A = BD^{-1}$.

Remark 5. *The matrix $B = AD$ for $D \in \hat{D}$ is such that $Re(\lambda_i(B)) > 0, \forall i$. One can prove this result by using Gersgorin's circle theorem [21, 25].*

A sufficient condition to $D-$stability to $A \in \mathbb{C}^{n,n}$ is that A is a triangular matrix and further the real part of all elements of A is strictly positive, that is, $Re(a_{ii}) > 0, \forall i = 1 : n$.

Theorem 1. *[16] Any condition to a class of structured matrices implying structured stability, and remain preserve under multiplication of structured positive diagonal matrices is a sufficient condition to structured D-stable matrices.*

Following theorem [16] guarantees that not a single condition is the necessary condition for the structured D-stability of matrices.

Theorem 2. *Each of below class of structured matrices is a structured D-stable matrix.*
1. The diagonally stable structured matrices are structured D-stable matrices.
2. The structured M-matrices are structured D-stable matrices.
3. The strictly diagonally dominant matrices having principal diagonal positive are $D-$stable matrices.
4. The triangular matrices with main (principal) diagonal entries to be positive are structured D-stable matrices.
5. The sign-stable matrices are $D-$stable matrices.
6. The tri-diagonal structured P-matrices are structured D-stable.
7. The oscillatory matrices are structured D-stable matrices.
8. The Hadamard H-stable structured matrices are structured D-stable.
9. The structured P-matrices (sign symmetric) are structured D-stable matrices.

2.1. The μ-values and D-stability:

The stability of a dynamical system

$$\dot{x} = Ax, x \in \mathbb{R}^{n,1},\ A \in \mathbb{K}^{m,n},\ \mathbb{K} = \mathbb{R}(or\ \mathbb{C}),$$

demands that for given A, and for all $\Delta \in \hat{\Delta}$, $\lambda_i(I_n + M\Delta) \neq 0, \forall i$. The notation $\hat{\Delta}$ denotes a set of block-diagonal matrices,

$$\hat{\Delta} = \{diag(\delta_1 I_{r_1}, \delta_2 I_{r_2}, \ldots, \delta_S I_{r_S}; \Delta_1, \Delta_2, \ldots, \Delta_F) : \delta_i \in \mathbb{K} \ \ \forall i = 1 : S; \Delta_j \in \mathbb{K}^{m_j,m_j} \ \ \forall j = 1 : F\}.$$

The matrix problem to determine the necessary and sufficient conditions so that $\lambda_i(I_n + M\Delta) \neq 0, \forall i = 1 : n$ is a key problem in control engineering. These discussions leads us to the definition of structured singular value.

Definition 1. *[26] The structured singular value ($\mu-$value) of $M \in \mathbb{K}^{m,n}$, $m = n$ with respect to $\hat{\Delta}$ the set of block-diagonal matrices is defined as*

$$\mu_{\hat{\Delta}}(M) := \begin{cases} 0, & \text{if } det(I_n - M\Delta) \neq 0 \ \forall \Delta \in \hat{\Delta} \\ \frac{1}{min\{\|\Delta\|_2 : \Delta \in \hat{\Delta},\ det(I_n - M\Delta) = 0\}}, & \text{else} \end{cases}$$

where ***min*** *is taken over $\Delta \in \hat{\Delta}$, $\|\cdot\|_2$ represent maximum singular value, that is, σ_{max}.*

2.1.1. Properties of μ-values

The following properties of $\mu-$values are easily proven and the proofs are available in literature.

1. For $\alpha \in \mathbb{C}$, $\mu_{\hat{\Delta}}(\alpha M) = |\alpha| \mu_{\hat{\Delta}}(M)$.
2. For an identity matrix I_n, $\mu_{\hat{\Delta}}(I_n) = 1$.
3. For given matrices A, B, $\mu_{\hat{\Delta}}(AB) \leq \|A\|_2 \, \mu_{\hat{\Delta}}(B)$.
4. $\mu(\Delta) = \|\Delta\|_2$, $\forall \Delta \in \hat{\Delta}$.

An alternative expression to $\mu-$values can be easily follows from the definition.

Lemma 1. *[26] For $M \in \mathbb{K}^{n,n}$,*

$$\mu_{\hat{\Delta}_1}(M) = max \ \rho(\Delta M),$$

where max is taken over $\Delta \in \hat{\Delta}_1$ with

$$\hat{\Delta}_1 := \{\Delta \in \hat{\Delta} : \sigma_{max}(\Delta) \leq 1\},$$

and the mathematical notation $\rho(\cdot)$ represents spectral radius (max of absolute value of an eigenvalue) of a matrix.

Remark 6. *The above lemma applies the continuity of $\mu \colon \mathbb{C}^{n,n} \to \mathbb{R}$ which encompass the continuity property of spectral radius as well as the max functions. It can be shown that for $S = 1, F = 0, r_1 = n$,*

$$\mu_{\hat{\Delta}}(M) = \rho(M),$$

where $\hat{\Delta} = \{\delta I_n : \delta \in \mathbb{C}\}$. Furthermore, if $S = 0, F = 1$, then

$$\mu_{\hat{\Delta}}(M) = \sigma_{max}(M).$$

The interconnection between structured singular values, the largest singular value, and the spectral radius of $M \in \mathbb{K}^{n,n}$ is given by

$$\rho(M) \leq \mu_{\hat{\Delta}}(M) \leq \sigma_{max}(M).$$

3. New results for stability analysis, and D-stability analysis of first and second order dynamical systems

In this particular section of the paper, we present new results for structured stability, and structured D-stability analysis of first and second order dynamical systems. Furthermore, we establish the necessary conditions on the interconnection among the structured stability, structured D-stability, and μ-values. We make use of various tools from linear algebra, matrix analysis and system theory to derive new results.

3.1. Stability and D-stability of first order dynamical systems

First, we discuss and provide results on the stability of $\dot{x} = Ax,\ x \in \mathbb{R}^{n,1},\ A \in \mathbb{R}^{n,n}$.

