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On a New Stochastic Space with Applications to Nonlinear Economic Model

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Abstract. This article will utilize a weighted regular matrix composed of Fibonacci numbers and variable exponent sequence spaces to create a novel stochastic space with certain geometric and topological properties. This area demonstrates the new form of the Kannan contraction operator with a fixed point. In mathematical economics, we represent economic entities, processes, and phenomena by mathematically structured functional equations, either as summable equations or integral equations. We investigate a category of Volterra-type non-linear dynamical systems, similar to an economic model. We employ our acquired results to formulate new solvability criteria for a unique solution of these non-linear discrete economic dynamical systems. Ultimately, we illustrate our findings with specific instances and applications related to the presence of solutions in non-linear dynamical systems of Volterra-type.

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Key Words and Phrases: Fibonacci numbers, variable exponent, extended $s-$soft numbers, new type of Kannan contraction.

Abbreviations

(i) $\boldsymbol{p\text{-}q.N}$: pre-quasi norm.

(ii) $\mathfrak{pssf}$: private sequence space of soft functions.

(iii) $\boldsymbol{p\text{-}m}$: pre-modular.

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(iv) ***p-q.B***: pre-quasi Banach.

(v) ***Bs***: Banach space.

(vi) ***Cs***: Cauchy sequence.

(vii) ***CMs***: complete metric space.

(viii) ***NAT***: non-absolute type.

(ix) ***FP***: Fatou property.

(x) ***NTK***-$\|.\|_{p-qN}$-***C***: New type of Kannan $\|.\|_{p-qN}$-contraction.

(xi) $\|.\|_{p-qN}$-***Seq.C***: $\|.\|_{p-qN}$-sequentially continuous.

(xii) ***ufp***: unique fixed point.

(xiii) ***NLDEs***: non-linear difference equations.

Notations

(i) $\mathcal{N} := \{0, 1, 2, ...\}$ and $\mathbf{R}$ is the set of real numbers.

(ii) $\mathbf{R}^{+\mathcal{N}}$: The space of all sequences of positive reals.

(iii) ℓ_m, ℓ_∞, and c_0: The spaces of m-absolutely summable, bounded, and convergent to zero sequences of reals, respectively.

(iv) $\mathfrak{B}(\mathbf{R})$ and $\mathbb{E}$: The collection of all nonempty bounded subsets of $\mathbf{R}$ and the set of parameters, respectively.

(v) $\mathbf{R}(\mathbb{A})^*$ and $\mathbf{R}(\mathbb{A})$: The set of nonnegative and all soft real numbers (corresponding to $\mathbb{A}$), where $\mathbb{A} \subset \mathbb{E}$, respectively.

(vi) $\widetilde{0}$ and $\widetilde{1}$: The additive identity and multiplicative identity in $\mathbf{R}(\mathbb{A})$, respectively, see [6].

(vii) $\mu^{\mathbf{S}}$: The space of all sequences of soft reals.

(viii) $\mathfrak{G}$, $\mathfrak{V}$: Infinite dimensional Banach spaces.

(ix) $\mathcal{E}^{\mathbf{S}}$: The linear space of sequences of soft functions.

(x) $\widehat{\mathfrak{e}_d} := (\widehat{0}, \widehat{0}, ..., \widehat{1}, \widehat{0}, \widehat{0}, \cdots)$, while $\widehat{1}$ displays at the d^{th} place.

(xi) $[d]$: The integral part of real number d.

(xii) $\widehat{\vartheta} := (\widehat{0}, \widehat{0}, \widehat{0}, \ldots)$.

(xiii) $\mathcal{F}$: The space of finite sequences of soft numbers.

(xiv) $\mathfrak{N}_+$ and $\mathfrak{D}_-$: The space of all monotonic increasing and decreasing sequences of positive reals, respectively.

(xv) I_r: The identity mapping on ℓ_2^d.

(xvi) J_r: The natural embedding mapping from M_r into $\mathfrak{V}$.

(xvii) T_r: The quotient mapping from $\mathfrak{G}$ onto $\mathfrak{G}/Y_r$.

(xviii) Dvoretzky's Theorem [19]: For any $d \in \mathcal{N}$, we have quotient spaces $\mathfrak{G}/Y_d$ and subspaces M_d of $\mathfrak{V}$ which can be transformed onto ℓ_2^d by isomorphisms V_d and X_d such that $\|W_d\|\|W_d^{-1}\| \leq 2$ and $\|X_d\|\|X_d^{-1}\| \leq 2$.

1. Introduction

The ability to mathematically simulate non-Newtonian fluids in hydrodynamics is drawing more and more attention to the study of variable exponent Lebesgue spaces, as discussed by Rużîĉka [20]. A variety of disciplines, including orthopedics, civil engineering, and military science, make use of electrorheological fluids, a class of non-Newtonian fluids. The work of Diening et al. [7] covered the topic of Lebesgue and Sobolev spaces involving variable exponents. A particular sequence space contains the solutions of discrete dynamical systems. According to [16], the construction of new sequence spaces is a subject of significant mathematical interest. The domain of the Cesàro mean of order one in certain spaces of double sequences was developed and studied by Mursaleen and Başar [15], while Noman and Mursaleen [17] investigated novel non-absolute sequence spaces that are connected to ℓ_p and ℓ_∞. The notion of $\mathfrak{psssf}$ was first proposed by Alsolmi and Bakery [4]. In [14], the researchers examined the distinctiveness and presence of solutions inside a novel complex function space for Kannan nonlinear dynamical systems. Supposing that $(\mathfrak{f}_k)_{k=0}^{\infty}$ is the sequence of Fibonacci numbers defined by the recurrence relation $\mathfrak{f}_v = \mathfrak{f}_{v-1} + \mathfrak{f}_{v-2}$, $v \geq 2$, so that $\mathfrak{f}_0 = 1$ and $\mathfrak{f}_1 = 1$. Note that in [12] that $\sum_{z=0}^{l} \mathfrak{f}_z^2 = \mathfrak{f}_l \mathfrak{f}_{l+1}$, and $\sum_{z=0}^{\infty} \frac{1}{\mathfrak{f}_z} < \infty$. Kara and Başarır further strengthen the studies on Fibonacci sequence spaces [11]. They defined and studied the matrix domains $\ell_p(\gamma_\mathfrak{f}) := (\ell_p)_{\gamma_\mathfrak{f}}$, $c_0(\gamma_\mathfrak{f}) := (c_0)_{\gamma_\mathfrak{f}}$, $c(\gamma_\mathfrak{f}) := (c)_{\gamma_\mathfrak{f}}$ and $\ell_\infty(\gamma_\mathfrak{f}) := (\ell_\infty)_{\gamma_\mathfrak{f}}$.

Assume that $(t_l), (q_l) \in \mathbf{R}^{+\mathcal{N}}$. We have presented a novel stochastic space $\left(\gamma_\mathfrak{f}^{\mathbf{S}}(q,t)\right)_{\|\cdot\|_{p-qN}}$ of soft functions as:

$$\left(\gamma_\mathfrak{f}^{\mathbf{S}}(q,t)\right)_{\|\cdot\|_{p-qN}} := \left\{\widehat{d} = (\widehat{d_b}) \in \mu^{\mathbf{S}} : \|\delta\widehat{d}\|_{p-qN} < \infty, \text{ for some } \delta > 0\right\},$$

where $\|\widehat{d}\|_{p-qN} = \sum_{l \in \mathcal{N}} \left(\frac{\widehat{\hbar}\left(\sum_{z=0}^{l} \mathfrak{f}_z^2 q_z \widehat{d}_z, \widehat{0}\right)}{\mathfrak{f}_l \mathfrak{f}_{l+1}}\right)^{t_l}$, $\hbar : \mathbf{R}(\mathbb{A}) \times \mathbf{R}(\mathbb{A}) \to \mathbf{R}(\mathbb{A})^*$, with $\hbar(\widehat{f}, \widehat{g}) = |\widehat{f} - \widehat{g}|$, for all $\widehat{f}, \widehat{g} \in \mathbf{R}(\mathbb{A})$, and $\widehat{\hbar} : \mathbf{R}(\mathbb{A}) \times \mathbf{R}(\mathbb{A}) \to \mathbf{R}^+$ is defined by

$$\widehat{\hbar}(\widehat{f}, \widehat{g}) = \max_{\lambda \in A} \hbar(\widehat{f}, \widehat{g})(\lambda).$$

If $(t_l) \in \mathbf{R}^{+\mathcal{N}} \cap \ell_\infty$, then

$$\left(\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)\right)_{\|.\|_{p-qN}} = \left\{\widehat{d} = (\widehat{d_b}) \in \mu^{\mathbf{S}} : \|\delta\widehat{d}\|_{p-qN} < \infty, \text{ for any } \delta > 0\right\}.$$

Volterra-type summable equations are fundamental in investigating dynamical systems [1] and stochastic processes [9, 13]. Assuming that $\widehat{\nu} \in \gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)$, $\Gamma : \mathcal{N}^2 \to \mathfrak{R}$, $\Psi : \mathcal{N} \times \mathbf{R}(\mathbb{A}) \to \mathbf{R}(\mathbb{A})$, $\widehat{\nu} : \mathcal{N} \to \mathbf{R}(\mathbb{A})$, and $\widehat{\beta} : \mathcal{N} \to \mathbf{R}(\mathbb{A})$.
Consider the Volterra-type summable equations of soft functions [4]:

$$\widehat{\nu_d} = \widehat{\beta_d} + \sum_{v \in \mathcal{N}} \Gamma_{d,v} \Psi_{v,\widehat{\nu_v}}, \tag{1}$$

and when $\Phi : \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)\right)_{\|.\|_{p-qN}} \to \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)\right)_{\|.\|_{p-qN}}$ is defined as

$$\Phi(\widehat{\nu_d})_{d \in \mathcal{N}} = \left(\widehat{\beta_d} + \sum_{v \in \mathcal{N}} \Gamma_{d,v} \Psi_{v,\widehat{\nu_v}}\right)_{d \in \mathcal{N}}. \tag{2}$$

Considering the multitude of fixed point theorems within a specific space, it is necessary to either enlarge the space itself or to augment the self-mapping that operates within it; both alternatives are feasible. Examples include granular systems, sweeping processes, oscillation issues, control challenges, and decision-making problems, among others. A particular sequence space encompasses the solutions of summable equations. So, there is significant interest in mathematics to develop new sequence spaces. This work aims to create a new stochastic space by utilizing a weighted regular matrix based on Fibonacci numbers and variable exponent sequence spaces. We have applied specific geometric and topological structures to soft functions represented as $\left(\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)\right)_{\|.\|_{p-qN}}$. The fixed point of the new type of pre-quasi-Kannan contraction operator is verified in this context. We conclude by illustrating our findings with several examples and applications related to the existence of solutions to non-linear difference equations.

