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Another Look at Hop Independence in Graphs

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Abstract. A set $S \subseteq V(G)$ is a hop independent set in an undirected graph G if $d_G(v, w) \neq 2$ for any two distinct vertices $v, w \in S$. The maximum cardinality among the hop independent sets in G, denoted by $\alpha_h(G)$, is called the hop independence number of G. The hop independent sets in the shadow graph, complementary prism, edge corona and disjunctive products of two graphs are characterized. These characterizations are used to determine the exact or sharp bounds of the hop independence numbers of these graphs. Furthermore, we show that the hop independent set decision problem (HISP) is NP-complete.

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Key Words and Phrases: Hop independence number, shadow graph, complementary prism, edge corona, disjunction, strong product

1. Introduction

Hop domination is a domination-related concept introduced and studied by Natarajan and Ayyaswamy in [7]. This parameter has been widely studied since its introduction and some variations of the concept have been defined and investigated (see for example [1], [2], [5], [8], [9], and [10]).

Recently, Hassan et al. [4] introduced an independent-type parameter called hop independence. As mentioned in the paper, the motivation of such study is the ever increasing figure of research on hop-domination related topics. It's worth noting that the hop independence number provides a sharp upper bound for the hop domination number of a graph. The authors also showed that the absolute difference of the (ordinary) independence number and the hop independence number can be made arbitrarily large. Moreover, hop independent sets in the join, corona, lexicographic product, and Cartesian product of two graphs have been characterized. Subsequently, sharp bounds (exact values for others)

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of the hop independence numbers of these graphs have been obtained. In [3], the authors used the concept of hop independence to define a variation of hop domination.

Karp in [6], as one of his many original problems, showed that the clique decision problem is NP-complete. We shall use this result to show that the hop independence decision problem is also NP-complete.

2. Terminology and Notation

For any two vertices u and v in an undirected connected graph G, the distance $d_G(u,v)$ is the length of a shortest path joining u and v. Any u-v path of length $d_G(u,v)$ is called a *u-v geodesic*. The distance between two subsets A and B of $V(G)$ is given by $d_G(A,B) = \min\{d_G(a,b) : a \in A \text{ and } b \in B\}$. The *open neighborhood* of a point u is the set $N_G(u)$ consisting of all points v which are adjacent to u. The *closed neighborhood* of u is $N_G[u] = N_G(u) \cup \{u\}$. For any $A \subseteq V(G)$, $N_G(A) = \bigcup_{v\in A} N_G(v)$ is called the *open neighborhood of A* and $N_G[A] = N_G(A) \cup A$ is called the *closed neighborhood of A*. A vertex v of G is isolated if $|N_G(v)| = 0$.

The *open hop neighborhood* of a point u is the set $N^2_G(u) = \{v \in V(G) : d_G(v,u) = 2\}$. The *closed hop neighborhood* of u is $N^2_G[u] = N^2_G(u) \cup \{u\}$. For any $A \subseteq V(G)$, $N^2_G(A) = \bigcup_{v\in A} N^2_G(v)$ is called the *open hop neighborhood of A* and $N^2_G[A] = N^2_G(A) \cup A$ is called the *closed hop neighborhood of A*.

A set $S \subseteq V(G)$ is a *hop dominating* set if $N^2_G[S] = V(G)$. The minimum cardinality of a hop dominating set of a graph G, denoted by $\gamma_h(G)$, is called the *hop domination number* of G. A set $S \subseteq V(G)$ is an *independent* set of G if no two pair of distinct vertices of S are adjacent. The maximum cardinality of an independent set of G, denoted by $\alpha(G)$, is called the *independence number* of G. Set S is a *hop independent* set of G if $d_G(v,w) \neq 2$ for any two distinct vertices v and w of S. The maximum cardinality of a hop independent set of G, denoted by $\alpha_h(G)$, is called the *hop independence number* of G. Any independent (hop independent) set with cardinality $\alpha(G)$ (resp. $\alpha_h(G)$) is referred to as a *maximum independent* set or α-set (resp. *maximum hop independent* set or α_h-set) of G.