Assumption 1. *The given matrix $A \in \mathbb{R}^{n,n}$ is a symmetric matrix, that is, $A^T = A$.*

Theorem 3 provides mathematical results on structured stability dynamical system which means that the real part of its largest eigenvalue is strictly greater than real part of the eigenvalues of all remaining eigenvalues.

Theorem 3. *The first order dynamic model $\dot{x} = Ax$, $x \in \mathbb{R}^{n,1}$ is stable if*

$$Re(\lambda_1(A)) > |Re(\mu_i(A))| > 0,\ \forall i,$$

with $\lambda_1(A)$, the largest eigenvalue, and $\mu_i(A), \forall i$ are the eigenvalues other than $\lambda_1(A)$.

Proof. The $Re(\lambda_1(A)) > 0$ because $\lambda_i(A) \in \mathbb{R}, \forall i$ and $\sum_i(\lambda_i(A)) = Trace(A) > 0$, and hence it follows that $\lambda_1(A) > 0$. Now, we aim to show that

$$\mu_i(A) \neq \lambda_1(A), Re(\lambda_1(A)) > |Re(\mu_i(A))| > 0, \forall i.$$

Let $\vec{v}_j$ be the normalized eigenvector for $\lambda_1 > \mu_i, \forall i$, then,

$$\sum_i a_{ij}\vec{v}_j = \mu_i \vec{v}_j,\ \forall i.$$

We set $|v_j| = x_j$, then

$$0 < Re(\lambda_1(A)) = \sum_{ij} a_{ij}\vec{v}_i\vec{v}_j = |\sum_{ij} a_{ij}\vec{v}_i\vec{v}_j| \leq \sum_{ij} a_{ij}x_ix_j.$$

Let x_j be an eigenvector for $\lambda_1(A)$, then

$$\sum_j a_{ij}x_j = \lambda_1 x_i, \forall i.$$

If we take $x_i = 0$, for some i, then $a_{ij} > 0, \forall j$. From this we have that each of $x_j = 0$, this is not possible, and hence each $x_j > 0$, the non-decreasing condition of $Re(\lambda_1(A))$. The non-decreasing condition allows us to have that

$$Re(\lambda_1(A)) > \sum_{ij} a_{ij}|v_i||v_j| \geq |\sum_{ij} a_{ij}v_i^* v_j| = |\mu_i(A)| = |Re(\mu_i(A))| > 0.$$

Theorem 4 shows that first order dynamical system is stable if the quadratic form $x^T Ax$ is strictly positive if and only if $Re(\lambda_i)$, that is, real part of all the eigenvalues of a real valued matrix A be strictly positive. Here, $x \in \mathbb{R}^{n,1}$ is a non-zero vector.

Theorem 4. *Let dynamical system $\dot{x} = Ax, x \in \mathbb{R}^{n,1}$ is stable, then $x^T Ax > 0, \forall x \in \mathbb{R}^{n,1}$ iff $Re(\lambda_i(A)) > 0, \forall i$.*

Proof. For the stability of dynamical system $\dot{x} = Ax, x \in \mathbb{R}^{n,1}$, we aim to show that for $z = x + \mathbf{i}y, x, y \in \mathbb{R}^{n,1}$, the quantity $z^* Az > 0, z \in \mathbb{C}^{n,1}$. Now, we take $(y^T Ax)^T = x^T Ay, A^T = A$. Furthermore,

$$z^* Az = (x + \mathbf{i}y)^* A(x + \mathbf{i}y) = x^T Ax + y^T Ay + \mathbf{i}(x^T Ay - y^T Ax) = x^T Ax + y^T Ay > 0.$$

The strict inequality holds true if one of x and y be non-zero. This imply that $Re(\lambda_i(A)) > 0, \forall i$.

Theorem 5 shows that first order dynamical system is stable if the quadratic form $x^T Ax$ is strictly positive if and only if $Re(\lambda_i)$, that is, real part of all the eigenvalues of a complex valued matrix A is strictly positive. Here $x \in \mathbb{C}^{n,1}$ is a non-zero vector corresponding to the eigenvalues of A.

Theorem 5. *The dynamical system $\dot{x} = Ax$, $x \in \mathbb{C}^{n,1}$ is stable, then $x^* Ax > 0 \Leftrightarrow Re(\lambda_i(A)) > 0, \forall i$.*

Proof. The quantity $x^* Ax$ is a real valued and positive, then for all $x \in \mathbb{C}^{n,1}$, $x^* Ax$ is real valued. Furthermore,

$$Re(\lambda_i(A)) = v_i^* (\lambda_i v_i) = v_i^* A v_i, v_i \in \mathbb{C}^{n,1}, \forall i$$

being unit eigenvectors corresponding to λ_i of A. This ensures that $Re(\lambda_i(A)) > 0, \forall i$. On the other hand if we take A has only one eigenvalue to be positive, then

$$A = U \Lambda U^*,$$

with U being a unitary matrix, $U^* U = I_n$ and $\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$. From this we have that

$$x^* Ax = x^* U \Lambda U^* x = (U^* x)^* \Lambda (U^* x) = \sum_i Re(\lambda_i) |v_i^* x|^2 \geq 0,$$

for $Re(\lambda_i(A)) > 0$, and for some non-zero $v_i^* x$ for a non-zero x. This complete the proof.

Theorem 6 shows structured D-stability analysis of the first order dynamical model. For this, we aim to prove that A is structured D-stable if for each positive structured diagonal matrix D, the matrix-product DA is structured stable, that is, the real part of each of the eigenvalue (spectrum) of DA is strictly positive.

