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Generalization of bi-antiideals in semigroups

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Abstract. Algebraic structure consisting of a set together with an associative internal binary operation on it, so called semigroup has applications in different fields of science. For a better understanding of these applications, semigroups are characterized through their subsets. Fuzzy sets deal with uncertainties, and because many real-life problems have an associated algebraic structure, fuzzification of these structures makes sense and is useful. This paper investigates the generalization of bi-antiideals in semigroups and their fuzzification to enhance understanding of algebraic structures with uncertainties. Building upon prior research, we define and explore (m, n)-bi-antiideals as an extension of bi-antiideals, studying their properties through theoretical analysis and illustrative examples. We introduce fuzzy (m, n)-bi-antiideals by leveraging fuzzy set theory to model uncertainties, establishing a connection with (m, n)-bi-antiideals via level sets.

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1. Introduction

The initial paper on semigroups emerged in 1905 as a concise work by L.E. Dickson. However, the true inception of the theory occurred in 1928 when A.K. Suschkewitsch [20] published a paper of paramount significance. In contemporary language, he demonstrated that within any finite semigroup, there exists a "kernel" (referred to as a simple ideal), and he comprehensively characterized the structure of finite simple semigroups. Semigroups provide a foundational framework for understanding how elements combine under certain operations, and their applications span across multiple branches of mathematics and various interdisciplinary fields such as Coding theory, Automata, etc. For more details about semigroup terminology and history, we refer to [6].

The history of fuzzy sets can be traced back to the mid-20th century when Lotfi Zadeh [21] introduced the concept of fuzzy logic in 1965. Zadeh's groundbreaking idea

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challenged the traditional binary approach of classical set theory by allowing elements to possess degrees of membership in sets, rather than being strictly classified as either inside or outside a set. This innovative notion found its roots in the observation that many real-world concepts are not easily definable in precise terms. Fuzzy sets quickly garnered attention across various disciplines, including algebraic structures. The combination of the two concepts led to the launch of fuzzy algebraic structures. The latter was established by Rosenfeld [17] in 1971 when he introduced fuzzy groups.

There are many research items in the literature characterizing semigroups through its (fuzzy) subsets. For example, (fuzzy) filters of a semigroup were studied in [3, 4, 10], and (fuzzy) ideals of a semigroup were studied in [12–15]. For further details, we refer to the work cited in [7, 9, 11, 16]. Inspired by the literature, our present work sets out on an exploration of specific subsets within semigroups and fuzzifies them. The remaining part is constructed as follows. After an Introduction, in Section 2 we present some results about (fuzzy) antiideals and (fuzzy) bi-antiideals of semigroups that are used in the subsequent sections. In Section 3 we generalizes bi-antiideals to (m, n)-bi-antiideals, discuss some of their properties, and give some non-trivial examples. In Section 4 we fuzzify (m, n)-bi-antiideals by introducing fuzzy (m, n)-bi-antiideals of a semigroup. Moreover, we link the two new notions by means of level sets.

2. (Fuzzy) Left(right) antiideals and bi-antiideals of a semigroup

In this section, we present some definitions and results that are used throughout the paper.

Antiideals were introduced by Schwarz [19] and were studied and generalized by Iseksi [8, 9]. Other antiideals of semigroups were introduced. For example, Al-Tahan and Sarka [5] introduced interior antiideals of a semigroup and investigated their properties and Al-Kaseasbeh et al. [18] studied antiideals of a semiring.

A non-empty set X with an associative binary operation is called a *semigroup* and a non-empty subset A of X is a *subsemigroup* of X if it is a semigroup. If X has an identity, then it is called a *monoid*. As simple examples, the set of non-negative even integers under standard addition is a semigroup and the set of positive real numbers under standard multiplication is a semigroup.

Definition 1. [8] *Let* $(X, \cdot)$ *be a semigroup and* $A \neq \emptyset \subseteq X$. *Then*

(i) A *is a left antiideal of* X *if* $XA \cap A = \emptyset$;

(ii) A *is a right antiideal of* X *if* $AX \cap A = \emptyset$;

(iii) A *is an antiideal of* X *if it is both a left and right antiideal of* X.