2. Definitions and Preliminaries

Definition 1. *[4] $\mathcal{E}^{\mathbf{S}}$ is referred to as a $\mathfrak{psssf}$ if it meets the following criteria:*

(1c) *$\mathcal{E}^{\mathbf{S}}$ is linear space and $\widehat{\mathfrak{e}_r} \in \mathcal{E}^{\mathbf{S}}$, for $r \in \mathcal{N}$,*

(2c) *$\mathcal{E}^{\mathbf{S}}$ is solid i.e., if $\widehat{m} = (\widehat{m_r}) \in \mu^{\mathbf{S}}$, $|\widehat{k}| = (|\widehat{k_r}|) \in \mathcal{E}^{\mathbf{S}}$ and $|\widehat{m_r}| \leq |\widehat{k_r}|$, where $r \in \mathcal{N}$, then $|\widehat{m}| \in \mathcal{E}^{\mathbf{S}}$,*

(3c) *$\left(\left|\widehat{k_{[\frac{r}{2}]}}\right|\right)_{r \in \mathcal{N}} \in \mathcal{E}^{\mathbf{S}}$, if $\left(\left|\widehat{k_x}\right|\right)_{r \in \mathcal{N}} \in \mathcal{E}^{\mathbf{S}}$.*

Definition 2. *[4] A subspace $\mathfrak{psssf}$ $\mathcal{E}^{\mathbf{S}}_{\|.\|_{p-qN}}$ is called a **p-m** $\mathfrak{psssf}$, if $\|.\|_{p-qN} : \mathcal{E}^{\mathbf{S}} \to [0, \infty)$ meets the following criteria for all $\widehat{m}, \widehat{k} \in \mathcal{E}^{\mathbf{S}}$, and $\delta \in \mathbf{R}$:*

(a1) $\widehat{k}=\widehat{\vartheta} \Longleftrightarrow \|(|\widehat{k}|)\|_{p-qN}=0$, *and* $\|\widehat{k}\|_{p-qN}\geq 0$,

(a2) *there are* $C_1\geq 1$ *so that* $\|\delta\widehat{m}\|_{p-qN}\leq|\delta|C_1\|\widehat{m}\|_{p-qN}$,

(a3) $\|\widehat{m}+\widehat{k}\|_{p-qN}\leq C_2(\|\widehat{m}\|_{p-qN}+\|\widehat{k}\|_{p-qN})$ *verifies so that* $C_2\geq 1$,

(a4) *if* $|\widehat{m_r}|\leq|\widehat{k_r}|$, *then* $\|(|\widehat{m_r}|)\|_{p-qN}\leq\|(|\widehat{k_r}|)\|_{p-qN}$,

(a5) *the inequality,* $\|(|\widehat{k_r}|)\|_{p-qN}\leq\|(|\widehat{k_{[\frac{r}{2}]}}|)\|_{p-qN}\leq C_3\|(|\widehat{k_r}|)\|_{p-qN}$ *holds, for* $C_3\geq 1$,

(a6) *the closure* $\overline{\mathcal{F}}$ *of* $\mathcal{F}=\mathcal{E}^{\mathbf{S}}_{\|\cdot\|_{p-qN}}$,

(a7) *the inequality,* $\|(\widehat{m},\widehat{0},\widehat{0},\widehat{0},...)\|_{p-qN}\geq\alpha|m|\|\widehat{\mathfrak{e}_1}\|_{p-qN}$ *verifies for* $\alpha>0$.

Definition 3. *[4] The space* $\mathfrak{pssf}$ $\mathcal{E}^{\mathbf{S}}_{\|\cdot\|_{p-qN}}$ *is called a* ***p-q.N*** $\mathfrak{pssf}$ *when* $\|.\|_{p-qN}$ *verifies the parts (a1)-(a3) of Definition 2. The space* $\mathfrak{pssf}$ $\mathcal{E}^{\mathbf{S}}_{\|\cdot\|_{p-qN}}$ *is said to be* ***p-q.B*** $\mathfrak{pssf}$ *if the space* $\mathfrak{pssf}$ $\mathcal{E}^{\mathbf{S}}_{\|\cdot\|_{p-qN}}$ *is complete equipped with* $\|.\|_{p-qN}$.

Theorem 2.1. *[3] Every* ***p-m*** $\mathfrak{pssf}$ $\mathcal{E}^{\mathbf{S}}_{\|\cdot\|_{p-qN}}$ *is a* ***p-q.N*** $\mathfrak{pssf}$.

Lemma 2.2. *[5] Suppose* $r_m>1$ *and* $\alpha_m,\delta_m\in\mathbf{R}$, *for all* $m\in\mathcal{N}$, *and* $\beth=\sup_m r_m$, *one has*

$$|\alpha_m+\delta_m|^{r_m}\leq 2^{\beth-1}\left(|\alpha_m|^{r_m}+|\delta_m|^{r_m}\right). \tag{3}$$

Definition 4. *[3] A function* $\|.\|_{p-qN}$ *on* $\mathcal{E}^{\mathbf{S}}_{\|\cdot\|_{p-qN}}$ *satisfies the* ***FP*** *when for each* $\{\widehat{k_r}\}\subseteq\mathcal{E}^{\mathbf{S}}_{\|\cdot\|_{p-qN}}$ *such that* $\lim_{r\to\infty}\|\widehat{k_r}-\widehat{k}\|_{p-qN}=0$ *and all* $\widehat{u}\in\mathcal{E}^{\mathbf{S}}_{\|\cdot\|_{p-qN}}$, *then* $\|\widehat{u}-\widehat{k}\|_{p-qN}\leq\sup_m\inf_{r\geq m}\|\widehat{u}-\widehat{k_r}\|_{p-qN}$.

Definition 5. *[4] Supposing that* $\mathcal{E}^{\mathbf{S}}_{\|\cdot\|_{p-qN}}$ *is a* ***p-q.N*** $\mathfrak{pssf}$, $M:\mathcal{E}^{\mathbf{S}}_{\|\cdot\|_{p-qN}}\to\mathcal{E}^{\mathbf{S}}_{\|\cdot\|_{p-qN}}$ *and* $\widehat{d}\in\mathcal{E}^{\mathbf{S}}_{\|\cdot\|_{p-qN}}$. *The mapping* M *is said to be* $\|.\|_{p-qN}$*-**Seq.C** at* $\widehat{d}$, *if and only if, for any* $\{\widehat{k_r}\}\subseteq\mathcal{E}^{\mathbf{S}}_{\|\cdot\|_{p-qN}}$ *such that* $\lim_{r\to\infty}\|\widehat{k_r}-\widehat{d}\|_{p-qN}=0$ *then* $\lim_{r\to\infty}\|M\widehat{k_r}-M\widehat{d}\|_{p-qN}=0$.

3. Configuration and properties of $\left(\gamma^{\mathbf{S}}_{\mathfrak{f}}(q,t)\right)_{\|\cdot\|_{p-qN}}$

This section introduces the definition and inclusion relations of the sequence space $\left(\gamma^{\mathbf{S}}_{\mathfrak{f}}(q,t)\right)_{\|\cdot\|_{p-qN}}$ with the function $\|.\|_{p-qN}$.

Theorem 3.1. *The space* $\left(\gamma^{\mathbf{S}}_{\mathfrak{f}}(q,t)\right)_{\|\cdot\|_{p-qN}}$ *is a* ***NAT***, *if* $(t_l)\in(0,\infty)^{\mathcal{N}}\cap\ell_\infty$.

Proof. Clearly, as

$$\begin{aligned}\|\widehat{\mathfrak{e}_0}-\widehat{\mathfrak{e}_1}\|_{p-qN}&=(q_0)^{t_0}+\left(\frac{|q_0-q_1|}{2}\right)^{t_1}+\left(\frac{|q_0-q_1|}{6}\right)^{t_2}+\cdots\\&\neq(q_0)^{t_0}+\left(\frac{|q_0+q_1|}{2}\right)^{t_1}+\left(\frac{|q_0+q_1|}{6}\right)^{t_2}+\cdots=\|\widehat{\mathfrak{e}_0}+\widehat{\mathfrak{e}_1}\|_{p-qN}.\end{aligned}$$

Definition 6. *Supposing that* $(t_l) \in [0.5, \infty)^{\mathcal{N}}$. *The absolute type space* $\left(|\gamma_{\mathfrak{f}}^{\mathbf{S}}|(q,t)\right)_{\varphi}$ *is defined as*

$$\left(|\gamma_{\mathfrak{f}}^{\mathbf{S}}|(q,t)\right)_{\varphi} := \left\{\widehat{J} = (\widehat{J_k}) \in \mu^{\mathbf{S}} : \varphi(\delta f) < \infty, \text{ for some } \delta > 0\right\},$$

where $\varphi(\widehat{J}) = \sum_{l=0}^{\infty} \left(\frac{\widehat{\hbar}\left(\sum_{z=0}^{l} \mathfrak{f}_z^2 q_z |\widehat{J}_z|, \widehat{0}\right)}{\mathfrak{f}_l \mathfrak{f}_{l+1}} \right)^{t_l}$.

Theorem 3.2. *If* $(t_l) \in [0.5, \infty)^{\mathcal{N}} \cap \ell_\infty$ *with* $\left(\frac{l+1}{\mathfrak{f}_l \mathfrak{f}_{l+1}}\right) \notin \ell_{(t_l)}$, *one gets* $\left(|\gamma_{\mathfrak{f}}^{\mathbf{S}}|(q,t)\right)_{\varphi} \subsetneqq \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)\right)_{\|\cdot\|_{p-qN}}$.

Proof. If $\widehat{d} \in \left(|\gamma_{\mathfrak{f}}^{\mathbf{S}}|(q,t)\right)_{\varphi}$, then

$$\sum_{l \in \mathcal{N}} \left(\frac{\widehat{\hbar}\left(\sum_{z=0}^{l} \mathfrak{f}_z^2 q_z \widehat{d_z}, \widehat{0}\right)}{\mathfrak{f}_l \mathfrak{f}_{l+1}} \right)^{t_l} \leq \sum_{l \in \mathcal{N}} \left(\frac{\widehat{\hbar}\left(\sum_{z=0}^{l} \mathfrak{f}_z^2 q_z |\widehat{d_z}|, \widehat{0}\right)}{\mathfrak{f}_l \mathfrak{f}_{l+1}} \right)^{t_l} < \infty.$$

Therefore, $\widehat{d} \in \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)\right)_{\|\cdot\|_{p-qN}}$. If $\widehat{J} = \left(\frac{(\widehat{-1})^z}{\mathfrak{f}_z^2 q_z}\right)_{z \in \mathcal{N}}$, one has $\widehat{J} \in \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)\right)_{\|\cdot\|_{p-qN}}$ and $\widehat{J} \notin \left(|\gamma_{\mathfrak{f}}^{\mathbf{S}}|(q,t)\right)_{\varphi}$.

We provide the sufficient conditions on $\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)$ to form a ***p-q.B*** $\mathfrak{pssf}$.

Theorem 3.3. $\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)$ *is a **p-m*** $\mathfrak{pssf}$, *if*

(o1) $(t_l) \in \mathfrak{N}_+ \cap \ell_\infty$ *and* $t_0 \geq 0.5$.

(o2) $\left(\mathfrak{f}_z^2 q_z\right)_{z \in \mathcal{N}} \in \mathfrak{D}_-$ *or,* $\left(\mathfrak{f}_z^2 q_z\right)_{z \in \mathcal{N}} \in \mathfrak{N}_+ \cap \ell_\infty$ *and one has* $A \geq 1$ *such that* $\mathfrak{r}_{2z+1} q_{2z+1} \leq A \mathfrak{f}_z^2 q_z$.