Theorem 6. *The dynamical system $\dot{x} = Ax$, $x \in \mathbb{R}^{n,1}$ is structured D-stable if DA is structured stable matrix for each $D = diag(d_{ii}),\ d_{ii} > 0,\ \forall\ i = 1 : n$.*

Proof. We aim to show that $Re(\lambda_i(DA)) > 0, \forall\, i = 1 : n$. In turn this will imply that dynamical system $\dot{x} = Ax, x \in \mathbb{R}^{n,1}$ is D-stable. The Jordan canonical form of DA is

$$\hat{D}^{-1} DA\hat{D} = J.$$

Let λ_i be an eigenvalues of DA, then

$$J = \begin{pmatrix} J_1 & & 0 \\ & & \\ 0 & & J_m \end{pmatrix},$$

with

$$J_j = \begin{pmatrix} \lambda_i & 1 & 0 & & & 0 \\ 0 & \lambda_i & 1 & & & \\ 0 & 0 & \lambda_i & & & \\ & & & & & 0 \\ & & & & & 1 \\ 0 & & & 0 & 0 & \lambda_i \end{pmatrix}.$$

Let $x = \hat{D}y$, then $\dot{x} = Ax$, $x \in \mathbb{R}^{n,1}$ becomes $\dot{y} = Jy$. For y_0 being an initial condition, we have that, $y(t) = e^{Jt}y_0$. Now,

$$e^{\dot{J}t} = \begin{pmatrix} e^{J_1 t} & & 0 \\ & & \\ 0 & & e^{J_m t} \end{pmatrix}.$$

The Jordan blocks $J_j,\ j = 1 : m$ can be written as $J_j = \lambda_i I_{\mu_i} + M_{\mu_i}$, where

$$M_{\mu_i} = \begin{pmatrix} 0 & 1 & 0 & & & 0 \\ 0 & 0 & 1 & & & \\ 0 & 0 & 0 & & & \\ & & & & & 0 \\ & & & & & 1 \\ 0 & & & 0 & 0 & 0 \end{pmatrix},$$

with I_{μ_i}, an identity matrix with order μ_i, $i = 1 : m$. Since, $e^{\lambda_i I_{\mu_i} t} = e^{\lambda_i t} I_{\mu_i}$, which can be rewritten as

$$e^{J_i t} = e^{\lambda_i t} e^{M_{\mu_i} t}.$$

Thus,

$$e^{\dot{J}_i t} = e^{Re(\lambda_i)t} e^{Img(\lambda_i)t} R,$$

where

$$R = \begin{pmatrix} 1 & t & \frac{t^2}{2!} & & \frac{t^{\mu_i - 1}}{(\mu_i - 1)} \\ & & & & \\ & & & & \frac{t^2}{2!} \\ & & & & t \\ & & & & 1 \end{pmatrix}.$$

Define $\beta_i = \sum_i \mu_i;\ \hat{\beta}_i = \{\beta_{i-1} + t, \ldots, \beta_i\}$, then we have that

$$y_k(t) = e^{Re(\lambda_i)t} e^{Img(\lambda_i)t} \sum_{\hat{q}} M_{\tilde{p}_k \hat{q}}(\mu_i, t) y_{\circ \hat{q}}$$

where $\tilde{p}_k = k - \beta_{i-1}$.

Consider that $\mu := \max \mu_i, i = 1 : m$, then we have that

$$y_k(t) = e^{Re(\lambda_i)t} e^{Img(\lambda_i)t} t^{\mu} \sum_{\hat{q}} M_{\tilde{p}_k \hat{q}}(\mu_i, t) t^{-\mu} y_{\circ \hat{q}}.$$

Thus,

$$|y_k(t)| = e^{Re(\lambda_i)t} t^{\mu} | \sum_{\hat{q}} M_{\tilde{p}_k \hat{q}}(\mu_i, t) t^{-\mu} y_{\circ \hat{q}}|.$$

For $t \to \infty$, $| \sum_{\hat{q}} M_{\tilde{p}_k \hat{q}}(\mu_i, t) t^{-\mu} y_{\circ \hat{q}}| < \epsilon$, $\epsilon > 0$. Since, $Re(\lambda_i) > 0,\ \forall i$, then we have that

$$e^{-Re(\lambda_i)t} > t^{\mu},\ i = 1 : m.$$

This further implies that $y_k(t) \to 0$ as $t \to \infty$. Further, consider that $\lambda_{i\circ} = Re(\lambda_{i\circ}) > 0$. The initial state $\hat{y}_0$ so that $\hat{y}_{\circ \hat{q}} = \alpha > 0$, $\hat{y}_{\circ \gamma} = 0$, $\gamma \neq \hat{q}$.

If $Re(\lambda_{i\circ}) > 0$. Then

$$|y_k(t)| = e^{Re(\lambda_{i\circ})t} | \sum_{\hat{q}} M_{\tilde{p}_k \hat{q}}(\mu_{i\circ}, t) \hat{y}_{\circ \hat{q}}| \to \infty,$$

as $t \to 0$. This ensure that $|y_k(t)| = \hat{y}_{\circ k} = \alpha > 0$. Thus, this concludes that $Re(\lambda_i) > 0$.

Theorem 7. *The linear dynamical model* $\dot{x} = Ax$, $x \in \mathbb{R}^{n,1}$ *is structured D-stable if* $\lambda_j(A + \boldsymbol{i}D) \neq 0$, $D = diag(d_{ii}) > 0$, $\forall\ j = 1 : n$.

Proof. Consider that A in $\dot{x} = Ax$ is D-stable, means that, for all positive diagonal matrices P, $Re(\lambda_j(PA)) > 0, \forall j$. The matrix-product PA does not have $\mathbf{i} = \sqrt{-1}$ as it's one of eigenvalue. Also, we have that

$$A + \mathbf{i}D = D(D^{-1}A + \mathbf{i}I_n),$$

such that

$$\lambda_j(D(D^{-1}A + \mathbf{i}I_n)) \neq 0,\ \forall j,$$

where I_n being $n \times n$ identity matrix.

On the other hand, let A is not structured D-stable. This further implies that $Re(\lambda_j(PA)) \leq 0, \forall j$. Then for a positive structured diagonal matrix $\hat{D}$, $\hat{D}PA$ is not structured stable but $\hat{D}A$ is stable, that is, $Re(\lambda_j(\hat{D}A)) > 0, \forall j$. This follows form the fact that for $0 < t \leq 1$, and for some $\beta > 0, \mathbf{i} = \sqrt{-1}$ is an eigenvalue of $\frac{1}{\beta}(tP + (1-t)I_n)\hat{D}A$, where $D = \beta(tP + (1-t)I_n)^{-1}$.

Theorem 8. *The dynamical system $\dot{x} = Ax, \ x \in \mathbb{R}^{n,1}$ is D-stable if*

$$\prod_{k=1}^{n} (\lambda_k(AD^{-1} + DA^{-1})) > 0, \ \forall k.$$

Proof.