Example 1. *Let* $(K, \cdot)$ *be the semigroup of integers greater than* 1 *under standard multiplication of integers and* $A = \{2, 3\}$. *Then* A *is an antiideal of* K. *This is clear as*

$$KA \cap A = AK \cap A = \{4, 6, 8, 9, \ldots\} \cap \{2, 3\} = \emptyset.$$

We present an example of an infinite antiideal.

Example 2. *Let $M = \{1, 2, 3, 4, \ldots\}$ and define the semigroup $(M, \star)$ as follows.*

$$x \star y = \begin{cases} 1 & \text{if } x \text{ is an odd number;} \\ y & \text{otherwise.} \end{cases}$$

Then $A = \{3, 5, 7, 9, \ldots\}$ is an antiideal of M. This is clear as $AM \cap A = \{1\} \cap A = \emptyset$.

Al-Tahan et al. [2] introduced bi-antiideals of a semigroup. We present some of their results.

Definition 2. [2] *Let $(X, \cdot)$ be a semigroup and $A \neq \emptyset \subseteq X$. Then A is a bi-antiideal of X if $AXA \cap A = \emptyset$.*

Example 3. *Let $(K, \cdot)$ be the semigroup defined in Example 1 and $B = \{2, 3, 4\}$. Then B is a bi-antiideal of K. Moreover, it is not a left(right) antiideal of K. This is clear as $BK \cap B = \{4\} \neq \emptyset$.*

Proposition 1. *[4] Evey left(right) antiideal of a semigroup X is a bi-antiideal of X.*

Fuzzy sets were introduced by Zadeh [21] in 1965 to accommodate uncertainties that classical sets fail to deal with. In a fuzzy set, the element's membership is a real number in the unit interval.

Definition 3. [21] *Let X be a universal set, $I = [0, 1]$, and $\mu : X \to I$. Then a fuzzy set of X is given as: $A = \{(x, \mu(x)) : x \in X\}$. Here $\mu(x)$ denotes the membership's grade of the element x in X.*

Definition 4. [7] *For the fuzzy sets μ_1, μ_2 of X, the fuzzy sets $\mu_1 \wedge \mu_2, \mu_1 \vee \mu_2$ of X are defined as follows.*

$$(\mu_1 \wedge \mu_2)(x) = \min\{\mu_1(x), \mu_2(x)\} \qquad \text{for all } x \in X.$$

$$(\mu_1 \vee \mu_2)(x) = \max\{\mu_1(x), \mu_2(x)\} \qquad \text{for all } x \in X.$$

Definition 5. [7] *Let X_1, X_2 be non-empty sets and μ_1, μ_2 be fuzzy sets of X_1, X_2 respectively. Then the fuzzy set $\mu = \mu_1 \times \mu_2$ of $X_1 \times X_2$ is defined as follows.*

$$\mu((x_1, x_2)) = \min\{\mu_1(x_1), \mu_2(x_2)\} \qquad \text{for all } x_1 \in X_1, x_2 \in X_2.$$

Definition 6. [2] *Let $(X, \cdot)$ be a semigroup and $\mu : X \to [0, 1]$ be a non-zero fuzzy set of X. Then*

(i) μ is a fuzzy left antiideal of X if $\mu(ra) \wedge \mu(a) = 0$ for all $r, a \in X$;

(ii) μ is a fuzzy right antiideal of X if $\mu(ar) \wedge \mu(a) = 0$ for all $r, a \in X$;

(iii) *μ is a fuzzy antiideal of X if μ is a fuzzy left antiideal of X and a fuzzy right antiideal of X;*

(iv) *μ is a fuzzy bi-antiideal of X if $\mu(xry) \wedge \mu(x) \wedge \mu(y) = 0$ for all $r, x, y \in X$.*