Proof. Assuming that $\widehat{d}, \widehat{k} \in \gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)$, and $\delta \in \mathbf{R}$. Assume the setups (o1) and (o2) are verified.
The condition (a1): Clearly, $\|\widehat{d}\|_{p-qN} \geq 0$ and $\|(|\widehat{d}|)\|_{p-qN} = 0 \Leftrightarrow \widehat{d} = \widehat{\vartheta}$.
The conditions (1c) and (a3):

$$\|\widehat{d} + \widehat{k}\|_{p-qN} = \sum_{l \in \mathcal{N}} \left(\frac{\widehat{\hbar}\left(\sum_{z=0}^{l} \mathfrak{f}_z^2 q_z \left(\widehat{d_z} + \widehat{k_z}\right), \widehat{0}\right)}{\mathfrak{f}_l \mathfrak{f}_{l+1}} \right)^{t_l}$$
$$\leq 2^{\beth-1} \left(\sum_{l \in \mathcal{N}} \left(\frac{\widehat{\hbar}\left(\sum_{z=0}^{l} \mathfrak{f}_z^2 q_z \widehat{d_z}, \widehat{0}\right)}{\mathfrak{f}_l \mathfrak{f}_{l+1}} \right)^{t_l} + \sum_{l \in \mathcal{N}} \left(\frac{\widehat{\hbar}\left(\sum_{z=0}^{l} \mathfrak{f}_z^2 q_z \widehat{k_z}, \widehat{0}\right)}{\mathfrak{f}_l \mathfrak{f}_{l+1}} \right)^{t_l} \right) = C_2(\|\widehat{d}\|_{p-qN} + \|\widehat{k}\|_{p-qN}) < \infty,$$

therefore, $\widehat{d}+\widehat{k}\in\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)$.
The conditions (1c) and (a2):

$$\|\delta\widehat{d}\|_{p-qN}=\sum_{l\in\mathcal{N}}\left(\frac{\widehat{\hbar}\left(\sum_{z=0}^{l}\mathfrak{f}_z^2q_z\delta\widehat{d}_z,\widehat{0}\right)}{\mathfrak{f}_l\mathfrak{f}_{l+1}}\right)^{t_l}\leq\sup_l|\delta|^{t_l}\sum_{l\in\mathcal{N}}\left(\frac{\widehat{\hbar}\left(\sum_{z=0}^{l}\mathfrak{f}_z^2q_z\widehat{d}_z,\widehat{0}\right)}{\mathfrak{f}_l\mathfrak{f}_{l+1}}\right)^{t_l}=C_1\|\widehat{d}\|_{p-qN}<\infty.$$

Hence, $\delta\widehat{d}\in\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)$. Hence $\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)$ is a linear space. Also

$$\sum_{l\in\mathcal{N}}\left(\frac{\widehat{\hbar}\left(\sum_{z=0}^{l}\mathfrak{f}_z^2q_z\widehat{(e_b)_z},\widehat{0}\right)}{\mathfrak{f}_l\mathfrak{f}_{l+1}}\right)^{t_l}=\sum_{l=b}^{\infty}\left(\frac{\mathfrak{f}_b^2q_b}{\mathfrak{f}_l\mathfrak{f}_{l+1}}\right)^{t_l}\leq\sup_{l=b}^{\infty}\left(\mathfrak{f}_b^2q_b\right)^{t_l}\sum_{l=b}^{\infty}\left(\frac{1}{\mathfrak{f}_l\mathfrak{f}_{l+1}}\right)^{t_l}<\infty.$$

So, $\widehat{\mathfrak{e}}_b\in\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)$, for every $b\in\mathcal{N}$.
The conditions (2c) and (a4): Assume $|\widehat{d_b}|\leq|\widehat{k_b}|$, for $b\in\mathcal{N}$ and $|\widehat{k}|\in\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)$. Hence

$$\|(|\widehat{d}|)\|_{p-qN}=\sum_{l\in\mathcal{N}}\left(\frac{\widehat{\hbar}\left(\sum_{z=0}^{l}\mathfrak{f}_z^2q_z|\widehat{d_z}|,\widehat{0}\right)}{\mathfrak{f}_l\mathfrak{f}_{l+1}}\right)^{t_l}\leq\sum_{l\in\mathcal{N}}\left(\frac{\widehat{\hbar}\left(\sum_{z=0}^{l}\mathfrak{f}_z^2q_z|\widehat{k_z}|,\widehat{0}\right)}{\mathfrak{f}_l\mathfrak{f}_{l+1}}\right)^{t_l}=\|(|\widehat{k}|)\|_{p-qN}<\infty,$$

then $|\widehat{d}|\in\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)$.
The conditions (3c) and (a5): Let $(|\widehat{d_z}|)\in\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)$ and $\left(\mathfrak{f}_z^2q_z\right)_{z\in\mathcal{N}}\in\mathfrak{D}_-$, one can see that

$$\begin{aligned}
\|(|\widehat{d_{[\frac{z}{2}]}}|)\|_{p-qN}&=\sum_{l\in\mathcal{N}}\left(\frac{\widehat{\hbar}\left(\sum_{z=0}^{l}\mathfrak{f}_z^2q_z|\widehat{d_{[\frac{z}{2}]}}|,\widehat{0}\right)}{\mathfrak{f}_l\mathfrak{f}_{l+1}}\right)^{t_l}\\
&=\sum_{l\in\mathcal{N}}\left(\frac{\widehat{\hbar}\left(\sum_{z=0}^{2l}\mathfrak{f}_z^2q_z|\widehat{d_{[\frac{z}{2}]}}|,\widehat{0}\right)}{\mathfrak{f}_{2l}\mathfrak{f}_{2l+1}}\right)^{t_{2l}}+\sum_{l\in\mathcal{N}}\left(\frac{\widehat{\hbar}\left(\sum_{z=0}^{2l+1}\mathfrak{f}_z^2q_z|\widehat{d_{[\frac{z}{2}]}}|,\widehat{0}\right)}{\mathfrak{f}_{2l+1}\mathfrak{f}_{2l+2}}\right)^{t_{2l+1}}
\end{aligned}$$

$$\leq\sum_{l\in\mathcal{N}}\left(\frac{\widehat{\hbar}\left(\sum_{z=0}^{2l}\mathfrak{f}_z^2q_z|\widehat{d_{[\frac{z}{2}]}}|,\widehat{0}\right)}{\mathfrak{f}_l\mathfrak{f}_{l+1}}\right)^{t_l}+\sum_{l\in\mathcal{N}}\left(\frac{\widehat{\hbar}\left(\sum_{z=0}^{2l+1}\mathfrak{f}_z^2q_z|\widehat{d_{[\frac{z}{2}]}}|,\widehat{0}\right)}{\mathfrak{f}_l\mathfrak{f}_{l+1}}\right)^{t_l}$$

$$\leq\sum_{l\in\mathcal{N}}\left(\frac{\widehat{\hbar}\left(\mathfrak{f}_{2l}^2q_{2l}|\widehat{d_l}|+\sum_{z=0}^{l}\left(\mathfrak{f}_{2z}^2q_{2z}+\mathfrak{f}_{2z+1}^2q_{2z+1}\right)|\widehat{d_z}|,\widehat{0}\right)}{\mathfrak{f}_l\mathfrak{f}_{l+1}}\right)^{t_l}+\sum_{l\in\mathcal{N}}\left(\frac{\widehat{\hbar}\left(\sum_{z=0}^{l}\left(\mathfrak{f}_{2z}^2q_{2z}+\mathfrak{f}_{2z+1}^2q_{2z+1}\right)|\widehat{d_z}|,\widehat{0}\right)}{\mathfrak{f}_l\mathfrak{f}_{l+1}}\right)^{t_l}$$

$$\leq 2^{\beth-1}\left(\sum_{l\in\mathcal{N}}\left(\frac{\widehat{\hbar}\left(\sum_{z=0}^{l}\mathfrak{f}_z^2q_z|\widehat{d_z}|,\widehat{0}\right)}{\mathfrak{f}_l\mathfrak{f}_{l+1}}\right)^{t_l}+\sum_{l\in\mathcal{N}}\left(\frac{2\widehat{\hbar}\left(\sum_{z=0}^{l}\mathfrak{f}_z^2q_z|\widehat{d_z}|,\widehat{0}\right)}{\mathfrak{f}_l\mathfrak{f}_{l+1}}\right)^{t_l}\right)+\sum_{l\in\mathcal{N}}\left(\frac{2\widehat{\hbar}\left(\sum_{z=0}^{l}\mathfrak{f}_z^2q_z|\widehat{d_z}|,\widehat{0}\right)}{\mathfrak{f}_l\mathfrak{f}_{l+1}}\right)^{t_l}$$

$$\leq\left(2^{2\beth-1}+2^{\beth-1}+2^{\beth}\right)\sum_{l\in\mathcal{N}}\left(\frac{\widehat{\hbar}\left(\sum_{z=0}^{l}\mathfrak{f}_z^2q_z|\widehat{d_z}|,\widehat{0}\right)}{\mathfrak{f}_l\mathfrak{f}_{l+1}}\right)^{t_l}=C_3\|(|\widehat{d_z}|)\|_{p-qN}<\infty,$$

then $(|\widehat{d_{[\frac{z}{2}]}}|) \in \gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)$.
Evidently, the conditions (a6) and (a7) can be easily shown.

Assume here and after the parts of theorem 3.3 are verified.

Theorem 3.4. *$(\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t))_{\|\cdot\|_{p-qN}}$ is a **p-q.B** $\mathfrak{psssf}$.*

Proof. In view of Theorem 2.1, then $(\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t))_{\|\cdot\|_{p-qN}}$ is a ***p-q.N*** $\mathfrak{psssf}$. To prove that $(\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t))_{\|\cdot\|_{p-qN}}$ is a ***p-q.B*** $\mathfrak{psssf}$, assume $\widehat{f^a} = (\widehat{f_z^a})_{z\in\mathcal{N}}$ is a ***Cs*** in $(\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t))_{\|\cdot\|_{p-qN}}$, then for $\lambda \in (0,1)$, we get $m_0 \in \mathcal{N}$ for every $m, j \geq m_0$, so

$$\|\widehat{d^m} - \widehat{d^j}\|_{p-qN} = \sum_{l\in\mathcal{N}} \left(\frac{\widehat{\hbar}\left(\sum_{z=0}^{l} \mathfrak{f}_z^2 q_z \left(\widehat{d_z^z} - \widehat{d_z^j}\right), \widehat{0}\right)}{\mathfrak{f}_l \mathfrak{f}_{l+1}} \right)^{t_l} < \lambda^{\beth}.$$

So $\widehat{\hbar}\left(\sum_{z=0}^{l} \mathfrak{f}_z^2 q_z \left(\widehat{d_z^m} - \widehat{d_z^j}\right), \widehat{0}\right) < \lambda$. Since $(\mathbf{R}(\mathbb{A}), \widehat{\hbar})$ is a ***CMs***. Therefore, $(\widehat{d_z^j})$ is a ***Cs*** in $\mathbf{R}(\mathbb{A})$, for fixed $z \in \mathcal{N}$. Therefore, $\|\widehat{d^m} - \widehat{d^0}\|_{p-qN} < \lambda^{\beth}$, for any $m \geq m_0$. Obviously from the linearity, $\widehat{d^0} \in (\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t))_{\|\cdot\|_{p-qN}}$.

4. *NTK*-$\|.\|_{p-qN}$-*C*

We will explore the existence and uniqueness of the fixed point of ***NTK***-$\|.\|_{p-qN}$-***C*** defined on $\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)$ in this section.
Supposing that the conditions of theorem 3.3 are confirmed. We will use in this part the following two equivalent ***p-q.Ns***:

$$\|\widehat{f}\|_{p-qN} = \left[\sum_{l\in\mathcal{N}} \left(\frac{\widehat{\hbar}\left(\sum_{z=0}^{l} \mathfrak{f}_z^2 q_z \widehat{f}_z, \widehat{0}\right)}{\mathfrak{f}_l \mathfrak{f}_{l+1}} \right)^{t_l} \right]^{\frac{1}{\beth}} \text{ and } \|\widehat{f}\|_{p-qN}^{\beth} = \sum_{l\in\mathcal{N}} \left(\frac{\widehat{\hbar}\left(\sum_{z=0}^{l} \mathfrak{f}_z^2 q_z \widehat{f}_z, \widehat{0}\right)}{\mathfrak{f}_l \mathfrak{f}_{l+1}} \right)^{t_l},$$

for every $\widehat{f} \in \gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)$.