Consider the partitioned matrix $\begin{pmatrix} A & -D \\ D & A \end{pmatrix}$. The Schur complement of the partitioned matrix is

$$A + DA^{-1}D = AD^{-1}D + DA^{-1}D = (AD^{-1} + DA^{-1})D.$$

By computing the kth eigenvalue of $(A + DA^{-1}D)$, we have that

$$\prod_{k=1}^{n} (\lambda_k(A + DA^{-1}D)) = \prod_{k=1}^{n} \lambda_k((AD^{-1} + DA^{-1})D) == \prod_{k=1}^{n} \lambda_k(A) \prod_{k=1}^{n} \lambda_k(AD^{-1} + DA^{-1}) \prod_{k=1}^{n} \lambda_k(D).$$

Since,

$$\prod_{k=1}^{n} \lambda_k(A + DA^{-1}D) > 0 \Leftrightarrow \prod_{k=1}^{n} \lambda_k(AD^{-1} + DA^{-1}) > 0.$$

As $\lambda_k(A) \leq 0, \lambda_k(D) \leq 0, \ \ \forall\, k$, and hence $\lambda_k(AD^{-1}) \leq 0, \lambda_k(DA^{-1}) \leq 0$.

3.2. Stability and D-stability of second order dynamical systems

In [27], some close interconnections between μ-values and structured D-stability of real squared matrices were developed. It was further shown that real-valued square matrix is structured D-stable if and only if $\mu_{\hat{\Delta}}(M) < 1$ for a given $M \in \mathbb{C}^{n,n}$. The following theorem gives an interconnection between $\mu-$value and $D-$stability.

Theorem 9. *[27] Consider $M \in \mathbb{R}^{n,n}$. Then structured matrix M is structured D-stable matrix $\Leftrightarrow$ M is structured stable matrix and*

$$0 \leq \mu_{\hat{\Delta}}((\boldsymbol{i}I_n + M)^{-1}(\boldsymbol{i}I_n - M)) < 1.$$

An extension to strong structured D-stability condition for a given structured matrix was derived in [27].

Theorem 10. *[27] Let $M \in \mathbb{R}^{n,n}$, and let $\mu_{\hat{\Delta}}(\cdot)$ is the μ-value of a matrix. Then structured matrix M is strongly D-stable iff $Re(\lambda_i(M)) > 0, \forall i$ and $\exists\, \varepsilon > 0$ so that $0 \leq \mu_{\hat{\Delta}}(M) < 1$, where*

$$M = \begin{pmatrix} (\boldsymbol{i}I_n + M)^{-1}(\boldsymbol{i}I_n - M) & 2\boldsymbol{i}(\boldsymbol{i}I_n + M)^{-1} \\ \varepsilon(\boldsymbol{i}I_n + M)^{-1} & -\varepsilon(\boldsymbol{i}I_n + M)^{-1} \end{pmatrix}.$$

Lemma 2. *[28] Let* $M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$. *If* M *is strongly D-stable matrix,* M_{22} *and* M^c_{22} *are both strongly D-stable matrices, that is, for* M_{22} *or* M^c_{22}, $\exists\ \delta > 0$, *such that, to each* $E_m \in \mathbb{R}^{m,m}$, $\|E_m\| < \delta$,

$$\lambda_i(I_n - (\boldsymbol{i}I_n + M^c_{22} + E_m)^{-1}(\boldsymbol{i}I_n - M^c_{22} - E_m)Q_m) \neq 0,$$

for $Q_m \in \hat{\Delta}$.

Remark 7. *The matrix* M^c_{22} *is defined by Schur complement, that is,* $M^c_{22} = M_{22} - M_{21}M_{11}^{-1}M_{12}$ *with* M_{11} *such that* $\lambda_i(M_{11}) \neq 0\ \forall i$.

Remark 8. *A sufficient condition [29, 30] for structured matrix* M *to be strong D-stable structured matrix is that*

$$\inf \sigma_1(D(I_n + M)^{-1}(I_n - M)D^{-1}) < 1,$$

where ***inf*** *is taken over* $D \in \hat{D}$.

Theorem 11 shows that second order dynamical system is structured D-stable for matrices A, B such that $\begin{pmatrix} A & B \\ I_n & O \end{pmatrix}$ is stable, and its non-negative structured singular value is bounded by 1.

Theorem 11. *The dynamical system* $\ddot{x} = A\dot{x} + Bx, x \in \mathbb{R}^{n,1}$ *with* $A, B \in \mathbb{R}^{n,n}$, *is structured D-stable if the matrix* $\begin{pmatrix} A & B \\ I_n & O \end{pmatrix}$ *is structured stable and*

$$0 \leq \mu_{\hat{\Delta}}\left(\begin{pmatrix} A & B \\ I_n & O \end{pmatrix}^{-2}\right) < 1.$$

Proof.

We assume that $\begin{pmatrix} A & B \\ I_n & O \end{pmatrix}$ is stable matrix and aim to show that

$$0 \leq \mu_{\hat{\Delta}}\left(\begin{pmatrix} A & B \\ I_n & O \end{pmatrix}\right) < 1.$$

From [31] it follows that $\begin{pmatrix} A & B \\ I_n & O \end{pmatrix}$ is D-stable if and only if $Re\left(\lambda_i\begin{pmatrix} A & B \\ I_n & O \end{pmatrix}\right) > 0\ \forall i$ and

$$\prod_{i=1}^{n} \lambda_i\left(\begin{pmatrix} X & -P \\ P & X \end{pmatrix}\right) \neq 0\ \forall i,\ \forall P \in \hat{D},\ where X = \begin{pmatrix} A & B \\ I_n & O \end{pmatrix}.$$

Thus,

$$\prod_{i=1}^{n} \lambda_i\left(\begin{pmatrix} X & -P \\ P & X \end{pmatrix}\right) \neq 0 \Longrightarrow \prod_{i=1}^{n} \lambda_i(X^2 - PX^{-1}PX) \neq 0,\ \forall i.$$