Definition 7. [7] *Let $(X, \cdot)$ be a semigroup, μ be a non-zero fuzzy set of X, and $t \in [0, 1]$. Then the level set μ_t is defined as follows.*

$$\mu_t = \{x \in X : \mu(x) \geq t\}.$$

Example 4. *Let $(K, \cdot)$ be the semigroup defined in Example 1 and define the fuzzy sets μ_1, μ_2 on K as follows.*

$$\mu_1(k) = \begin{cases} 0.54 & \textit{if } k = 2; \\ 0 & \textit{otherwise.} \end{cases} \qquad \textit{and} \qquad \mu_2(k) = \begin{cases} 0.65 & \textit{if } k = 4; \\ 0.6 & \textit{if } k = 3; \\ 0.55 & \textit{if } k = 2; \\ 0 & \textit{otherwise.} \end{cases}$$

Then μ_1 is a fuzzy antiideal of K and μ_2 is a fuzzy bi-antiideal of K.

Theorem 1. [2] *Let X be a semigroup, $t \in]0, 1]$, and μ a non-zero fuzzy set of X. Then the following statements hold.*

(i) *μ is a fuzzy left(right) antiideal of X if and only of $\mu_t \neq \emptyset$ is a left(right) antiideal of X.*

(ii) *μ is a fuzzy bi-antiideal of X if and only of $\mu_t \neq \emptyset$ is a bi-antiideal of X.*

3. (m, n)-bi-antiideals of a semigroup

In this section and inspired by (m, n)-antiideals [1, 8] and by bi-antiideals [2], we introduce (m, n)-bi-antiideals of a semigroup as a generalization of bi-antiideals and study their properties. The results of this section are considered as a generalization of some results in [2].

Definition 8. *Let $(X, \cdot)$ be a semigroup, m, n be positive integers, and $A \neq \emptyset \subseteq X$. Then A is an (m, n)-bi-antiideal of X if $A^m X A^n \cap A = \emptyset$.*

Remark 1. *A monoid can have (m, n)-bi-antiideals. (See Example 5.)*

Example 5. *Let $(P_0, +)$ be the monoid of non-negative integers under standard addition of integers. Then $\{1, 2\}$ is a $(2, 1)$-bi-antiideal of P_0. This is clear as*

$$(\{1, 2\} + \{1, 2\} + P_0 + \{1, 2\}) \cap \{1, 2\} = \{x \in P_0 : x \geq 3\} \cap \{1, 2\} = \emptyset.$$

Proposition 2. *Let $(X, \cdot)$ be a semigroup and $A \neq \emptyset \subseteq X$ be a bi-antiideal of X. Then A is an (m, n)-bi-antiideal of X.*

Proof. The proof results form having

$$A^m X A^n \cap A = A(A^{m-1} X A^{n-1}) A \cap A \subseteq AXA \cap A = \emptyset.$$

Remark 2. *The converse of Proposition 2 may not hold. (See Example 6.)*

Example 6. *Let $M_2(P_0)$ be the semigroup of all two by two matrices with non-negative integral entries under multiplication of matrices and $A = \{\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}\}$. Then A is a $(2,1)$-bi-antiideal of $M_2(P_0)$. Furthermore, it is not a bi-antiideal of $M_2(P_0)$.*

Proposition 3. *Let $(X, \cdot)$ be a semigroup and $A \neq \emptyset \subseteq X$ be an (m,n)-bi-antiideal of X. If $k \geq m, andl \geq n$, then A is a (k,l)-bi-antiideal of X.*

Proof. The proof results form having

$$A^k X A^l \cap A = A^m (A^{k-m} X A^{l-n}) A^n \cap A \subseteq A^m X A^n \cap A = \emptyset.$$

Example 7. *Let $\mathbb{N}$ be the senigroup of natural numbers under standard multiplication and $A = \{2, 3, 6, 12\}$. Then A is a $(3,1)$-bi-antiideal of $\mathbb{N}$ that is not a $(2,1)$-bi-antiideal of $\mathbb{N}$. This is clear as $2(3)(1)(2) \in A^2 \mathbb{N} A \cap A$.*

Proposition 4. *Let $(X, \cdot)$ be a semigroup and $A \neq \emptyset \subseteq X$ be an (m,n)-bi-antiideal of X. Then A is not a subsemigroup of X.*

Proof. Let A be an (m,n)-bi-antiideal of X that is subsemigroup of A. Then $A^{m+n+1} = A^m A A^n \neq \emptyset \subseteq A^m X A^n \cap A = \emptyset$.