Theorem 4.1. *The **p-q.N** $\|\widehat{f}\|_{p-qN}$ satisfies the **FP**.*

Proof. Assume $\{\widehat{u^d}\} \subseteq \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)\right)_{\|\cdot\|_{p-qN}}$ with $\lim_{d\to\infty} \|\widehat{u^d} - \widehat{u}\|_{p-qN} = 0$. As $\left(\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)\right)_{\|\cdot\|_{p-qN}}$ is a ***p-q.B***, then $\widehat{u} \in \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)\right)_{\|\cdot\|_{p-qN}}$. Hence

$$\begin{aligned}
\|\widehat{f} - \widehat{u}\|_{p-qN} &= \left[\sum_{l\in\mathcal{N}} \left(\frac{\widehat{\hbar}\left(\sum_{z=0}^{l} \mathfrak{f}_z^2 q_z(\widehat{f_z} - \widehat{u_z}), \widehat{0}\right)}{\mathfrak{f}_l \mathfrak{f}_{l+1}}\right)^{t_l}\right]^{\frac{1}{\beth}} \\
&\leq \left[\sum_{l\in\mathcal{N}} \left(\frac{\widehat{\hbar}\left(\sum_{z=0}^{l} \mathfrak{f}_z^2 q_z(\widehat{f_z} - \widehat{u_z^d}), \widehat{0}\right)}{\mathfrak{f}_l \mathfrak{f}_{l+1}}\right)^{t_l}\right]^{\frac{1}{\beth}} + \left[\sum_{l\in\mathcal{N}} \left(\frac{\widehat{\hbar}\left(\sum_{z=0}^{l} \mathfrak{f}_z^2 q_z(\widehat{u_z^d} - \widehat{u_z}), \widehat{0}\right)}{\mathfrak{f}_l \mathfrak{f}_{l+1}}\right)^{t_l}\right]^{\frac{1}{\beth}} \\
&\leq \sup_v \inf_{d\geq v} \|\widehat{f} - \widehat{u^d}\|_{p-qN}.
\end{aligned}$$

Theorem 4.2. *The **p-q.N** $\|\widehat{f}\|_{p-qN}^{\beth}$ does not hold the **FP** under $t_0 > 1$.*

Proof. When $\{\widehat{u^d}\} \subseteq \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)\right)_{\|\cdot\|_{p-qN}^{\beth}}$ with $\lim_{d\to\infty} \|\widehat{u^d} - \widehat{u}\|_{p-qN}^{\beth} = 0$. As $\left(\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)\right)_{\|\cdot\|_{p-qN}^{\beth}}$ is a ***p-q.B***, then $\widehat{u} \in \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)\right)_{\|\cdot\|_{p-qN}^{\beth}}$. So

$$\begin{aligned}
\|\widehat{f} - \widehat{u}\|_{p-qN}^{\beth} &= \sum_{l\in\mathcal{N}} \left(\frac{\widehat{\hbar}\left(\sum_{z=0}^{l} \mathfrak{f}_z^2 q_z(\widehat{f_z} - \widehat{u_z}), \widehat{0}\right)}{\mathfrak{f}_l \mathfrak{f}_{l+1}}\right)^{t_l} \\
&\leq 2^{\beth-1} \left[\sum_{l\in\mathcal{N}} \left(\frac{\widehat{\hbar}\left(\sum_{z=0}^{l} \mathfrak{f}_z^2 q_z(\widehat{f_z} - \widehat{u_z^d}), \widehat{0}\right)}{\mathfrak{f}_l \mathfrak{f}_{l+1}}\right)^{t_l} + \sum_{l\in\mathcal{N}} \left(\frac{\widehat{\hbar}\left(\sum_{z=0}^{l} \mathfrak{f}_z^2 q_z(\widehat{u_z^d} - \widehat{u_z}), \widehat{0}\right)}{\mathfrak{f}_l \mathfrak{f}_{l+1}}\right)^{t_l}\right] \\
&\leq 2^{\beth-1} \sup_j \inf_{d\geq j} \|\widehat{f} - \widehat{u^d}\|_{p-qN}^{\beth}.
\end{aligned}$$

Definition 7. *A mapping $M : \mathcal{E}_{\|\cdot\|_{p-qN}}^{\mathbf{S}} \to \mathcal{E}_{\|\cdot\|_{p-qN}}^{\mathbf{S}}$ is said to be a **NTK**-$\|.\|_{p-qN}$-**C** if there are $\{\alpha_i\}_{i=1}^{3} \subset [0,1)$ with $\alpha_1 + \alpha_2 + \alpha_3 \in [0,1)$ so that*

$$\|M\widehat{u} - M\widehat{k}\|_{p-qN} \leq \alpha_1\|M\widehat{u} - \widehat{u}\|_{p-qN} + \alpha_2\|M\widehat{k} - \widehat{k}\|_{p-qN} + \alpha_3\|\widehat{u} - \widehat{k}\|_{p-qN}$$

for any $\widehat{u}, \widehat{k} \in \mathcal{E}_{\|\cdot\|_{p-qN}}^{\mathbf{S}}$.

If $M(\widehat{u}) = \widehat{u}$, we say $\widehat{u} \in \mathcal{E}_{\|\cdot\|_{p-qN}}^{\mathbf{S}}$ is a fixed point of M.

Theorem 4.3. *Assume $W : \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)\right)_{\|\cdot\|_{p-qN}} \to \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)\right)_{\|\cdot\|_{p-qN}}$ is **NTK**-$\|.\|_{p-qN}$-**C**, then W has a **ufp**.*

Proof. When $\widehat{d} \in \gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)$, then $W^m\widehat{d} \in \gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)$. Since W is a ***NTK***-$\|.\|_{p-qN}$-***C***, then

$$\|W^{m+1}\widehat{d} - W^m\widehat{d}\|_{p-qN} \leq \alpha_1\|W^{m+1}\widehat{d} - W^m\widehat{d}\|_{p-qN} + \alpha_2\|W^m\widehat{d} - W^{m-1}\widehat{d}\|_{p-qN} + \alpha_3\|W^m\widehat{d} - W^{m-1}\widehat{d}\|_{p-qN} \Rightarrow$$

$$\|W^{m+1}\widehat{d} - W^m\widehat{d}\|_{p-qN} \leq \frac{\alpha_2+\alpha_3}{1-\alpha_1}\|W^m\widehat{d} - W^{m-1}\widehat{d}\|_{p-qN} \leq \left(\frac{\alpha_2+\alpha_3}{1-\alpha_1}\right)^2 \|W^{m-1}\widehat{d} - W^{m-2}\widehat{d}\|_{p-qN} \leq \dots$$
$$\leq \left(\frac{\alpha_2+\alpha_3}{1-\alpha_1}\right)^m \|W\widehat{d} - \widehat{d}\|_{p-qN}.$$

For every $m, n \in \mathcal{N}$ so that $n > m$, we obtain

$$\begin{aligned}\|W^m\widehat{d} - W^n\widehat{d}\|_{p-qN} &\leq \alpha_1\|W^m\widehat{d} - W^{m-1}\widehat{d}\|_{p-qN} + \alpha_2\|W^n\widehat{d} - W^{n-1}\widehat{d}\|_{p-qN} + \alpha_3\|W^{m-1}\widehat{d} - W^{n-1}\widehat{d}\|_{p-qN} \\ &\leq \left(\alpha_1\left(\frac{\alpha_2+\alpha_3}{1-\alpha_1}\right)^{m-1} + \alpha_2\left(\frac{\alpha_2+\alpha_3}{1-\alpha_1}\right)^{n-1}\right)\|W\widehat{d} - \widehat{d}\|_{p-qN} + \alpha_3\|W^{m-1}\widehat{d} - W^{n-1}\widehat{d}\|_{p-qN}\end{aligned}$$

Hence, $\{W^m\widehat{d}\}$ is a ***Cs*** in $\left(\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)\right)_{\|\cdot\|_{p-qN}}$. Since $\left(\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)\right)_{\|\cdot\|_{p-qN}}$ is ***p-q.B***. One has $\widehat{v} \in \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)\right)_{\|\cdot\|_{p-qN}}$ with $\lim_{m\to\infty} W^m\widehat{d} = \widehat{v}$. As $\|.\|_{p-qN}$ has the ***FP***, so

$$\|W\widehat{v} - \widehat{v}\|_{p-qN} \leq \sup_i \inf_{m\geq i} \|W^{m+1}\widehat{d} - W^m\widehat{d}\|_{p-qN} \leq \sup_i \inf_{m\geq i} \left(\frac{\alpha_2+\alpha_3}{1-\alpha_1}\right)^m \|W\widehat{d} - \widehat{d}\|_{p-qN} = 0,$$

then $W\widehat{v} = \widehat{v}$. Hence, $\widehat{v}$ is a ***fp*** of W. Next, when we have two ***fp*** $\widehat{a}, \widehat{v} \in \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)\right)_{\|\cdot\|_{p-qN}}$ of W with $\widehat{a} \neq \widehat{v}$. Then

$$(1-\alpha_3)\|\widehat{a} - \widehat{v}\|_{p-qN} \leq \alpha_1\|W\widehat{a} - \widehat{a}\|_{p-qN} + \alpha_2\|W\widehat{v} - \widehat{v}\|_{p-qN} = 0.$$

So $\widehat{a} = \widehat{v}$.

Corollary 4.4. *Suppose* $W : \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)\right)_{\|\cdot\|_{p-qN}} \to \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)\right)_{\|\cdot\|_{p-qN}}$ *is* ***NTK***-$\|.\|_{p-qN}$-***C****, then* W *has a* ***ufp*** $\widehat{a}$ *under* $\|W^m\widehat{d} - \widehat{a}\|_{p-qN} \leq \alpha_1\left(\frac{\alpha_2+\alpha_3}{1-\alpha_1}\right)^{m-1}\|W\widehat{d} - \widehat{d}\|_{p-qN} + \alpha_3\|W^{m-1}\widehat{d} - \widehat{a}\|_{p-qN}$.

Proof. By Theorem 4.3, we have a ***ufp*** $\widehat{a}$ of W. Hence

$$\begin{aligned}\|W^m\widehat{d} - \widehat{a}\|_{p-qN} &= \|W^m\widehat{d} - W\widehat{a}\|_{p-qN} \leq \alpha_1\|W^m\widehat{d} - W^{m-1}\widehat{d}\|_{p-qN} + \alpha_2\|W\widehat{a} - \widehat{a}\|_{p-qN} + \alpha_3\|W^{m-1}\widehat{d} - \widehat{a}\|_{p-qN} \\ &= \alpha_1\left(\frac{\alpha_2+\alpha_3}{1-\alpha_1}\right)^{m-1}\|W\widehat{d} - \widehat{d}\|_{p-qN} + \alpha_3\|W^{m-1}\widehat{d} - \widehat{a}\|_{p-qN}.\end{aligned}$$

Theorem 4.5. *Assuming that* $W : \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)\right)_{\|.\|_{p-qN}^{\beth}} \to \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)\right)_{\|.\|_{p-qN}^{\beth}}$, *where* $\|\widehat{d}\|_{p-qN}^{\beth} = \sum_{l\in\mathcal{N}} \left(\frac{\hbar\left(\sum_{z=0}^{l} \mathfrak{f}_z^2 q_z \widehat{d}_z, \widehat{0}\right)}{\mathfrak{f}_l \mathfrak{f}_{l+1}} \right)^{t_l}$, *for all* $\widehat{d} \in \gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)$ *under* $t_0 > 1$. *If the following conditions*

(k1) *W is **NTK**-$\|.\|_{p-qN}^{\beth}$-**C**,*

(k2) *W is $\|.\|_{p-qN}^{\beth}$-**Seq.C** at $\widehat{v} \in \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)\right)_{\|.\|_{p-qN}^{\beth}}$, and*

(k3) *there is an element $\widehat{d} \in \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)\right)_{\|.\|_{p-qN}^{\beth}}$ so that the sequence of iterates $\{W^m \widehat{d}\}$ has a subsequence $\{W^{m_i}\widehat{d}\}$ converges to $\widehat{v}$,*

*are satisfied, then the vector $\widehat{v} \in \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)\right)_{\|.\|_{p-qN}^{\beth}}$ is the **ufp** of W.*

Proof. Let $W\widehat{v} \neq \widehat{v}$. From parts (k2) and (k3), one obtains

$$\lim_{m_i\to\infty} \|W^{m_i}\widehat{d} - \widehat{v}\|_{p-qN}^{\beth} = 0 \text{ and } \lim_{m_i\to\infty} \|W^{m_i+1}\widehat{d} - W\widehat{v}\|_{p-qN}^{\beth} = 0.$$

Since W is ***NTK***-$\|.\|_{p-qN}^{\beth}$-***C***, then

$$\begin{aligned} 0 < \|W\widehat{v} - \widehat{v}\|_{p-qN}^{\beth} &= \|(W\widehat{v} - W^{m_i+1}\widehat{d}) + (W^{m_i}\widehat{d} - \widehat{v}) + (W^{m_i+1}\widehat{d} - W^{m_i}\widehat{d})\|_{p-qN}^{\beth} \\ &\leq 2^{2\beth-2}\|W^{m_i+1}\widehat{v} - W\widehat{v}\|_{p-qN}^{\beth} + 2^{2\beth-2}\|W^{m_i}\widehat{v} - \widehat{v}\|_{p-qN}^{\beth} + 2^{\beth-1}\left(\frac{\alpha_2+\alpha_3}{1-\alpha_1}\right)^{m_i}\|W\widehat{d} - \widehat{d}\|_{p-qN}^{\beth} \end{aligned}$$

When $m_i \to \infty$, we get a contradiction. So, $W\widehat{v} = \widehat{v}$. For the uniqueness, assume $W\widehat{v} = \widehat{v}$ and $W\widehat{a} = \widehat{a}$, where $\widehat{v}, \widehat{a} \in \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)\right)_{\|.\|_{p-qN}^{\beth}}$ and $\widehat{v} \neq \widehat{a}$. Hence

$$\|\widehat{v} - \widehat{a}\|_{p-qN}^{\beth} \leq \|W\widehat{v} - W\widehat{a}\|_{p-qN}^{\beth} \leq \alpha_1\|W\widehat{v} - \widehat{v}\|_{p-qN}^{\beth} + \alpha_2\|W\widehat{a} - \widehat{a}\|_{p-qN}^{\beth} + \alpha_3\|\widehat{v} - \widehat{a}\|_{p-qN}^{\beth}.$$

So, $\widehat{v} = \widehat{a}$.