A set S is *clique* of a graph G if the graph $\langle S\rangle$ induced by S is a complete graph. The maximum size or cardinality of a clique of G, denoted by $\omega(G)$, is called the *clique number* of G. Any clique in G with cardinality $\omega(G)$ is called an ω-set in G. A *matching* of a graph G is a set $M = \{e_1, e_2, \cdots, e_n\} \subseteq E(G)$ such that each vertex $v \in V(G)$ appears in at most one edge in M (i.e., the edges in M do not have a common vertex). A matching M is *maximum* if $M \cup \{e\}$ is not a matching for any $e \in E(G) \setminus M$. The *matching number* of a graph G, denoted by $\nu(G)$ is the size of a maximum matching in G.

Let G and H be undirected graphs. The *shadow graph* $D_2(G)$ of G is the graph obtained by taking two copies of G, say G_1 and G_2, and then joining each vertex $v \in V(G_1)$ to the neighbors of $v' \in V(G_2)$, where v' is the vertex in $V(G_2)$ corresponding to v, i.e., v and v' represent the same vertex in G. The *complementary prism* of graph G, denoted by $G\overline{G}$, is the graph obtained from the disjoint union of G and $\overline{G}$ by adding the edges $v\overline{v}$, where $v \in V(G)$ and $\overline{v}$ is the vertex of $\overline{G}$ corresponding to vertex v. The *disjunction* of graphs G and H, denoted by $G \vee H$, is the graph with $V(G \vee H) = V(G) \times V(H)$ and $(x,p)(y,q) \in E(G \vee H)$ if and only if $xy \in E(G)$ or $pq \in E(H)$. The *edge corona* $G \diamond H$ of G and H is the graph obtained by taking one copy of G and $|E(G)|$ copies of H, and then joining two end-vertices of the i-th edge of G to every vertex in the i-th copy of H. The *strong product* $G \boxtimes H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and (u,v) is adjacent with (u',v') whenever $[uu' \in E(G)$ and $v = v']$ or $[vv' \in E(H)$ and $u = u']$ or $[uu' \in E(G)$ and $vv' \in E(H)]$.

3. Results

Proposition 1 ([4]). *Let G be any graph on n vertices. If S is a maximun hop independent set of G, then S is a hop dominating set. In particular, $\gamma_h(G) \leq \alpha_h(G)$.*

Theorem 1 ([4]). *Let G be any graph on n vertices. If S is a hop independent set of G, then every component of $\langle S \rangle$ is complete. Moreover,*

(*i*) *$\alpha_h(G) = n$ if and only if every component of G is complete; and*

(*ii*) *for $n \geq 3$, $\alpha_h(G) = n - 1$ if and only if all but a single component C of G are complete and $C \setminus v$ is a complete graph for some vertex $v \in V(C)$.*

Corollary 1 ([4]). *Let G be a connected graph on n vertices. Then*

(*i*) *$\alpha_h(G) = n$ if and only if $G = K_n$; and*

(*ii*) *for $n \geq 3$, $\alpha_h(G) = n - 1$ if and only if $G \neq K_n$ and there exists $v \in V(G)$ such that $G \setminus v = K_{n-1}$.*

Observation 1: Let G be a graph of order n.

(*i*) $\varnothing$ and cliques in G are hop independent sets.

(*ii*) If S is a hop independent set in G and $I \subseteq I(G)$, where $I(G)$ denotes the set consisting of the isolated vertices of G, then $S \cup I$ is also hop independent in G. Moreover, if S is an α_h-set in G, then $I(G) \subseteq S$.

Observation 2: Let $G_1, G_2, \cdots, G_k$ be the components of graph G, where $k \geq 1$. Then S is hop independent in G if and only if $S_j = S \cap V(G_j)$ is hop independent in G_j for each $j \in \{1, 2, \cdots, k\}$. Moreover, $\alpha_h(G) = \sum_{i=1}^{k} \alpha_h(G_j)$.

4. Shadow Graph

If G_1 and G_2 are the copies of graph G in the definition of the shadow graph $D_2(G)$ and if $S_{G_1} \subseteq V(G_1)$ and $S_{G_2} \subseteq V(G_2)$, then the sets S'_{G_1} and S'_{G_2} are the sets given by

$$S'_{G_1} = \{a' \in V(G_2) : a \in S_{G_1}\} \text{ and } S'_{G_2} = \{a \in V(G_1) : a' \in S_{G_2}\}.$$

We denote by $I(G)$ the set containing all the isolated vertices of G.