Al-Tahan et al. [2] proved that every left(right) antiideal of a semigroup X is a bi-antiideal of X. Example 8 shows that the converse may not hold.

Example 8. *Let $M_2(P_0)$ be the semigroup of all two by two matrices with non-negative integer entries and $A = \{\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}\}$. Then A is a bi-antiideal of $M_2(P_0)$. Furthermore, it is not a left(right) antiideal of P_0.*

Proposition 5. *Let $(X, \cdot)$ be a semigroup and $A \neq \emptyset \subseteq X$ be an (m,n)-bi-antiideal of X. Then every non-empty subset of A is an (m,n)-bi-antiideal of X.*

Proof. The proof is straightforward.

Corollary 1. *Let $(X, \cdot)$ be a semigroup and $A_i \neq \emptyset \subseteq X$ for $i \in \mathbb{N}$. If A_i is an (m,n)-bi-antiideal of X for some $i \in \mathbb{N}$, then every non-empty intersection of A_i is an (m,n)-bi-antiideal of X.*

Theorem 2. *Let X_1, X_2 be semigroups, $f : X_1 \to X_2$ be an onto semigroup homomorphism, and $A_1 \neq \emptyset \subseteq X_1$ an (m,n)-bi-antiideal of X_1. Then $f(A_1)$ is an (m,n)-bi-antiideal of X_2.*

Proof. Let $y \in f(A_1)^m X_2 f(A_1)^n \cap f(A_1)$. Then there exist $x_1, \ldots, x_m, z_1, \ldots, z_m \in A_1, x \in X_1$, $r = f(x) \in X_2$ with $y = f(x_1) \ldots f(x_m) f(r) f(y_1) \ldots f(y_n) \in f(A_1)$. Having f a semigroup homomorphism implies that $y = f(x_1 \ldots x_m r y_1 \ldots y_n) \in f(A_1)$ and hence, $x_1 \ldots x_m r y_1 \ldots y_n \in A_1^m X_1 A_1^n \cap A_1 = \emptyset$.

Theorem 3. *Let X_1, X_2 be semigroups, $f : X_1 \to X_2$ be a semigroup homomorphism, and $A_2 \neq \emptyset \subseteq X_2$ an (m,n)-bi-antiideal of X_2. Then $f^{-1}(A_2) \neq \emptyset$ is an (m,n)-bi-antiideal of X_1.*

Proof. Let $x \in f^{-1}(A_2)^m X_1 f^{-1}(A_2)^n \cap f^{-1}(A_2)$. Then there exist $x_i, z_j \in f^{-1}(A_2)$ with $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n\}$, $r \in X_1$ satisfying $x = x_1 \ldots x_m r y_1 \ldots y_n \in f^{-1}(A_2)$ and hence

$$f(x) = f(x_1 \ldots x_m r y_1 \ldots y_n) \in A_2.$$

Having f a semigroup homomorphism implies that

$$y = f(x_1) \ldots f(x_m) f(r) f(y_1) \ldots f(y_n) \in A_2$$

and hence, $y \in A_2^m X_2 A_2^n \cap A_2 = \emptyset$.

Example 9. *Let $M_2(P_0)$, $M_2(2P_0)$ be the semigroups of all two by two matrices with non-negative integer entries and with non-negative even integral entries under multiplication of matrices respectively defined in Example 6 and $f : M_2(P_0) \to M_2(2P_0)$ be defined as follows. For every matrix $M \in M_2(P_0)$, $f(M) = 2M$. From Example 8, we have $A = \{\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}\}$ is a $(2,1)$-bi-antiideal of $M_2(P_0)$. Having f an onto semigroup homorphism implies that $f(A) = \{\begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 8 & 0 \\ 0 & 0 \end{pmatrix}\}$ is a $(2,1)$-bi-antiideal of $M_2(2P_0)$.*

4. Fuzzy (m,n)-bi-antiideals of a semigroup

In this section, we introduce new fuzzy algebraic structures and study their properties. More precisely and inspired by fuzzy interior antiideals introduced in [2] and fuzzy antiideals of a semiring [18], we define fuzzy (m,n)-bi-antiideals of a semigroup.