Example 4.6. *Supposing that*
$\Phi : \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}\left(\left(\frac{1}{(l+5)\mathfrak{f}_l^2}\right)_{l\in\mathcal{N}}, \left(\frac{2l+3}{l+2}\right)_{l\in\mathcal{N}}\right)\right)_{\|\cdot\|_{p-qN}} \to \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}\left(\left(\frac{1}{(l+5)\mathfrak{f}_l^2}\right)_{l\in\mathcal{N}}, \left(\frac{2l+3}{l+2}\right)_{l\in\mathcal{N}}\right)\right)_{\|\cdot\|_{p-qN}}$, *where*

$\|\widehat{d}\|_{p-qN} = \sqrt{\sum_{l\in\mathcal{N}} \left(\frac{\hbar\left(\sum_{z=0}^{l} \frac{\widehat{d_z}}{z+5}, \widehat{0}\right)}{\mathfrak{f}_l \mathfrak{f}_{l+1}} \right)^{\frac{2l+3}{l+2}}}$, *with* $\widehat{v}, \widehat{d} \in \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}\left(\left(\frac{1}{(l+5)\mathfrak{f}_l^2}\right)_{l\in\mathcal{N}}, \left(\frac{2l+3}{l+2}\right)_{l\in\mathcal{N}}\right)\right)_{\|\cdot\|_{p-qN}}$

and

$$\Phi(\widehat{d}) = \begin{cases} \frac{\widehat{d}}{4}, & \|\widehat{d}\|_{p-qN} \in [0,1), \\ \frac{\widehat{d}}{5}, & \|\widehat{d}\|_{p-qN} \in [1,\infty). \end{cases}$$

If $\|\widehat{d}\|_{p-qN}, \|\widehat{v}\|_{p-qN} \in [0,1)$*, one has*

$$\begin{aligned}\|\Phi\widehat{d}-\Phi\widehat{v}\|_{p-qN} &= \|\frac{\widehat{d}}{4}-\frac{\widehat{v}}{4}\|_{p-qN} \leq \frac{1}{\sqrt[4]{27}}\Big(\|\frac{3\widehat{d}}{4}\|_{p-qN}+\|\frac{3\widehat{v}}{4}\|_{p-qN}\Big)+0.1\|\widehat{d}-\widehat{v}\|_{p-qN}\\ &= \frac{1}{\sqrt[4]{27}}\Big(\|\Phi\widehat{d}-\widehat{d}\|_{p-qN}+\|\Phi\widehat{v}-\widehat{v}\|_{p-qN}\Big)+0.1\|\widehat{d}-\widehat{v}\|_{p-qN}.\end{aligned}$$

If $\|\widehat{d}\|_{p-qN}, \|\widehat{v}\|_{p-qN} \in [1,\infty)$*, then*

$$\begin{aligned}\|\Phi\widehat{d}-\Phi\widehat{v}\|_{p-qN} &= \|\frac{\widehat{d}}{5}-\frac{\widehat{v}}{5}\|_{p-qN} \leq \frac{1}{\sqrt[4]{64}}\Big(\|\frac{4\widehat{d}}{5}\|_{p-qN}+\|\frac{4\widehat{v}}{5}\|_{p-qN}\Big)+0.2\|\widehat{d}-\widehat{v}\|_{p-qN}\\ &= \frac{1}{\sqrt[4]{64}}\Big(\|\Phi\widehat{d}-\widehat{d}\|_{p-qN}+\|\Phi\widehat{v}-\widehat{v}\|_{p-qN}\Big)+0.2\|\widehat{d}-\widehat{v}\|_{p-qN}.\end{aligned}$$

Suppose $\|\widehat{d}\|_{p-qN} \in [0,1)$ *and* $\|\widehat{v}\|_{p-qN} \in [1,\infty)$*, then*

$$\begin{aligned}\|\Phi\widehat{d}-\Phi\widehat{v}\|_{p-qN} &= \|\frac{\widehat{d}}{4}-\frac{\widehat{v}}{5}\|_{p-qN} \leq \frac{1}{\sqrt[4]{27}}\|\frac{3\widehat{d}}{4}\|_{p-qN}+\frac{1}{\sqrt[4]{64}}\|\frac{4\widehat{v}}{5}\|_{p-qN}+0.1\|\widehat{d}-\widehat{v}\|_{p-qN}\\ &= \frac{1}{\sqrt[4]{27}}\|\Phi\widehat{d}-\widehat{d}\|_{p-qN}+\frac{1}{\sqrt[4]{64}}\|\Phi\widehat{v}-\widehat{v}\|_{p-qN}+0.1\|\widehat{d}-\widehat{v}\|_{p-qN}.\end{aligned}$$

Therefore, Φ *is* ***NTK***-$\|.\|_{p-qN}$-***C****. Since* $\|.\|_{p-qN}$ *verifies the* ***FP****. By Theorem 4.3, one obtains* Φ *has a* ***ufp*** ϑ.
Assume $\{\widehat{d^{(a)}}\} \subseteq \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}\left(\left(\frac{1}{(l+5)\mathfrak{f}_l^2}\right)_{l\in\mathcal{N}}, \left(\frac{2l+3}{l+2}\right)_{l\in\mathcal{N}}\right)\right)_{\|\cdot\|_{p-qN}}$ *under* $\lim_{a\to\infty}\|\widehat{d^{(a)}}-\widehat{d^{(0)}}\|_{p-qN} = 0$*, where*
$\widehat{d^{(0)}} \in \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}\left(\left(\frac{1}{(l+5)\mathfrak{f}_l^2}\right)_{l\in\mathcal{N}}, \left(\frac{2l+3}{l+2}\right)_{l\in\mathcal{N}}\right)\right)_{\|\cdot\|_{p-qN}}$ *and* $\|\widehat{d^{(0)}}\|_{p-qN} = 1$*. As the* ***p-q.N*** $\|.\|_{p-qN}$ *is continuous, one gets*

$$\lim_{a\to\infty}\|\Phi\widehat{d^{(a)}}-\Phi\widehat{d^{(0)}}\|_{p-qN} = \lim_{a\to\infty}\|\frac{\widehat{d^{(a)}}}{4}-\frac{\widehat{d^{(0)}}}{5}\|_{p-qN} = \|\frac{\widehat{d^{(0)}}}{20}\|_{p-qN} > 0.$$

Hence, Φ *is not* $\|.\|_{p-qN}$-***Seq.C*** *at* $\widehat{d^{(0)}}$*. Therefore,* Φ *is not continuous at* $\widehat{d^{(0)}}$*.*
Assume $\|\widehat{d}\|_{p-qN}^2 = \sum_{l\in\mathcal{N}}\left(\frac{\widehat{\hbar}\left(\sum_{z=0}^{l}\frac{\widehat{d_z}}{z+5}, \widehat{0}\right)}{\mathfrak{f}_l\mathfrak{f}_{l+1}}\right)^{\frac{2l+3}{l+2}}$*, so that* $\widehat{d}, \widehat{v} \in \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}\left(\left(\frac{1}{(l+5)\mathfrak{f}_l^2}\right)_{l\in\mathcal{N}}, \left(\frac{2l+3}{l+2}\right)_{l\in\mathcal{N}}\right)\right)_{\|\cdot\|_{p-qN}^2}$.
For $\|\widehat{d}\|_{p-qN}^2, \|\widehat{v}\|_{p-qN}^2 \in [0,1)$*, one gets*

$$\begin{aligned}\|\Phi\widehat{d}-\Phi\widehat{v}\|_{p-qN}^2 &= \|\frac{\widehat{d}}{4}-\frac{\widehat{v}}{4}\|_{p-qN}^2 \leq \frac{2}{\sqrt{27}}\Big(\|\frac{3\widehat{d}}{4}\|_{p-qN}^2+\|\frac{3\widehat{v}}{4}\|_{p-qN}^2\Big)+0.05\|\widehat{d}-\widehat{v}\|_{p-qN}^2\\ &= \frac{2}{\sqrt{27}}\Big(\|\Phi\widehat{d}-\widehat{d}\|_{p-qN}^2+\|\Phi\widehat{v}-\widehat{v}\|_{p-qN}^2\Big)+0.05\|\widehat{d}-\widehat{v}\|_{p-qN}^2.\end{aligned}$$

If $\|\widehat{d}\|^2_{p-qN}, \|\widehat{v}\|^2_{p-qN} \in [1,\infty)$*, then*

$$\begin{aligned}\|\Phi\widehat{d}-\Phi\widehat{v}\|^2_{p-qN} &= \|\frac{\widehat{d}}{5}-\frac{\widehat{v}}{5}\|^2_{p-qN} \leq \frac{1}{4}\Big(\|\frac{4\widehat{d}}{5}\|^2_{p-qN}+\|\frac{4\widehat{v}}{5}\|^2_{p-qN}\Big)+0.01\|\widehat{d}-\widehat{v}\|^2_{p-qN}\\ &= \frac{1}{4}\Big(\|\Phi\widehat{d}-\widehat{d}\|^2_{p-qN}+\|\Phi\widehat{v}-\widehat{v}\|^2_{p-qN}\Big)+0.01\|\widehat{d}-\widehat{v}\|^2_{p-qN}.\end{aligned}$$

When $\|\widehat{d}\|^2_{p-qN} \in [0,1)$ *and* $\|\widehat{v}\|^2_{p-qN} \in [1,\infty)$*, one obtains*

$$\begin{aligned}\|\Phi\widehat{d}-\Phi\widehat{v}\|^2_{p-qN} &= \|\frac{\widehat{d}}{4}-\frac{\widehat{v}}{5}\|^2_{p-qN} \leq \frac{2}{\sqrt{27}}\|\frac{3\widehat{d}}{4}\|^2_{p-qN}+\frac{1}{4}\|\frac{4\widehat{v}}{5}\|^2_{p-qN}+0.01\|\widehat{d}-\widehat{v}\|^2_{p-qN}\\ &= \frac{2}{\sqrt{27}}\|\Phi\widehat{d}-\widehat{d}\|^2_{p-qN}+\frac{1}{4}\|\Phi\widehat{v}-\widehat{v}\|^2_{p-qN}+0.01\|\widehat{d}-\widehat{v}\|^2_{p-qN}.\end{aligned}$$