Theorem 2. *Let G be a graph. Then a subset S of $V(D_2(G))$ is hop independent in $D_2(G)$ if and only if one of the following conditions holds:*

(i) *S is a hop independent set in G_1.*

(ii) *S is a hop independent set in G_2.*

(iii) *$S = S_{G_1} \cup S_{G_2} \cup I_1 \cup I_2$ and satisfies the following conditions:*

  (a) *$I_1 \subseteq I(G_1)$ and $I_2 \subseteq I(G_2)$.*

  (b) *S_{G_1} and S_{G_2} are hop independent sets in G_1 and G_2, respectively, not containing isolated vertices.*

  (c) *$S_{G_1} \cap S'_{G_2} = \varnothing$ and $S'_{G_1} \cap S_{G_2} = \varnothing$.*

  (d) *$S_{G_1} \cup S'_{G_2}$ and $S'_{G_1} \cup S_{G_2}$ are hop independent sets in G_1 and G_2, respectively.*

Proof. Suppose S is a hop independent set in $D_2(G)$. If $S \cap V(G_2) = \varnothing$, then $S \subseteq V(G_1)$. Since S is hop independent in $D_2(G)$, it follows that S is hop independent in G_1. This shows that (i) holds. Similarly, (ii) holds if $S \cap V(G_1) = \varnothing$. Next, suppose that $S \cap V(G_1) \neq \varnothing$ and $S \cap V(G_2) \neq \varnothing$. Let $I_1 = I(G_1) \cap S$, $I_2 = I(G_2) \cap S$, $S_{G_1} = S \cap (V(G_1) \setminus I_1)$, and $S_{G_2} = S \cap (V(G_2) \setminus I_2)$. Then $S = S_{G_1} \cup S_{G_2} \cup I_1 \cup I_2$. Clearly, (a) holds. Since S is a hop independent set in $D_2(G)$, property (b) also holds. Now, if $S_{G_1} = \varnothing$ or $S_{G_2} = \varnothing$, then (c) and (d) hold. So suppose $S_{G_1} \neq \varnothing$ and $S_{G_2} \neq \varnothing$. Suppose further that $S_{G_1} \cap S'_{G_2} \neq \varnothing$, say $v \in S_{G_1} \cap S'_{G_2}$. Then $v, v' \in S$. Since $v \notin I(G_1)$, there exists $w \in N_{G_1}(v)$. It follows that $d_{D_2(G)}(w, v') = 1$. Thus, $d_{D_2(G)}(v, v') = 2$, contradicting the assumption that S is hop independent in $D_2(G)$. Therefore, $S_{G_1} \cap S'_{G_2} = \varnothing$. Similarly, $S'_{G_1} \cap S_{G_2} = \varnothing$, showing that (c) holds. Finally, suppose that $S_{G_1} \cup S'_{G_2}$ is not hop independent in G_1. Since S_{G_1} and S_{G_2} are hop independent in G_1 and G_2, respectively, according to (b), there exist $x \in S_{G_1}$ and $y \in S'_{G_2}$ such that $d_{D_2(G)}(x, y) = 2$. This, however, would imply that $d_{D_2(G)}(x, y') = 2$, contradicting the fact that $y' \in S_{G_2}$ and S is a hop independent set in $D_2(G)$. Hence, $S_{G_1} \cup S'_{G_2}$ is hop independent in G_1. Similarly, $S'_{G_1} \cup S_{G_2}$ is hop independent in G_2. This shows that (d) holds.

Conversely, if (i) or (ii) holds, then S is a hop independent set in $D_2(G)$. Suppose now that (iii) holds. Since (b) holds, $S_{G_1} \cup I_1$ and $S_{G_2} \cup I_2$ are hop independent sets in G_1 and G_2, respectively (and hence, in $D_2(G)$). Let $x \in S_{G_1}$ and $y' \in S_{G_2}$. Then $y \in S'_{G_2}$ and by (c), $x \neq y$. By property (d), $d_{D_2(G)}(x, y') = d_{D_2(G)}(x, y) \neq 2$. Therefore, S is a hop independent set in $D_2(G)$. □