Definition 9. *Let $(X, \cdot)$ be a semigroup, m, n be positive integers, and $\mu : X \to [0,1]$ be a non-zero fuzzy set of X. Then μ is a fuzzy (m,n)-bi-antiideal of X if for all $x_i, y_j, r \in X$,*

$$\mu(x_1 \ldots x_m r y_1 \ldots y_n) \wedge \mu(x_1) \wedge \ldots \wedge \mu_(x_m) \wedge \mu(y_1) \wedge \ldots \wedge \mu(y_n) = 0.$$

Example 10. *Let P_0 be the semigroup of non-negative integers under standard addition and μ be the fuzzy set on P_0 defined as follows.*

$$\mu(k) = \begin{cases} 0.65 & \textit{if k=1;} \\ 0.54 & \textit{if k=2;} \\ 0 & \textit{otherwise.} \end{cases}$$

Then μ is a fuzzy $(2,1)$-bi-antiideal of P_0. Moreover, it is not a fuzzy bi-antiideal of P_0. This is clear as $0.54 = \mu(2) \wedge \mu(1) = \mu(1+0+1) \wedge \mu(1)$.

Next, we study fuzzy (m,n)-bi-antiideals of a semigroup under some operations of fuzzy sets such as the intersection, union, and product of fuzzy sets.

Theorem 4. *Let $(X, \cdot)$ be a semigroup and μ_i be a non-zero fuzzy set of X for $i = 1, \ldots, k$. If μ_i is a fuzzy (m,n)-bi-antiideal of X for some $i \in \{1, \ldots, k\}$, then so is $\mu = \mu_1 \wedge \mu_2 \wedge \ldots \wedge \mu_k$.*

Proof. Let $x_1, \ldots, x_m, r, y_1, \ldots, y_n \in X$. Without loss of generality, let μ_1 be a fuzzy (m,n)-bi-antiideal of X. Then $\mu(x_1 \ldots x_m r y_1 \ldots y_n) \wedge \mu(x_1) \wedge \ldots \mu(x_m) \wedge \mu(y_1) \wedge \ldots \wedge \mu(y_n) \leq \mu_1(x_1 \ldots x_m r y_1 \ldots y_n) \wedge \mu_1(x_1) \wedge \ldots \mu_1(x_m) \wedge \mu_1(y_1) \wedge \ldots \wedge \mu_1(y_n) = 0$.

Remark 3. *The union of fuzzy (m,n)-bi-antiideals is not necessarily a fuzzy (m,n)-bi-antiideal. (See Example 11.)*

Example 11. *Let $M_2(P_0)$ be the semigroup defined in Example 6 and μ_1, μ_2 be defined as follows.*

$$\mu_1(B) = \begin{cases} 0.6 & \textit{if } B = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}; \\ 0 & \textit{otherwise.} \end{cases} \quad \textit{and} \quad \mu_2(B) = \begin{cases} 0.8 & \textit{if } B = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}; \\ 0.7 & \textit{if } B = \begin{pmatrix} 16 & 0 \\ 0 & 0 \end{pmatrix}; \\ 0 & \textit{otherwise.} \end{cases}$$

Then μ_1, μ_2 are fuzzy $(2,1)$-bi-antiideals of $M_2(P_0)$. The fuzzy set $\mu = \mu_1 \vee \mu_2$ of $M_2(P_0)$ is given by:

$$\mu(B) = \begin{cases} 0.6 & \textit{if } B = M_1 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}; \\ 0.8 & \textit{if } B = M_2 = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}; \\ 0.7 & \textit{if } B = M_3 = \begin{pmatrix} 16 & 0 \\ 0 & 0 \end{pmatrix}; \\ 0 & \textit{otherwise.} \end{cases}$$