Therefore, Φ *is* ***NTK-***$\|.\|^2_{p-qN}$***-C*** *and* $\Phi^m(\widehat{d}) = \begin{cases}\frac{\widehat{d}}{4^m}, & \|\widehat{d}\|^2_{p-qN} \in [0,1),\\ \frac{\widehat{d}}{5^m}, & \|\widehat{d}\|^2_{p-qN} \in [1,\infty).\end{cases}$

Clearly, Φ *is* $\|.\|^2_{p-qN}$***-Seq.C*** *at* $\widehat{\vartheta} \in \left(\gamma^{\mathbf{S}}_{\mathfrak{f}}\left(\left(\frac{1}{(l+5)\mathfrak{f}_l^2}\right)_{l\in\mathcal{N}}, \left(\frac{2l+3}{l+2}\right)_{l\in\mathcal{N}}\right)\right)_{\|.\|^2_{p-qN}}$ *and* $\{\Phi^m\widehat{d}\}$ *includes a* $\{\Phi^{m_j}\widehat{d}\}$ *converges to* $\widehat{\vartheta}$*. According Theorem 4.5, the element* $\widehat{\vartheta}$ *is the* ***ufp*** *of* Φ*.*

Example 4.7. *Assuming that*
$\Phi : \left(\gamma^{\mathbf{S}}_{\mathfrak{f}}\left(\left(\frac{1}{(l+5)\mathfrak{f}_l^2}\right)_{l\in\mathcal{N}}, \left(\frac{2l+3}{l+2}\right)_{l\in\mathcal{N}}\right)\right)_{\|\cdot\|^2_{p-qN}} \to \left(\gamma^{\mathbf{S}}_{\mathfrak{f}}\left(\left(\frac{1}{(l+5)\mathfrak{f}_l^2}\right)_{l\in\mathcal{N}}, \left(\frac{2l+3}{l+2}\right)_{l\in\mathcal{N}}\right)\right)_{\|\cdot\|^2_{p-qN}}$,
where
$\|\widehat{d}\|^2_{p-qN} = \sum_{l\in\mathcal{N}}\left(\frac{\widehat{\hbar}\left(\sum_{z=0}^{l}\frac{\widehat{d_z}}{z+5},\widehat{0}\right)}{\mathfrak{f}_l\mathfrak{f}_{l+1}}\right)^{\frac{2l+3}{l+2}}$*, so that* $\widehat{d} \in \left(\gamma^{\mathbf{S}}_{\mathfrak{f}}\left(\left(\frac{1}{(l+5)\mathfrak{f}_l^2}\right)_{l\in\mathcal{N}}, \left(\frac{2l+3}{l+2}\right)_{l\in\mathcal{N}}\right)\right)_{\|\cdot\|^2_{p-qN}}$
and for all $t \in \mathbb{A}$*,*

$$\Phi(\widehat{d}) = \begin{cases}\frac{1}{4}(\widehat{\mathfrak{e}}_1+\widehat{d}), & \widehat{d}_0(t) \in [0,\frac{1}{3}),\\ \frac{1}{3}\widehat{\mathfrak{e}}_1, & \widehat{d}_0(t) = \frac{1}{3},\\ \frac{1}{4}\widehat{\mathfrak{e}}_1, & \widehat{d}_0(t) \in (\frac{1}{3},1].\end{cases}$$

If $\widehat{f},\widehat{g} \in \left(\gamma^{\mathbf{S}}_{\mathfrak{f}}\left(\left(\frac{1}{(l+5)\mathfrak{f}_l^2}\right)_{l=0}^{\infty}, \left(\frac{2l+3}{l+2}\right)_{l=0}^{\infty}\right)\right)_{\|\cdot\|^2_{p-qN}}$ *with* $\widehat{f}_0(t), \widehat{g}_0(t) \in [0,\frac{1}{3})$*, then*

$$\begin{aligned}\|\Phi\widehat{f}-\Phi\widehat{g}\|^2_{p-qN} &= \|\frac{1}{4}(\widehat{f}_0-\widehat{g}_0,\widehat{f}_1-\widehat{g}_1,\widehat{f}_2-\widehat{g}_2,\ldots)\|^2_{p-qN}\\ &\leq \frac{2}{\sqrt{27}}\Big(\|\frac{3\widehat{f}}{4}\|^2_{p-qN}+\|\frac{3\widehat{g}}{4}\|^2_{p-qN}\Big)+0.03\|\widehat{f}-\widehat{g}\|^2_{p-qN}\\ &\leq \frac{2}{\sqrt{27}}\Big(\|\Phi\widehat{f}-\widehat{f}\|^2_{p-qN}+\|\Phi\widehat{g}-\widehat{g}\|^2_{p-qN}\Big)+0.03\|\widehat{f}-\widehat{g}\|^2_{p-qN}.\end{aligned}$$

For every $\widehat{f}, \widehat{g} \in \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}\left(\left(\frac{1}{(l+5)\mathfrak{f}_l^2}\right)_{l=0}^{\infty}, \left(\frac{2l+3}{l+2}\right)_{l=0}^{\infty}\right)\right)_{\|\cdot\|_{p-qN}^2}$ *under* $\widehat{f_0}(t), \widehat{g_0}(t) \in (\frac{1}{3}, 1]$, *hence for all* $\varepsilon_i > 0$ *and* $i = 1, 2,$ *and* 3, *we have*

$$\|\Phi\widehat{f} - \Phi\widehat{g}\|_{p-qN}^2 = 0 \leq \varepsilon_1\|\Phi\widehat{f} - \widehat{f}\|_{p-qN}^2 + \varepsilon_2\|\Phi\widehat{g} - \widehat{g}\|_{p-qN}^2 + \varepsilon_3\|\widehat{f} - \widehat{g}\|_{p-qN}^2.$$

If $\widehat{f}, \widehat{g} \in \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}\left(\left(\frac{1}{(l+5)\mathfrak{f}_l^2}\right)_{l=0}^{\infty}, \left(\frac{2l+3}{l+2}\right)_{l=0}^{\infty}\right)\right)_{\|\cdot\|_{p-qN}^2}$ *with* $\widehat{f_0}(t) \in [0, \frac{1}{3})$ *and* $\widehat{g_0}(t) \in (\frac{1}{3}, 1]$, *then*

$$\begin{aligned}\|\Phi\widehat{f} - \Phi\widehat{g}\|_{p-qN}^2 &= \|\frac{\widehat{f}}{4}\|_{p-qN}^2 \leq \frac{1}{\sqrt{27}}\|\frac{3\widehat{f}}{4}\|_{p-qN}^2 = \frac{1}{\sqrt{27}}\|\Phi\widehat{f} - \widehat{f}\|_{p-qN}^2 + 0.2\|\widehat{f} - \widehat{g}\|_{p-qN}^2 \\ &\leq \frac{1}{\sqrt{27}}\left(\|\Phi\widehat{f} - \widehat{f}\|_{p-qN}^2 + \|\Phi\widehat{g} - \widehat{g}\|_{p-qN}^2\right) + 0.2\|\widehat{f} - \widehat{g}\|_{p-qN}^2.\end{aligned}$$

Hence, Φ *is* ***NTK-***$\|.\|_{p-qN}^2$***-C****,* $\|.\|_{p-qN}^2$***-Seq.C*** *at* $\frac{1}{3}\widehat{\mathfrak{e}_1} \in \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}\left(\left(\frac{1}{(l+5)\mathfrak{f}_l^2}\right)_{l\in\mathcal{N}}, \left(\frac{2l+3}{l+2}\right)_{l\in\mathcal{N}}\right)\right)_{\|\cdot\|_{p-qN}^2}$, *discontinuous at* $\frac{1}{3}\widehat{\mathfrak{e}_1}$, *and we have* $\widehat{d}$ *so that* $\widehat{d_0} \in [0, \frac{1}{3})$ *under* $\{\Phi^m\widehat{d}\} = \left\{\sum_{a=1}^{m} \frac{1}{4^a}\widehat{\mathfrak{e}_1} + \frac{1}{4^m}\widehat{d}\right\}$ *has a* $\{\Phi^{m_j}\widehat{d}\} = \left\{\sum_{a=1}^{m_j} \frac{1}{4^a}\widehat{\mathfrak{e}_1} + \frac{1}{4^{m_j}}\widehat{d}\right\}$ *converges to* $\frac{1}{3}\widehat{\mathfrak{e}_1}$. *By Theorem 4.5,* Φ *has* ***ufp*** *at* $\frac{1}{3}\widehat{\mathfrak{e}_1}$.

For $\Phi : \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}\left(\left(\frac{1}{(l+5)\mathfrak{f}_l^2}\right)_{l\in\mathcal{N}}, \left(\frac{2l+3}{l+2}\right)_{l\in\mathcal{N}}\right)\right)_{\|\cdot\|_{p-qN}} \to \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}\left(\left(\frac{1}{(l+5)\mathfrak{f}_l^2}\right)_{l\in\mathcal{N}}, \left(\frac{2l+3}{l+2}\right)_{l\in\mathcal{N}}\right)\right)_{\|\cdot\|_{p-qN}}$,

where $\|\widehat{d}\|_{p-qN} = \sqrt{\sum_{l\in\mathcal{N}}\left(\frac{\hbar\left(\sum_{z=0}^{l}\frac{\widehat{d_z}}{z+5}, \widehat{0}\right)}{\mathfrak{f}_l\mathfrak{f}_{l+1}}\right)^{\frac{2l+3}{l+2}}}$, *for all* $\widehat{d} \in \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}\left(\left(\frac{1}{(l+5)\mathfrak{f}_l^2}\right)_{l\in\mathcal{N}}, \left(\frac{2l+3}{l+2}\right)_{l\in\mathcal{N}}\right)\right)_{\|\cdot\|_{p-qN}}$.

If $\widehat{f}, \widehat{g} \in \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}\left(\left(\frac{1}{(l+5)\mathfrak{f}_l^2}\right)_{l=0}^{\infty}, \left(\frac{2l+3}{l+2}\right)_{l=0}^{\infty}\right)\right)_{\|\cdot\|_{p-qN}}$ *with* $\widehat{f_0}(t), \widehat{g_0}(t) \in [0, \frac{1}{3})$, *then*

$$\begin{aligned}\|\Phi\widehat{f} - \Phi\widehat{g}\|_{p-qN} &= \|\frac{1}{4}(\widehat{f_0} - \widehat{g_0}, \widehat{f_1} - \widehat{g_1}, \widehat{f_2} - \widehat{g_2}, \ldots)\|_{p-qN} \leq \frac{1}{\sqrt[4]{27}}\left(\|\frac{3\widehat{f}}{4}\|_{p-qN} + \|\frac{3\widehat{g}}{4}\|_{p-qN}\right) + 0.01\|\widehat{f} - \widehat{g}\|_{p-qN} \\ &\leq \frac{1}{\sqrt[4]{27}}\left(\|\Phi\widehat{f} - \widehat{f}\|_{p-qN} + \|\Phi\widehat{g} - \widehat{g}\|_{p-qN}\right) + 0.01\|\widehat{f} - \widehat{g}\|_{p-qN}.\end{aligned}$$

Suppose $\widehat{f}, \widehat{g} \in \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}\left(\left(\frac{1}{(l+5)\mathfrak{f}_l^2}\right)_{l=0}^{\infty}, \left(\frac{2l+3}{l+2}\right)_{l=0}^{\infty}\right)\right)_{\|\cdot\|_{p-qN}}$ *with* $\widehat{f_0}(t), \widehat{g_0}(t) \in (\frac{1}{3}, 1]$, *hence for all* $\varepsilon_i > 0$ *and* $i = 1, 2,$ *and* 3, *then*

$$\|\Phi\widehat{f} - \Phi\widehat{g}\|_{p-qN} = 0 \leq \varepsilon_1\|\Phi\widehat{f} - \widehat{f}\|_{p-qN} + \varepsilon_2\|\Phi\widehat{g} - \widehat{g}\|_{p-qN} + \varepsilon_3\|\widehat{f} - \widehat{g}\|_{p-qN}.$$