Corollary 2. *Let G be a graph. Then $\alpha_h(D_2(G)) = \alpha_h(G) + |I(G)|$.*

Proof. If G is an empty graph, then $D_2(G)$ is an empty graph. Hence,

$$\alpha_h(G) = |I(D_2(G))| = |I(G_1)| + |I(G_2)| = \alpha_h(G) + |I(G)|.$$

Suppose that G has an edge. Let S be an α_h-set in G_1. Then $S^* = S \cup I(G_2)$ is a hop independent set in $D_2(G)$. This implies that $\alpha_h(G) \geq |S^*| = \alpha_h(G) + |I(G)|$. Next, suppose that Q is α_h-set in $D_2(G)$. Then $I(D_2(G)) = I(G_1) \cup I(G_2) \subset Q$. Let $Q_{G_1} = (Q \cap V(G_1)) \setminus I(G_1)$ and $Q_{G_2} = (Q \cap V(G_2)) \setminus I(G_2)$. Then $Q = Q_{G_1} \cup Q_{G_2} \cup I(G_1) \cup I(G_2)$. From property (d), $Q_{G_1} \cup Q'_{G_2}$ is a hop independent set in G_1. It follows that $Q_{G_1} \cup Q'_{G_2} \cup I(G_1)$ is a hop independent set in G_1. Thus,

$$\begin{aligned}
\alpha_h(D_2(G)) &= |Q| \\
&= |Q_{G_1} \cup Q_{G_2} \cup I(G_1) \cup I(G_2)| \\
&= |Q_{G_1} \cup Q_{G_2} \cup I(G_1)| + |I(G_2)| \\
&= |Q_{G_1} \cup Q'_{G_2} \cup I(G_1)| + |I(G_2)| \\
&\leq \alpha_h(G) + |I(G)|.
\end{aligned}$$

This establishes the desired equality. □

5. Edge Corona of Two Graphs

For every edge $e = uv$ of G, denote by $H^e = H^{uv}$ the copy of H where the vertices are joined to vertices u and v.

Theorem 3. *Let G be a non-trivial connected graph and let H be any graph. Then S is a hop independent set in $G \diamond H$ if and only if $S = A \cup (\cup_{uv \in E(G)} S_{uv})$ and satisfies the following conditions:*

(i) A is a hop independent set in G.

(ii) If $S_{uv} \neq \varnothing$, then S_{uv} is a clique in H^{uv}.

(iii) $S_{uv} = \varnothing$ whenever any of the following holds:

- *(a) $\{u, v\} \cap N_G(A)) \neq \varnothing$*
- *(b) $S_{uw} \neq \varnothing$ for some $w \in N_G(u)$*
- *(c) $S_{vz} \neq \varnothing$ for some $z \in N_G(v)$*

Proof. Suppose S is a hop independent set in $G \diamond H$ and let $A = S \cap V(G)$ and $S_{uv} = S \cap V(H^{uv})$ for each $uv \in E(G)$. Then $S = A \cup (\cup_{uv \in E(G)} S_{uv})$. Since S is a hop independent set in $G \diamond H$, A is a hop independent set in G. This shows that (i) holds.

Next, let $uv \in E(G)$ and $S_{uv} \neq \varnothing$. If S_{uv} is not a clique in H^{uv}, then there exist $p, q \in S_{uv}$ such that $d_{H^{uv}}(p, q) \neq 1$. It follows that $d_{G \diamond H}(p, q) = 2$, contrary to the assumption that S is hop independent in $G \diamond H$. Thus, S_{uv} is a clique in H^{uv}, showing that (ii) holds. Finally, suppose that $uv \in V(G)$. Since S is hop independent in $G \diamond H$, $S_{uv} = \varnothing$ whenever (a) or (b) or (c) holds. This shows that (iii) holds.

For the converse, suppose that S has the given form and satisfies (i), (ii), and (iii). Let $a, b \in S$, where $a \neq b$, and let $uv, xy \in E(G)$ such that $a \in V(\langle \{u, v\} \rangle + H^{uv})$ and $b \in V(\langle \{x, y\} \rangle + H^{xy})$. If $a, b \in A$, then $d_{G \diamond H}(a, b) \neq 2$ because of (i). Suppose that a or b is not in A. Consider the following cases:

Case 1. $uv \neq xy$.