Having $M_3 = M_1 M_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} M_2$ *and* $0.7 = \mu(M3)$ *implies that*

$$\mu(M_1 M_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} M_2) \wedge \mu(M_1) \wedge \mu(M_2) \neq 0.$$

Theorem 5. *Let X_1, X_2 be semigroups and μ_1, μ_2 be non-zero fuzzy sets of X_1, X_2 respectively. If μ_1 or μ_2 is a fuzzy (m,n)-bi-antiideal of X_1, X_2, then $\mu = \mu_1 \times \mu_2$ is a fuzzy (m,n)-bi-antiideal of $X_1 \times X_2$.*

Proof. Let $x_1, \ldots, x_m, r, y_1, \ldots, y_n \in X_1, z_1, \ldots, z_m, r', w_1, \ldots, w_n \in X_2$. Without loss of generality, let μ_1 be a fuzzy (m,n)-bi-antiideal of X_1. Then
$\mu((x_1, z_1) \ldots (x_m, z_m)(r, r')(y_1, w_1) \ldots (y_n, w_n) \wedge \mu((x_1, z_1)) \wedge \ldots \mu((x_m, z_m)) \wedge \mu((y_1, w_1)) \wedge \ldots \wedge \mu((y_n, w_n)) \leq \mu_1(x_1 \ldots x_m r y_1 \ldots y_n) \wedge \mu_1(x_1) \wedge \ldots \mu_1(x_m) \wedge \mu_1(y_1) \wedge \ldots \wedge \mu_1(y_n) = 0.$

Theorem 6. *Let $(X_i, \cdot)$ be a semigroup for $i = 1, 2, \ldots, k$ and μ_i be a non-zero fuzzy set of X_i for $i = 1, \ldots, k$. If μ_i is a fuzzy (m,n)-bi-antiideal of X_i for some $i \in \{1, \ldots, k\}$, then $\mu = \mu_1 \times \mu_2 \times \ldots \times \mu_k$ is a fuzzy (m,n)-bi-antiideal of $X_1 \times \ldots \times X_k$.*

Proof. The proof is similar to that of Theorem 5.

Next, we link fuzzy (m,n)-bi-antiideals of a semigroup X to (m,n)-bi-antiideals of X.

Theorem 7. *Let $(X, \cdot)$ be a semigroup, μ be a non-zero fuzzy set of X, and $t \in [0,1]$. Then μ is a fuzzy (m,n)-bi-antiideal of X if and only if μ_t is either the empty set or an (m,n)-bi-antiideal of X.*

Proof. Let μ be a fuzzy (m,n)-bi-antiideal of X, and $\alpha \in \mu_t^m X \mu_t^n \cap \mu_t \neq \emptyset$. Then there exist $x_1, \ldots, x_m, y_1, \ldots, y_n \in \mu_t$, $r \in X$ with $\alpha = x_1 \ldots x_m r y_1 \ldots y_n$. Having $\alpha, x_1, \ldots, x_m, y_1, \ldots, y_n \in \mu_t$ implies that

$$0 = \mu(x_1 \ldots x_m r y_1 \ldots y_n) \wedge \mu(x_1) \wedge \ldots \wedge \mu(x_m) \wedge \mu(y_1) \wedge \ldots \wedge \mu(y_n) \geq t.$$

Conversely, let $\mu(x_1 \ldots x_m r y_1 \ldots y_n) \wedge \mu(x_1) \wedge \ldots \wedge \mu(x_m) \wedge \mu(y_1) \wedge \ldots \wedge \mu(y_n) = t > 0$. Then $x_1 \ldots x_m r y_1 \ldots y_n, x_1, \ldots, x_m, y_1, \ldots, y_n \in \mu_t \neq \emptyset$ and hence,

$$x_1 \ldots x_m r y_1 \ldots y_n \in \mu_t^m X \mu_t^n \cap \mu_t = \emptyset.$$