Furthermore, we have that

$$\prod_{i=1}^{n} \lambda_i(X^2 - PX^{-1}PX) \neq 0 \Longrightarrow \prod_{i=1}^{n} \lambda_i(I_n - X^{-2}\hat{P}) \neq 0,$$

where $\hat{P} = P \in \hat{D}$, a positive diagonal matrix. Thus, finally we have that

$$\prod_{i=1}^{n} \lambda_i(I_n - X^{-2}\hat{P}) \neq 0 \Longrightarrow 0 \leq \mu_{\hat{\Delta}} \left(\begin{pmatrix} A & B \\ I_n & O \end{pmatrix}^{-2} \right) < 1.$$

Lemma 3. *Let dynamical system* $\ddot{x} = A\dot{x} + Bx$, $x \in \mathbb{R}^{n,1}$, *with* $A \in \mathbb{R}^{n,n}$ *such that* $a_{ii} < 0$, $\forall i = 1 : n$. *Then dynamical system is structured D-stable if* $\begin{pmatrix} A & bI_n \\ I_n & O \end{pmatrix}$ *is stable for* $b < 0$ *and* $0 \leq \mu_{\hat{\Delta}} \left(\begin{pmatrix} A & B \\ I_n & O \end{pmatrix} \right) < 1$.

Proof. Assume $\begin{pmatrix} A & bI_n \\ I_n & O \end{pmatrix}$ is structured stable, and we aim to show that

$$0 \leq \mu_{\hat{\Delta}} \left(\begin{pmatrix} A & bI_n \\ I_n & O \end{pmatrix} \right) < 1.$$

The matrix $\begin{pmatrix} A & bI_n \\ I_n & O \end{pmatrix}$ is D-stable $\Leftrightarrow Re \left(\lambda_i \begin{pmatrix} A & bI_n \\ I_n & O \end{pmatrix} \right) > 0$, and

$$\prod_{i=1}^{n} \lambda_i \left(\begin{pmatrix} X & -P \\ P & X \end{pmatrix} \right) \neq 0, \forall i, \ \forall P \in \hat{D}, \ where X = \begin{pmatrix} A & bI_n \\ I_n & O \end{pmatrix}.$$

Thus,

$$\prod_{i=1}^{n} \lambda_i \left(\begin{pmatrix} X & -P \\ P & X \end{pmatrix} \right) \neq 0 \Longrightarrow \prod_{i=1}^{n} \lambda_i(X^2 - PX^{-1}PX) \neq 0, \ \forall i, \ \forall P \in \hat{D}.$$

Also,

$$\prod_{i=1}^{n} \lambda_i(X^2 - PX^{-1}PX) \neq 0 \Longrightarrow \prod_{i=1}^{n} \lambda_i(I_n - X^{-2}\hat{P}) \neq 0,$$

where $\hat{P} = P \in \hat{D}$, a positive diagonal matrix. So, finally we have that

$$\prod_{i=1}^{n} \lambda_i \left(I_n - \begin{pmatrix} A & bI_n \\ I_n & O \end{pmatrix}^{-2} \hat{P} \right) \neq 0 \Longrightarrow 0 \leq \mu_{\hat{\Delta}} \left(\begin{pmatrix} A & bI_n \\ I_n & O \end{pmatrix}^{-2} \right) < 1.$$

Theorem 12 shows that second order dynamical system is D-stable for matrices A, B such that $\begin{pmatrix} aI_n & B \\ I_n & O \end{pmatrix}$ is stable for $a < 0$, and the non-negative structured singular value is bounded by 1.

Theorem 12. *The dynamical system $\ddot{x} = A\dot{x} + Bx$, $x \in \mathbb{R}^{n,1}$, is structured D-stable if $\begin{pmatrix} aI_n & B \\ I_n & O \end{pmatrix}$, $a < 0$ is stable and $0 \le \mu_{\hat{\Delta}}(X) < 1$, where*

$$X = \left(\begin{pmatrix} \boldsymbol{i}I_n & O \\ O & \boldsymbol{i}I_n \end{pmatrix} + \begin{pmatrix} P_{11} & O \\ O & P_{22} \end{pmatrix} \begin{pmatrix} aI_n & B \\ I_n & O \end{pmatrix} + \begin{pmatrix} aI_n & B \\ I_n & O \end{pmatrix}^T \begin{pmatrix} P_{11} & O \\ O & P_{22} \end{pmatrix} \right)^{-1} \left(\begin{pmatrix} \boldsymbol{i}I_n & O \\ O & \boldsymbol{i}I_n \end{pmatrix} - \right.$$

$$\left. \begin{pmatrix} P_{11} & O \\ O & P_{22} \end{pmatrix} \begin{pmatrix} aI_n & B \\ I_n & O \end{pmatrix} - \begin{pmatrix} aI_n & B \\ I_n & O \end{pmatrix}^T \begin{pmatrix} P_{11} & O \\ O & P_{22} \end{pmatrix} \right).$$

Proof. To show that $\begin{pmatrix} aI_n & B \\ I_n & O \end{pmatrix}$ is D-stable matrix iff

$$Re \left(\lambda_i \left(\begin{pmatrix} P_{11} & O \\ O & P_{22} \end{pmatrix} \begin{pmatrix} aI_n & B \\ I_n & O \end{pmatrix} + \begin{pmatrix} aI_n & B \\ I_n & O \end{pmatrix}^T \begin{pmatrix} P_{11} & O \\ O & P_{22} \end{pmatrix} \right) \right) > 0, \ \forall i,$$

$P \in \hat{D}$ a positive diagonal matrix, one can follow Theorem-1 of [31].