If $\widehat{f}, \widehat{g} \in \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}\left(\left(\frac{1}{(l+5)\mathfrak{f}_l^2}\right)_{l=0}^{\infty}, \left(\frac{2l+3}{l+2}\right)_{l=0}^{\infty}\right)\right)_{\|\cdot\|_{p-qN}}$ *with* $\widehat{f_0}(t) \in [0, \frac{1}{3})$ *and* $\widehat{g_0}(t) \in (\frac{1}{3}, 1]$, *one obtains*

$$\begin{aligned}\|\Phi\widehat{f} - \Phi\widehat{g}\|_{p-qN} &= \|\frac{\widehat{f}}{4}\|_{p-qN} \leq \frac{1}{\sqrt[4]{27}}\|\frac{3\widehat{f}}{4}\|_{p-qN} = \frac{1}{\sqrt[4]{27}}\|\Phi\widehat{f} - \widehat{f}\|_{p-qN} \\ &\leq \frac{1}{\sqrt[4]{27}}\left(\|\Phi\widehat{f} - \widehat{f}\|_{p-qN} + \|\Phi\widehat{g} - \widehat{g}\|_{p-qN}\right) + 0.01\|\widehat{f} - \widehat{g}\|_{p-qN}.\end{aligned}$$

*Therefore, Φ is **NTK**-$\|.\|_{p-qN}$-**C**. As $\|.\|_{p-qN}$ verifies the **FP**. By Theorem 4.3, Φ has a **ufp** $\frac{1}{3}\widehat{\mathfrak{e}}_1$.*

5. Applications

Understanding economic models requires a firm grasp of summable equations, which offer a mathematical basis for investigating issues like producer and consumer surplus, total cost computation, and revenue functions, among others. The use of summable equations in the formulation and solution of economic problems has recently expanded substantially. Take a look at [2, 8, 10, 18, 21] and the citations that follow.
We present the existence and uniqueness of the soft dynamical systems (1) in $\left(\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)\right)_{\|\cdot\|_{p-qN}}$, where the conditions of theorem 3.3 are confirmed under the two equivalent ***p-q.Ns*** $\|\widehat{\nu}\|_{p-qN}$ and $\|\widehat{\nu}\|_{p-qN}^{\beth}$, for any $\widehat{\nu} \in \gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)$.

Theorem 5.1. *Assume that $\widehat{\xi} : \mathcal{N} \to \mathbf{R}(\mathbb{A})$. The dynamical systems (1) have a unique solution in $\left(\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)\right)_{\|\cdot\|_{p-qN}}$ whenever if there are $\varepsilon_i \in \mathbf{R}$ so that $\sum_{i=1}^{3} \sup_u |\varepsilon_i|^{\frac{t_u}{\beth}} \in [0,1)$ and for every $u \in \mathcal{N}$, then*

$$\left|\sum_{d=0}^{u}\left(\sum_{v\in\mathcal{N}}\Gamma_{d,v}[\Psi_{v,\widehat{\nu_v}} - \Psi_{v,\widehat{\xi_v}}]\right)\mathfrak{f}_d^2 q_d\right| \widehat{\leq} |\varepsilon_1| \left|\sum_{d=0}^{u}\left(\widehat{\beta_d} - \widehat{\nu}_d + \sum_{v\in\mathcal{N}}\Gamma_{d,v}\Psi_{v,\widehat{\nu_v}}\right)\mathfrak{f}_d^2 q_d\right|$$
$$+ |\varepsilon_2| \left|\sum_{d=0}^{u}\left(\widehat{\beta_d} - \widehat{\xi}_d + \sum_{v\in\mathcal{N}}\Gamma_{d,v}\Psi_{v,\widehat{\xi_v}}\right)\mathfrak{f}_d^2 q_d\right| + |\varepsilon_3| |\sum_{d=0}^{u}\left(\widehat{\nu}_d - \widehat{\xi}_d\right)\mathfrak{f}_d^2 q_d|.$$

Proof. Let $\Phi : \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)\right)_{\|\cdot\|_{p-qN}} \to \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)\right)_{\|\cdot\|_{p-qN}}$ is defined by equation (2). By Theorem 4.3 and

$$\|\Phi\widehat{\nu} - \Phi\widehat{\xi}\|_{p-qN} = \left[\sum_{u\in\mathcal{N}}\left(\frac{\widehat{\hbar}\left(\sum_{d=0}^{u}\mathfrak{f}_d^2 q_d(\Phi\widehat{\nu}_d - \Phi\widehat{\xi}_d), \widehat{0}\right)}{\mathfrak{f}_u\mathfrak{f}_{u+1}}\right)^{t_u}\right]^{\frac{1}{\beth}}$$
$$= \left[\sum_{u\in\mathcal{N}}\left(\frac{\widehat{\hbar}\left(\sum_{d=0}^{u}\left(\sum_{v\in\mathcal{N}}\Gamma_{d,v}[\Psi_{v,\widehat{\nu_v}} - \Psi_{v,\widehat{\xi_v}}]\right)\mathfrak{f}_d^2 q_d, \widehat{0}\right)}{\mathfrak{f}_u\mathfrak{f}_{u+1}}\right)^{t_u}\right]^{\frac{1}{\beth}}$$

$$
\begin{aligned}
&\leq \sup_u |\varepsilon_1|^{\frac{t_u}{\beth}} \left[\sum_{u\in\mathcal{N}} \left(\frac{\widehat{\hbar}\left(\sum_{d=0}^{u}\left(\widehat{\beta_d} - \widehat{\nu}_d + \sum_{v\in\mathcal{N}} \Gamma_{d,v}\Psi_{v,\widehat{\nu_v}}\right)\mathfrak{f}_d^2 q_d, \widehat{0}\right)}{\mathfrak{f}_u\mathfrak{f}_{u+1}}\right)^{t_u}\right]^{\frac{1}{\beth}} \\
&+ \sup_u |\varepsilon_2|^{\frac{t_u}{\beth}} \left[\sum_{u\in\mathcal{N}} \left(\frac{\widehat{\hbar}\left(\sum_{d=0}^{u}\left(\widehat{\beta_d} - \widehat{\xi}_d + \sum_{v\in\mathcal{N}} \Gamma_{d,v}\Psi_{v,\widehat{\xi_v}}\right)\mathfrak{f}_d^2 q_d, \widehat{0}\right)}{\mathfrak{f}_u\mathfrak{f}_{u+1}}\right)^{t_u}\right]^{\frac{1}{\beth}} \\
&+ \sup_u |\varepsilon_3|^{\frac{t_u}{\beth}} \left[\sum_{u\in\mathcal{N}} \left(\frac{\widehat{\hbar}\left(\sum_{d=0}^{u}\left(\widehat{\nu}_d - \widehat{\xi}_d\right)\mathfrak{f}_d^2 q_d, \widehat{0}\right)}{\mathfrak{f}_u\mathfrak{f}_{u+1}}\right)^{t_u}\right]^{\frac{1}{\beth}} \\
&= \sup_u |\varepsilon_1|^{\frac{t_u}{\beth}} \|\Phi\widehat{\nu} - \widehat{\nu}\|_{p-qN} + \sup_u |\varepsilon_2|^{\frac{t_u}{\beth}} \|\Phi\widehat{\xi} - \widehat{\xi}\|_{p-qN} + \sup_u |\varepsilon_3|^{\frac{t_u}{\beth}} \|\widehat{\nu} - \widehat{\xi}\|_{p-qN}.
\end{aligned}
$$

we have a unique solution of the dynamical systems (1) in $\left(\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)\right)_{\|\cdot\|_{p-qN}}$.

Example 5.2. *Supposing that* $\left(\gamma_{\mathfrak{f}}^{\mathbf{S}}\left(\left(\left(\frac{1}{(u+1)\mathfrak{f}_u^2}\right)_{u\in\mathcal{N}}, \left(\frac{2u+3}{u+2}\right)_{u\in\mathcal{N}}\right)\right)\right)_{\|\cdot\|_{p-qN}}$, *where*

$\|\widehat{\nu}\|_{p-qN} = \sqrt{\sum_{u\in\mathcal{N}} \left(\frac{\widehat{\hbar}\left(\sum_{d=0}^{u}\frac{\widehat{\nu_d}}{d+1}, \widehat{0}\right)}{\mathfrak{f}_u\mathfrak{f}_{u+1}}\right)^{\frac{2u+3}{u+2}}}$, *for all* $\widehat{\nu} \in \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}\left(\left(\left(\frac{1}{(u+1)\mathfrak{f}_u^2}\right)_{u\in\mathcal{N}}, \left(\frac{2u+3}{u+2}\right)_{u\in\mathcal{N}}\right)\right)\right)_{\|\cdot\|_{p-qN}}$.

Assume that the ***NLDEs****:*

$$
\widehat{\nu}_d = \log_2\widehat{(d^4+1)} + \sum_{v\in\mathcal{N}} \cosh d \cos^2 v \frac{\widehat{\nu_{d-2}^{x}}}{\widehat{\nu_{d-1}^{y}} + \tanh\widehat{(2v+3)}}, \tag{4}
$$

for every $x, y, \widehat{\nu_{-2}}(t), \widehat{\nu_{-1}}(t) > 0$, *under* $t \in \mathbb{A}$.
Let the mapping Φ *be defined as*

$$
\Phi : \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}\left(\left(\left(\frac{1}{(u+1)\mathfrak{f}_u^2}\right)_{u\in\mathcal{N}}, \left(\frac{2u+3}{u+2}\right)_{u\in\mathcal{N}}\right)\right)\right)_{\|\cdot\|_{p-qN}} \to \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}\left(\left(\left(\frac{1}{(u+1)\mathfrak{f}_u^2}\right)_{u\in\mathcal{N}}, \left(\frac{2u+3}{u+2}\right)_{u\in\mathcal{N}}\right)\right)\right)_{\|\cdot\|_{p-qN}},
$$

where

$$
\Phi(\widehat{\nu}_d)_{d\in\mathcal{N}} = \left(\log_2\widehat{(d^4+1)} + \sum_{v\in\mathcal{N}} \cosh d \cos^2 v \frac{\widehat{\nu_{d-2}^{x}}}{\widehat{\nu_{d-1}^{y}} + \tanh\widehat{(2v+3)}}\right)_{d\in\mathcal{N}}. \tag{5}
$$

Obviously, we have $\varepsilon_i \in \mathbf{R}$ *with* $\sum_{i=1}^{3} \sup_u |\varepsilon_i|^{\frac{2u+3}{2u+4}} \in [0,1)$ *and for any* $u \in \mathcal{N}$, *hence*

$$\left|\sum_{d=0}^{u}\left(\sum_{v\in\mathcal{N}} \frac{\cosh d\, \widehat{\nu_{d-2}^{x}}}{\widehat{\nu_{d-1}^{y}} + \widehat{\tanh(2v+3)}}\left(\cos^2 v - \cos^2 v\right)\right)\mathfrak{f}_d^2 q_d\right|$$

$$\widehat{\leq}|\varepsilon_1|\left|\sum_{d=0}^{u}\left(\log_2\widehat{(d^4+1)} - \widehat{\nu}_d + \sum_{v\in\mathcal{N}} \cosh d \cos^2 v \frac{\widehat{\nu_{d-2}^{x}}}{\widehat{\nu_{d-1}^{y}} + \widehat{\tanh(2v+3)}}\right)\mathfrak{f}_d^2 q_d\right| +$$

$$|\varepsilon_2|\left|\sum_{d=0}^{u}\left(\log_2\widehat{(d^4+1)} - \widehat{\eta}_d + \sum_{v\in\mathcal{N}} \cosh d \cos^2 v \frac{\widehat{\eta_{d-2}^{x}}}{\widehat{\eta_{d-1}^{y}} + \widehat{\tanh(2v+3)}}\right)\mathfrak{f}_d^2 q_d\right| + |\varepsilon_3|\left|\sum_{d=0}^{u}\left(\widehat{\nu}_d - \widehat{\eta}_d\right)\mathfrak{f}_d^2 q_d\right|.$$

By Theorem 5.1, the ***NLDEs*** *(4) include a unique solution in* $\left(\gamma_{\mathfrak{f}}^{\mathbf{S}}\left(\left(\frac{1}{(u+1)\mathfrak{f}_u^2}\right)_{u\in\mathcal{N}}, \left(\frac{2u+3}{u+2}\right)_{u\in\mathcal{N}}\right)\right)_{\|\cdot\|_{p-qN}}$.