Suppose first that uv and xy have a common vertex, say $x = v$. By (iii), S_{uv} and S_{xy} cannot be both nonempty. Assume that $S_{xy} = \varnothing$. Suppose $b = y$. Then $S_{uv} = \varnothing$ by (iii). It follows that $a \in \{u, v\}$, contrary to our assumption that a or b is not in A. Hence, $b = x$, $a \in S_{uv}$, and $ab \in E(G \diamond H)$. So suppose uv and xy do not have a common vertex. Assume first that one of a and b is in A, say $a = u \in A$. Then $b \in S_{xy}$. By (iii), $x, y \notin N_G(a)$. This implies that $d_{G \diamond H}(a, b) \neq 2$. Next, suppose that $a \in S_{uv}$ and $b \in S_{xy}$. Since uv and xy do not have a common vertex, $d_{G \diamond H}(a, b) \neq 2$.

Case 2. $uv = xy$.

If $a \in \{u, v\}$, then $b \in S_{uv}$ and $d_{G \diamond H}(a, b) = 1$. If $a, b \in S_{uv}$, then $d_{G \diamond H}(a, b) = 1$ by (ii).

Therefore, S is a hop independent set in $G \diamond H$. □

Lemma 1. *Let G be a connected graph of order $n \geq 3$. Then $\nu(G) = 1$ if and only if $G = K_3$ or $G = K_{1,n-1}$. Furthermore, if $\nu(G) = 1$ and $\alpha_h(G) > 2$, then $G = K_3$.*

Proof. Suppose that $\nu(G) = 1$. If $n = 3$, then $G \in \{K_3, P_3\}$. Suppose $n \geq 4$ and let $M = \{uv\}$ be a maximum matching in G. Let $x \in V(G) \setminus \{u, v\}$. Since $\nu(G) = 1$, $xu \in E(G)$ or $xv \in E(G)$. Assume that $xu \in E(G)$. Suppose further that $xv \in E(G)$. Then $\langle \{x, u, v\} \rangle$ is (isomorphic to) K_3. Next, let $y \in V(G) \setminus \{x, u, v\}$. Since G is connected and $n \geq 4$, pick any $y \in N_G(\{x, u, v\})$. We may assume that $xy \in E(G)$. Then xy and uv do not have a common vertex. This implies that $M \cup \{xy\}$ is a matching in G, contradicting the maximality of M. Thus, $xv \notin E(G)$. Now let $z \in V(G) \setminus \{u, x\}$. Since $M = \{uv\}$ is a maximum matching in G, $zu \in E(G)$ and $xz \notin E(G)$. Therefore, $G = K_{1,n-1}$.

The converse is clear.

For the second part, suppose that $\nu(G) = 1$ and $\alpha_h(G) > 2$. Since $\alpha_h(K_{1,n-1}) = 2$ for all $n \geq 3$, it follows from the first part that $G = K_3$. □

Corollary 3. *Let G be a non-trivial connected graph and let H be any graph.*

(i) *If $\nu(G) = 1$, then $\alpha_h(G \diamond H) = 2 + \omega(H)$.*

(ii) *If $\nu(G) \geq 2$, then $\alpha_h(G \diamond H) \geq \max\{\alpha_h(G), \nu(G)\omega(H)\}$.*

Proof. (i) Assume that $\nu(G) = 1$, say $M' = \{pq\}$ is a maximum matching in G. Clearly, $\alpha_h(G \diamond H) = 2 + \omega(H)$ if $G = \langle\{p, q\}\rangle = K_2$. Suppose $G \neq K_2$. Let $S_0 = \{p, q\} \cup S_{pq}$, where S_{pq} is a maximum clique in H. Then, by Theorem 3, S_0 is a hop independent set in $G \diamond H$. This implies that $\alpha_h(G \diamond H) \geq |S_0| = 2 + \omega(H)$.