To prove that $\begin{pmatrix} aI_n & B \\ I_n & O \end{pmatrix}$ is D-stable iff it's structured singular values are non-negative and bounded by 1, we let Δ, a block-diagonal structure as

$$\Delta = \left(\begin{pmatrix} \mathbf{i}I_n & O \\ O & \mathbf{i}I_n \end{pmatrix} - \begin{pmatrix} P_{11} & O \\ O & P_{22} \end{pmatrix} \right) \left(\begin{pmatrix} \mathbf{i}I_n & O \\ O & \mathbf{i}I_n \end{pmatrix} + \begin{pmatrix} P_{11} & O \\ O & P_{22} \end{pmatrix} \right)^{-1}, \Delta \in \hat{\Delta}$$

Since,

$$\lambda_i \left(\begin{pmatrix} P_{11} & O \\ O & P_{22} \end{pmatrix} \begin{pmatrix} aI_n & B \\ I_n & O \end{pmatrix} + \begin{pmatrix} aI_n & B \\ I_n & O \end{pmatrix}^T \begin{pmatrix} P_{11} & O \\ O & P_{22} \end{pmatrix} \right) \neq 0, \ \forall P \in \hat{D}.$$

This implies that

$$\lambda_i \left(\begin{pmatrix} P_{11} & O \\ O & P_{22} \end{pmatrix} \begin{pmatrix} aI_n & B \\ I_n & O \end{pmatrix} + \begin{pmatrix} aI_n & B \\ I_n & O \end{pmatrix}^T \begin{pmatrix} P_{11} & O \\ O & P_{22} \end{pmatrix} + \begin{pmatrix} \mathbf{i}P_{11} & O \\ O & \mathbf{i}P_{22} \end{pmatrix} \right) \neq 0, \ \forall i$$

iff $\lambda_i(Y) \neq 0$, $\forall i$, $\forall \Delta \in \hat{\Delta}$, where

$$Y = \left(\begin{pmatrix} P_{11} & O \\ O & P_{22} \end{pmatrix} \begin{pmatrix} aI_n & B \\ I_n & O \end{pmatrix} + \begin{pmatrix} aI_n & B \\ I_n & O \end{pmatrix}^T \begin{pmatrix} P_{11} & O \\ O & P_{22} \end{pmatrix} + i \left[\begin{pmatrix} \mathbf{i}I_n & O \\ O & \mathbf{i}I_n \end{pmatrix} + \begin{pmatrix} \Delta_{11} & O \\ O & \Delta_{22} \end{pmatrix} \right] \right)^{-1}$$

$$\left(\begin{pmatrix} \mathbf{i}I_n & O \\ O & \mathbf{i}I_n \end{pmatrix} - \begin{pmatrix} \Delta_{11} & O \\ O & \Delta_{22} \end{pmatrix} \right).$$

Also, $\lambda_i(Z) \neq 0$, $\forall \Delta \in \hat{\Delta}$, where

$$Z = \begin{pmatrix} \mathbf{i}I_n & O \\ O & \mathbf{i}I_n \end{pmatrix} + \begin{pmatrix} P_{11} & O \\ O & P_{22} \end{pmatrix} \begin{pmatrix} aI_n & B \\ I_n & O \end{pmatrix} + \begin{pmatrix} aI_n & B \\ I_n & O \end{pmatrix}^T \begin{pmatrix} P_{11} & O \\ O & P_{22} \end{pmatrix} - \left(i \begin{pmatrix} \mathbf{i}I_n & O \\ O & \mathbf{i}I_n \end{pmatrix} - \begin{pmatrix} P_{11} & O \\ O & P_{22} \end{pmatrix} \right.$$

$\begin{pmatrix} aI_n & B \\ I_n & O \end{pmatrix}$-$\begin{pmatrix} aI_n & B \\ I_n & O \end{pmatrix}^T \begin{pmatrix} P_{11} & O \\ O & P_{22} \end{pmatrix} \begin{pmatrix} \Delta_{11} & O \\ O & \Delta_{22} \end{pmatrix}$.

Finally, $\lambda_i(W) \neq 0, \ \forall i, \ \forall \Delta \in \hat{\Delta}$, where

$$W = \begin{pmatrix} I_n & O \\ O & I_n \end{pmatrix} - \left(\begin{pmatrix} \mathbf{i}I_n & O \\ O & \mathbf{i}I_n \end{pmatrix} + \begin{pmatrix} P_{11} & O \\ O & P_{22} \end{pmatrix} \begin{pmatrix} I_n & O \\ O & I_n \end{pmatrix} + \begin{pmatrix} aI_n & B \\ I_n & O \end{pmatrix}^T \begin{pmatrix} P_{11} & O \\ O & P_{22} \end{pmatrix} \right)^{-1} \left(\begin{pmatrix} \mathbf{i}I_n & O \\ O & \mathbf{i}I_n \end{pmatrix} - \right.$$

$\begin{pmatrix} P_{11} & O \\ O & P_{22} \end{pmatrix} \begin{pmatrix} aI_n & B \\ I_n & O \end{pmatrix}$ - $\begin{pmatrix} aI_n & B \\ I_n & O \end{pmatrix}^T \begin{pmatrix} P_{11} & O \\ O & P_{22} \end{pmatrix} \begin{pmatrix} \Delta_{11} & O \\ O & \Delta_{22} \end{pmatrix}$.

This is necessary condition for $\mu-$value to be less than 1. This complete the proof.

Theorem 13. *The dynamical system* $\ddot{x} = A\dot{x} + Bx, x \in \mathbb{R}^{n,1}$, *with* $A, B \in \mathbb{R}^{n,m}$ *is D-stable if* $\begin{pmatrix} aI_n & B \\ I_n & O \end{pmatrix}$ *is stable and* $0 \leq \mu_{\hat{\Delta}}(X) < 1$, *where*

$$X = \left(\begin{pmatrix} \mathbf{i}I_n & O \\ O & \mathbf{i}I_n \end{pmatrix} + \begin{pmatrix} aI_n & B \\ I_n & O \end{pmatrix} \right)^{-1} \left(\begin{pmatrix} \mathbf{i}I_n & O \\ O & \mathbf{i}I_n \end{pmatrix} - \begin{pmatrix} aI_n & B \\ I_n & O \end{pmatrix} \right).$$

Proof.

The matrix $\begin{pmatrix} aI_n & B \\ I_n & O \end{pmatrix}$ is D-stable iff $\begin{pmatrix} aI_n & B \\ I_n & O \end{pmatrix}$ is stable and

$$\lambda_i \left(\begin{pmatrix} aI_n & B \\ I_n & O \end{pmatrix} + \mathbf{i} \begin{pmatrix} P_{11} & O \\ O & P_{22} \end{pmatrix} \right) \neq 0, \forall i,$$

for any $P = \begin{pmatrix} P_{11} & O \\ O & P_{22} \end{pmatrix}$ positive diagonal matrix.