Theorem 8. *Let $(X, \cdot)$ be a semigroup. Then every (m,n)-bi-antiideal of X can be represented as a level set of a fuzzy (m,n)-bi-antiideal of X.*

Proof. Let A be an (m,n)-bi-antiideal of X and define the fuzzy set μ of X as follows.

$$\mu(x) = \begin{cases} 0.94 & \text{if } x \in A; \\ 0 & \text{otherwise.} \end{cases}$$

One can easily see that $\mu_{0.94} = A$ and that μ is a fuzzy (m,n)-bi-antiideal of X.

Corollary 2. *Every fuzzy bi-antiideal of a semigroup X is a fuzzy (m, n)-bi-antiideal of X.*

Proof. The proof follows from Theorem 1, Proposition 2 and Theorem 8.

Corollary 3. *Every fuzzy left(right) antiideal is a fuzzy (m, n)-bi-antiideal.*

Proof. The proof follows from Proposition 1, Proposition 2, and Theorem 8.

Theorem 9. *Let $(X, \cdot)$ be a semigroup and $A \neq \emptyset \subseteq X$. Then A is an (m, n)-bi-antiideal of X if and only if μ_A is a fuzzy (m, n)-bi-antiideal of X. Here, for all $x \in X$,*

$$\mu_A(x) = \begin{cases} 1 & \textit{if } x \in A; \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Let A be an (m, n)-bi-antiideal of X and $x_1, \ldots, x_m, r, y_1, \ldots, y_n \in X$. If there exist $i \in \{1, \ldots, m\}$ or $j \in \{1, \ldots, n\}$ with $x_i \notin A$ or $y_j \notin A$, then $\mu_A(x_1 \ldots x_m r y_1 \ldots y_n) \wedge \mu_A(x_1) \wedge \ldots \wedge \mu_A(x_m) \wedge \mu_A(y_1) \wedge \ldots \wedge \mu_A(y_n) = 0$. Otherwise and having $A^m X A^n \cap A = \emptyset$ implies that $x_1 \ldots x_m r y_1 \ldots y_n \notin A$ and hence, $\mu_A(x_1 \ldots x_m r y_1 \ldots y_n) \wedge \mu_A(x_1) \wedge \mu_A(x_m) \wedge \mu_A(y_1) \wedge \mu_A(y_n) = 0$.

Conversely, let μ_A be a fuzzy (m, n)-bi-antiideal of X and $\alpha \in A^m X A^n \cap A$. Then there exist $x_1, \ldots, x_m, y_1, \ldots, y_n \in A, r \in X$ with $\alpha = x_1 \ldots x_m r y_1 \ldots y_n \in A$. The latter implies that

$$\mu_A(x_1 \ldots x_m r y_1 \ldots y_n) \wedge \mu_A(x_1) \wedge \mu_A(x_m) \wedge \mu_A(y_1) \wedge \mu_A(y_n) = 1 \neq 0.$$

5. Conclusion

In this paper, we have extended the theory of semigroups by introducing and characterizing (m, n)-bi-antiideals and their fuzzy counterparts. Our exploration began with a review of the foundational concepts of antiideals and bi-antiideals, followed by the generalization to (m, n)-bi-antiideals. We demonstrated the properties of these new structures through various propositions and examples. Furthermore, we incorporated fuzzy set theory to handle uncertainties in semigroups, defining and analyzing fuzzy (m, n)-bi-antiideals. We established connections between fuzzy (m, n)-bi-antiideals and their classical counterparts using level sets, providing a comprehensive framework for understanding these concepts.

Our findings contribute to the broader understanding of semigroups and their applications in different fields of mathematics and science. Future research could focus on exploring additional properties of (m, n)-bi-antiideals, identifying all antiideals in specific semigroups, extending the theory to other algebraic structures, and finding practical applications for these theoretical concepts. By advancing the study of semigroups and their fuzzy generalizations, we hope to inspire further investigations and applications in both theoretical and applied mathematics.

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