Theorem 5.3. *Assume that* $\Phi : \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)\right)_{\|\cdot\|_{p-qN}^{\beth}} \to \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)\right)_{\|\cdot\|_{p-qN}^{\beth}}$ *is defined by (2). The dynamical systems (1) have a unique solution* $\widehat{Z} \in \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)\right)_{\|\cdot\|_{p-qN}^{\beth}}$, *when the following conditions are satisfied:*

(k1) *Suppose* $\Gamma : \mathcal{N}^2 \to \mathfrak{R}$, $\Psi : \mathcal{N} \times \mathbf{R}(\mathbb{A}) \to \mathbf{R}(\mathbb{A})$, $\widehat{\nu} : \mathcal{N} \to \mathbf{R}(\mathbb{A})$, $\widehat{\beta} : \mathcal{N} \to \mathbf{R}(\mathbb{A})$, $\widehat{\xi} : \mathcal{N} \to \mathbf{R}(\mathbb{A})$, *if one has* $\varepsilon_i \in \mathbf{R}$ *under* $2^{2\beth-2}\sum_{i=1}^{3}\sup_u |\varepsilon_i|^{t_u} \in [0,1)$ *and for every* $u \in \mathcal{N}$, *one gets*

$$\left|\sum_{d=0}^{u}\left(\sum_{v\in\mathcal{N}} \Gamma_{d,v}[\Psi_{v,\widehat{\nu_v}} - \Psi_{v,\widehat{\xi_v}}]\right)\mathfrak{f}_d^2 q_d\right| \widehat{\leq} |\varepsilon_1|\left|\sum_{d=0}^{u}\left(\widehat{\beta_d} - \widehat{\nu_d} + \sum_{v\in\mathcal{N}} \Gamma_{d,v}\Psi_{v,\widehat{\nu_v}}\right)\mathfrak{f}_d^2 q_d\right|$$
$$+ |\varepsilon_2|\left|\sum_{d=0}^{u}\left(\widehat{\beta_d} - \widehat{\xi_d} + \sum_{v\in\mathcal{N}} \Gamma_{d,v}\Psi_{v,\widehat{\xi_v}}\right)\mathfrak{f}_d^2 q_d\right| + |\varepsilon_3|\left|\sum_{d=0}^{u}\left(\widehat{\nu}_d - \widehat{\xi}_d\right)\mathfrak{f}_d^2 q_d\right|.$$

(k2) Φ *is* $\|.\|_{p-qN}^{\beth}$-***Seq.C*** *at* $\widehat{Z} \in \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)\right)_{\|\cdot\|_{p-qN}^{\beth}}$,

(k3) *there is* $\widehat{Y} \in \left(\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)\right)_{\|\cdot\|_{p-qN}^{\beth}}$ *with* $\{\Phi^m \widehat{Y}\}$ *has* $\{\Phi^{m_j}\widehat{Y}\}$ *converging to* $\widehat{Z}$.

Proof. By Theorem 4.5 and

$$\|\Phi\widehat{\nu} - \Phi\widehat{\xi}\|_{p-qN}^{\beth} = \sum_{u\in\mathcal{N}}\left(\frac{\widehat{\hbar}\left(\sum_{d=0}^{u}\mathfrak{f}_d^2 q_d(\Phi\widehat{\nu_d} - \Phi\widehat{\xi_d}), \widehat{0}\right)}{\mathfrak{f}_u\mathfrak{f}_{u+1}}\right)^{t_u}$$
$$= \sum_{u\in\mathcal{N}}\left(\frac{\widehat{\hbar}\left(\sum_{d=0}^{u}\left(\sum_{v\in\mathcal{N}}\Gamma_{d,v}[\Psi_{v,\widehat{\nu_v}} - \Psi_{v,\widehat{\xi_v}}]\right)\mathfrak{f}_d^2 q_d, \widehat{0}\right)}{\mathfrak{f}_u\mathfrak{f}_{u+1}}\right)^{t_u}$$

$$\leq 2^{2\beth-2}\sup_u|\varepsilon_1|^{t_u}\sum_{u\in\mathcal{N}}\left(\frac{\widehat{\hbar}\left(\sum_{d=0}^{u}\left(\widehat{\beta_d}-\widehat{\nu}_d+\sum_{v\in\mathcal{N}}\Gamma_{d,v}\Psi_{v,\widehat{\nu_v}}\right)\mathfrak{f}_d^2q_d,\widehat{0}\right)}{\mathfrak{f}_u\mathfrak{f}_{u+1}}\right)^{t_u}+$$

$$2^{2\beth-2}\sup_u|\varepsilon_2|^{t_u}\sum_{u\in\mathcal{N}}\left(\frac{\widehat{\hbar}\left(\sum_{d=0}^{u}\left(\widehat{\beta_d}-\widehat{\xi}_d+\sum_{v\in\mathcal{N}}\Gamma_{d,v}\Psi_{v,\widehat{\xi_v}}\right)\mathfrak{f}_d^2q_d,\widehat{0}\right)}{\mathfrak{f}_u\mathfrak{f}_{u+1}}\right)^{t_u}+$$

$$2^{\beth-1}\sup_u|\varepsilon_3|^{t_u}\sum_{u\in\mathcal{N}}\left(\frac{\widehat{\hbar}\left(\sum_{d=0}^{u}\left(\widehat{\nu}_d-\widehat{\xi}_d\right)\mathfrak{f}_d^2q_d,\widehat{0}\right)}{\mathfrak{f}_u\mathfrak{f}_{u+1}}\right)^{t_u}$$

$$\leq 2^{2\beth-2}\left(\sup_u|\varepsilon_1|^{t_u}\|\Phi\widehat{\nu}-\widehat{\nu}\|_{p-qN}^{\beth}+\sup_u|\varepsilon_2|^{t_u}\|\Phi\widehat{\xi}-\widehat{\xi}\|_{p-qN}^{\beth}+\sup_u|\varepsilon_3|^{t_u}\|\widehat{\nu}-\widehat{\xi}\|_{p-qN}^{\beth}\right).$$

That gives the required.

Example 5.4. *Supposing that* $\left(\gamma_{\mathfrak{f}}^{\mathbf{S}}\left(\left(\frac{1}{(u+1)\mathfrak{f}_u^2}\right)_{u\in\mathcal{N}},\left(\frac{2u+3}{u+2}\right)_{u\in\mathcal{N}}\right)\right)_{\|.\|_{p-qN}^2}$, *where*

$$\|\widehat{\nu}\|_{p-qN}^2=\sum_{u\in\mathcal{N}}\left(\frac{\widehat{\hbar}\left(\sum_{d=0}^{u}\frac{\widehat{\nu_d}}{d+1},\widehat{0}\right)}{\mathfrak{f}_u\mathfrak{f}_{u+1}}\right)^{\frac{2u+3}{u+2}},\ \text{for all } \widehat{\nu}\in\left(\gamma_{\mathfrak{f}}^{\mathbf{S}}\left(\left(\frac{1}{(u+1)\mathfrak{f}_u^2}\right)_{u\in\mathcal{N}},\left(\frac{2u+3}{u+2}\right)_{u\in\mathcal{N}}\right)\right)_{\|.\|_{p-qN}^2}.$$

Consider the dynamical system (4) and the mapping $\Phi:\left(\gamma_{\mathfrak{f}}^{\mathbf{S}}\left(\left(\frac{1}{(u+1)\mathfrak{f}_u^2}\right)_{u\in\mathcal{N}},\left(\frac{2u+3}{u+2}\right)_{u\in\mathcal{N}}\right)\right)_{\|.\|_{p-qN}^2}\to\left(\gamma_{\mathfrak{f}}^{\mathbf{S}}\left(\left(\frac{1}{(u+1)\mathfrak{f}_u^2}\right)_{u\in\mathcal{N}},\left(\frac{2u+3}{u+2}\right)_{u\in\mathcal{N}}\right)\right)_{\|.\|_{p-qN}^2}$ *as defined by (5). Assume* Φ *is* $\|.\|_{p-qN}^2$***-Seq.C*** *at* $\widehat{Z}\in\left(\gamma_{\mathfrak{f}}^{\mathbf{S}}\left(\left(\frac{1}{(u+1)\mathfrak{f}_u^2}\right)_{u\in\mathcal{N}},\left(\frac{2u+3}{u+2}\right)_{u\in\mathcal{N}}\right)\right)_{\|.\|_{p-qN}^2}$, *and one has* $\widehat{Y}\in\left(\gamma_{\mathfrak{f}}^{\mathbf{S}}\left(\left(\frac{1}{(u+1)\mathfrak{f}_u^2}\right)_{u\in\mathcal{N}},\left(\frac{2u+3}{u+2}\right)_{u\in\mathcal{N}}\right)\right)_{\|.\|_{p-qN}^2}$ *with* $\{\Phi^k\widehat{Y}\}$ *has* $\{\Phi^{k_r}\widehat{Y}\}$ *converging to* $\widehat{Z}$. *Clearly, there are* $\varepsilon_i\in\mathbf{R}$ *such that* $4\sum_{i=1}^{3}\sup_u|\varepsilon_i|^{\frac{2u+3}{u+2}}\in[0,1)$ *and for any* $u\in\mathcal{N}$, *hence*

$$\left|\sum_{d=0}^{u}\left(\sum_{v\in\mathcal{N}}\frac{\cosh d\,\widehat{\nu_{d-2}^{x}}}{\widehat{\nu_{d-1}^{y}}+\widehat{\tanh(2v+3)}}\left(\cos^2 v-\cos^2 v\right)\right)\mathfrak{f}_d^2q_d\right|$$

$$\widehat{\leq}|\varepsilon_1|\left|\sum_{d=0}^{u}\left(\widehat{\log_2(d^4+1)}-\widehat{\nu}_d+\sum_{v\in\mathcal{N}}\cosh d\cos^2 v\frac{\widehat{\nu_{d-2}^{x}}}{\widehat{\nu_{d-1}^{y}}+\widehat{\tanh(2v+3)}}\right)\mathfrak{f}_d^2q_d\right|+$$

$$|\varepsilon_2|\left|\sum_{d=0}^{u}\left(\widehat{\log_2(d^4+1)}-\widehat{\eta}_d+\sum_{v\in\mathcal{N}}\cosh d\cos^2 v\frac{\widehat{\eta_{d-2}^{x}}}{\widehat{\eta_{d-1}^{y}}+\widehat{\tanh(2v+3)}}\right)\mathfrak{f}_d^2q_d\right|+|\varepsilon_3|\left|\sum_{d=0}^{u}\left(\widehat{\nu}_d-\widehat{\eta}_d\right)\mathfrak{f}_d^2q_d\right|.$$

In view of Theorem 5.3, the dynamical systems (4) have a unique solution $\widehat{Z}\in\left(\gamma_{\mathfrak{f}}^{\mathbf{S}}\left(\left(\frac{1}{(u+1)\mathfrak{f}_u^2}\right)_{u\in\mathcal{N}},\left(\frac{2u+3}{u+2}\right)_{u\in\mathcal{N}}\right)\right)_{\|.\|_{p-qN}^2}$.

6. Conclusion

In this article, we discussed several topological and geometric characteristics of $\left(\gamma_{\mathfrak{f}}^{\mathbf{S}}(q,t)\right)_{\|\cdot\|_{p-qN}}$. Analyzed is the novel Kannan contraction operator in this space, along with the potential for a fixed point. We conducted numerous numerical experiments to validate our theories. Investigations are conducted on soft functions with non-linear uncertainty equation implementations. Future work uses the innovative soft function space to analyze the fixed points of the new type of Kannan contraction operator, providing a new universal solution space for a variety of stochastic non-linear dynamical systems.

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