Consider that $\begin{pmatrix} aI_n & B \\ I_n & O \end{pmatrix}$ is D-stable, that is

$$\lambda_i \left(\begin{pmatrix} aI_n & B \\ I_n & O \end{pmatrix} + \mathbf{i} \begin{pmatrix} P_{11} & O \\ O & P_{22} \end{pmatrix} \right) \neq 0, \forall i.$$

Let

$$\Delta = \left(\begin{pmatrix} \mathbf{i}I_n & O \\ O & \mathbf{i}I_n \end{pmatrix} - \begin{pmatrix} P_{11} & O \\ O & P_{22} \end{pmatrix} \right) \left(\begin{pmatrix} \mathbf{i}I_n & O \\ O & \mathbf{i}I_n \end{pmatrix} + \begin{pmatrix} P_{11} & O \\ O & P_{22} \end{pmatrix} \right)^{-1}$$

be a diagonal matrix and $\Delta \in \hat{\Delta}$. Then,

$$\begin{pmatrix} P_{11} & O \\ O & P_{22} \end{pmatrix} = \left(\begin{pmatrix} \mathbf{i}I_n & O \\ O & \mathbf{i}I_n \end{pmatrix} + \begin{pmatrix} \Delta_{11} & O \\ O & \Delta_{22} \end{pmatrix} \right)^{-1} \left(\begin{pmatrix} \mathbf{i}I_n & O \\ O & \mathbf{i}I_n \end{pmatrix} - \begin{pmatrix} \Delta_{11} & O \\ O & \Delta_{22} \end{pmatrix} \right)$$

is positive diagonal matrix if $\Delta \in \hat{\Delta}$. As,

$$\lambda_i \left(\begin{pmatrix} aI_n & B \\ I_n & O \end{pmatrix} + \mathbf{i} \begin{pmatrix} P_{11} & O \\ O & P_{22} \end{pmatrix} \right) \neq 0,$$

in turn this imples that $\lambda_i(X) \neq 0, \forall \Delta \in \hat{\Delta}$, where

$$X = \begin{pmatrix} aI_n & B \\ I_n & O \end{pmatrix} + \mathbf{i}\left(\begin{pmatrix} \mathbf{i}I_n & O \\ O & \mathbf{i}I_n \end{pmatrix} + \begin{pmatrix} \Delta_{11} & O \\ O & \Delta_{22} \end{pmatrix}\right)^{-1}\left(\begin{pmatrix} \mathbf{i}I_n & O \\ O & \mathbf{i}I_n \end{pmatrix} - \begin{pmatrix} \Delta_{11} & O \\ O & \Delta_{22} \end{pmatrix}\right).$$

The rank of

$$\begin{pmatrix} aI_n & B \\ I_n & O \end{pmatrix} + \mathbf{i}\left(\begin{pmatrix} \mathbf{i}I_n & O \\ O & \mathbf{i}I_n \end{pmatrix} + \begin{pmatrix} \Delta_{11} & O \\ O & \Delta_{22} \end{pmatrix}\right)^{-1}\left(\begin{pmatrix} \mathbf{i}I_n & O \\ O & \mathbf{i}I_n \end{pmatrix} - \begin{pmatrix} \Delta_{11} & O \\ O & \Delta_{22} \end{pmatrix}\right).$$

implies that $\lambda_i(Y) \neq 0$, $\forall \Delta \in \hat{\Delta}$, where

$$Y = \begin{pmatrix} I_n & O \\ O & I_n \end{pmatrix} - \left(\begin{pmatrix} \mathbf{i}I_n & O \\ O & \mathbf{i}I_n \end{pmatrix} + \begin{pmatrix} aI_n & B \\ I_n & O \end{pmatrix}\right)^{-1}\left(\begin{pmatrix} \mathbf{i}I_n & O \\ O & \mathbf{i}I_n \end{pmatrix} - \begin{pmatrix} aI_n & B \\ I_n & O \end{pmatrix}\right)\begin{pmatrix} \Delta_{11} & O \\ O & \Delta_{22} \end{pmatrix}.$$

Finally, this necessary condition guaratness that μ-value is less than 1.

3.3. Pseudo-spectrum:

The pseudo-spectrum of a matrix A is the set of which contains the spectrum of matrix A. The important question one can raise is about the singularity of A which does not appear as a robust in the sense that a small perturbation ϵ may vary the answer from yes to no in a dramatic way. This helps to think that either $||A^{-1}||$ is large enough or not? For λ, an eigenvalue of A, a much better question is to ask: Does $||(\lambda I_n - A)^{-1}||$ is large or not? such a pattern allows following definitions and results [32] of pseudo-spectrum.

Definition 2. *Let $A \in \mathbb{R}^{n,n}$, $\epsilon > 0$, a small perturbation. The ϵ-pseudo-spectrum $\sigma_\epsilon(A)$ which is set of eigenvalues (the spectrum) $\lambda \in \mathbb{C}$ such that*

$$||(\lambda I_n - A)^{-1}|| > \frac{1}{\epsilon}.$$

Remark 4. For $\lambda \in \sigma(A)$, $\sigma(A)$ being as the set of eigenvalues of A, $||(\lambda I_n - A)^{-1}|| = \infty$. The second definition of pseudo-spectrum is given as follows.

Definition 3. *Let $A \in \mathbb{R}^{n,n}$, $\epsilon > 0$, a small perturbation. The ϵ-pseudospectrum $\sigma_\epsilon(A)$ which is set of eigenvalues (the spectrum) $\lambda \in \mathbb{C}$ such that*

$$\lambda \in \sigma(A + E),$$

to some E having $||E|| < \epsilon$.

The third characterization of pseudo-spectrum is given as bellow.

Definition 4. *Let $A \in \mathbb{R}^{n,n}$, $\epsilon > 0$, a small perturbation. The ϵ-pseudospectrum $\sigma_\epsilon(A)$ which is the of eigenvalues (the spectrum) $\lambda \in \mathbb{C}$ such that*

$$||(\lambda I_n - A)v|| < \epsilon$$

for some $v \in \mathbb{C}^{n,1}$ having $||v|| = 1$.

4. Numerical Experimentation

We present a detailed comparison on the approximation of the bounds (from below) to μ-values. We consider some well-known algorithm for the approximation of μ-values: The **mussv** function **mussv**, Power Algorithm (PA) [33], Gain Based Algorithm (GBA) [34], Poles migration Algorithm (PMA) [35], Non-linear optimization Algorithm (NLA) [36], and Low-rank ODE's based algorithm (LRA) given by first author [37]. The structured matrices are taken from various economics models. We make use of EigTool [38] to compute and display pseudo-spectrum and eigenvalues.

The function **mussv** is being freely available in the Matlab Control ToolBox. This tool provides results on the numerical computation of bounds of μ-values. The proposed mathematical methodology give results on the numerics of bounds (from below) of μ-values of structured matrices from various economic models. The reasons for our results closer to the one obtained by existing methodologies are:
(i) Our mathematical methodology uses an approximation of singular values rather than computing eigenvalues for the structured matrices.
(ii) The approximation to singular values is obtained with singular value decomposition tool. The Matlab routine **svd** computes the singular values.
(iii) The approximation to largest singular value of structured matrices in the sense of low-rank provides the sharper bounds (from below) to μ-values. The computational cost is not high as compared with exiting techniques.
We give a number of numerical examples for economy models on the numerical approximation of bounds of μ-values. The obtained results on the computation of μ-values demonstrates the effectiveness of proposed methodology and existing techniques.

Example 1. We consider a first order dynamical model $\dot{x} = Ax$, with the coefficient matrix A, a 4-dimensional real valued matrix taken from [39].

$$A = \begin{bmatrix} -8 & 2 & 2 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 2 & -2 & 0 \\ 1 & 1 & 1 & -5 \end{bmatrix}.$$

The comparison on approximation to the bounds from below to μ-values is given in the following Table 1.

The approximated μ-values					
mussv	PA	GBA	PMA	NLA	LRA
8.6448	8.6421	8.6430	8.6442	8.6439	8.6448

Example 2. We consider a first order dynamical system $\dot{x} = Ax$, with the coefficient (D-stable) matrix A, a 3-dimensional real valued matrix taken from [39].

$$A = \begin{bmatrix} -1 & -1 & 0 \\ 1 & -2 & 1 \\ 0 & 2 & -1 \end{bmatrix}.$$

The comparison on approximation to the lower bounds from below to μ-values is given in the following Table 2.

The approximated μ-values					
mussv	PA	GBA	PMA	NLA	LRA
3.3181	3.3145	3.3167	3.3177	3.3120	3.3180

Example 3. We consider a second order dynamical system $\ddot{x} = A\dot{x} + Bx$, with A, B taken from [3].

$$A = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -3 & 2 \\ 2 & 2 & -2 \end{bmatrix}, \; B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -0.1 & 0 \\ 0 & 0 & -0.1 \end{bmatrix}.$$

The matrix M is:

$$M = \begin{bmatrix} A & B \\ I_3 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 & -1 & 0 & 0 & 0 \\ 8 & -3 & 2 & 0 & -0.1 & 0 \\ 2 & 2 & -2 & 0 & 0 & -0.1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

The comparison on approximation to bounds from below to μ-values is given in the following Table 3.

The approximated μ-values of matrix M					
mussv	PA	GBA	PMA	NLA	LRA
8.9841	8.4940	8.2361	8.2391	8.9821	9.9838

Example 4. We consider a second order dynamical system $\ddot{x} = A\dot{x} + Bx$, with A, B taken from [3].

$$A = \begin{bmatrix} -1 & 0.9 & 0.2 \\ 0.6 & -1.5 & 0.9 \\ 1 & 1 & -2 \end{bmatrix}, \; B = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

The matrix M is:

$$N = \begin{bmatrix} A & B \\ I_3 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0.9 & 0.2 & -2 & 0 & 0 \\ 0.6 & -1.5 & 0.9 & 0 & -1 & 0 \\ 1 & 1 & -2 & 0 & 0 & -2 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

The comparison on approximation to bounds from below to μ-values is given in the following Table 4.

The approximated μ-values of matrix M					
mussv	PA	GBA	PMA	NLA	LRA
3.4352	3.4189	3.4310	3.4329	3.4155	3.4350

5. Conclusion

This article considers a mathematical problem related with an interconnections among the notations of structured stability, structured D-stability analysis, and μ-values. The novel mathematical results are developed to link the bridge between stability, structured D-stability analysis, and μ-values. The numerical experimentation show the dynamic of the spectrum of structured matrices. The Matlab EigTool is used for the computation of pseud-spectrum of structured matrices across first, and second order dynamical models. The advantages and limitations of the proposed mathematical work are listed below.

Advantages of proposed work: The proposed mathematical technique provides an interesting interconnections between stability, D-stability of structured matrices and their structured singular values. The new results on structured stability, and structured D-stability may be applicable for robust analysis of nonlinear models which are subject to Lyapunov functions. The results on D-stability ensure the stable equilibrium points subject to various different constraints. The economic models involves structured or unstructured uncertainties among the tax policies, the rates of interest. The study of μ-values enable to identify the amount of structured or unstructured uncertainties affecting overall economy while allowing policymakers to design more robust policies.

Limitations of proposed work: The D-stability, and numerical computation of μ-values has limitations in the sense that the dynamic models subject to consideration are most often time linear or linear time varying in their nature. The application of stability, and D-stability the nonlinear systems may be overly simplistic. In general, the results for stability, and D-stability may not capture the true dynamics of the economy models. The μ-analysis is not cheap and is very costly. On the other hand, an exact determination of μ-values is infect an NP-hard problem. The economics models most often involves a large amount of parameters and hence it demands very high computational requirements for the analysis of robustness and performance. This leads the practical difficulties and complexities to scale μ-analysis to large dimensional macroeconomic models.

Acknowledgements

The authors thank the anonymous referees for their crucial and valuable comments and suggestions, which are essential for improving the manuscript.

Conflict of Interest Statement

The authors declare that they have no conflict of interest.

Authors contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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