On the other hand, if S^* is an α_h-set in $G\diamond H$, then $S^* = A \cup (\cup_{uv\in E(G)} S_{uv})$ and satisifes conditions (i), (ii), and (iii) of Theorem 3. Suppose there exists an $st \in E(G)$ such that $S_{st} \neq \varnothing$. Without loss of generality, we may assume that $st = pq$, i.e., $S_{st} = S_{pq} \neq \varnothing$. Then S_{pq} is a clique in H^{pq} by (ii). Let $kl \in E(G) \setminus M'$. Since M' is a maximum matching in G (or since $\nu(G) = 1$), pq and kl must have a common vertex. We may assume that $k = p$. Then $S_{kl} = \varnothing$ by (iii). Also, since $w \in N_G(\{p, q\})$ for all $w \in V(G) \setminus \{p, q\}$ and $S_{pq} \neq \varnothing$, it follows from (iii) that $w \notin A$ for all $w \in V(G) \setminus \{p, q\}$. This implies that $A \subseteq \{p, q\}$. Hence,

$$\alpha_h(G \diamond H) = |A| + |S_{pq}| \leq 2 + \omega(H).$$

Next, suppose that $S_{uv} = \varnothing$ for all $uv \in E(G)$. Then $S^* = A$. Since G is a non-trivial connected graph, $|A| \geq 2$. Clearly, $\alpha_h(G \diamond H) = |A| \leq 2 + \omega(H)$ if $|A| = 2$. So suppose $|A| > 2$. Since A is a hop independent set in G and $|A| > 2$, it follows from Lemma 1 that $G = K_3$. Hence, $\alpha_h(G \diamond H) = |A| = 3 \leq 2 + \omega(H)$.

Accordingly, $\alpha_h(G \diamond H) = 2 + \omega(H)$.

(ii) Suppose now that $\nu(G) \geq 2$. Let S be an α_h-set in G. Then S is a hop independent set in $G \diamond H$ by Theorem 3. It follows that $\alpha_h(G \diamond H) \geq |S| = \alpha_h(G)$. Next, let M be a maximum matching in G. Let S_{uv} be a maximum clique in H^{uv} for each $uv \in M$ and set $S_{xy} = \varnothing$ for each $xy \in E(G) \setminus M$. Then $S = \cup_{uv\in E(G)} S_{uv} = \cup_{uv\in M} S_{uv}$ is a hop independent set in $G \diamond H$ by Theorem 3. Thus, $\alpha_h(G \diamond H) \geq |S| = \nu(G)\omega(H)$. Therefore, $\alpha_h(G \diamond H) \geq \max\{\alpha_h(G), \nu(G)\omega(H)\}$. □

Remark 1. *The lower bound given in Corollary 3 is tight. However, strict inequality is also attainable.*

To see this, consider $K_5 \diamond P_3$, $C_4 \diamond P_3$, and $G \diamond P_3$, where G is the graph in Figure 3.

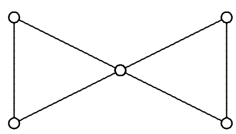

Figure 3

Clearly, $\alpha_h(K_5) = 5$, $\alpha_h(C_4) = 2$, $\alpha_h(G) = 3$, $\nu(K_5) = \nu(C_4) = \nu(G) = 2$, and $\omega(P_3) = 2$. It can be verified easily that $\alpha_h(K_5 \diamond P_3) = \alpha_h(K_5) = 5 > 4 = \nu(K_5)\omega(P_3)$, $\alpha_h(C_4 \diamond P_3) = \nu(C_4)\omega(P_3) = 4 > 2 = \alpha_h(C_4)$, and $\alpha_h(G \diamond P_3) = 6 > 4 = \max\{\alpha_h(G), \nu(G)\omega(P_3)\}$.

6. Complementary Prism

Theorem 4. *Let G be a graph. Then S is a hop independent set in $G\overline{G}$ if and only if one of the following holds.*

(i) *S is a hop independent set in G.*

(ii) *S is a hop independent set in $\overline{G}$.*

(iii) *$S = \{v, \overline{v}\}$ for some $v \in V(G)$.*

Proof. Suppose S is a hop independent set in $G\overline{G}$. If $S \subseteq V(G)$, then S is a hop independent set in G. If $S \subseteq V(\overline{G})$, then S is a hop independent set in $\overline{G}$. Hence, (i) or (ii) holds. Suppose $S \cap V(G) \neq \varnothing$ and $S \cap V(\overline{G}) \neq \varnothing$. Let $v \in S \cap V(G)$ and let $\overline{w} \in S \cap V(\overline{G})$. Suppose $\overline{w} \neq \overline{v}$. Then $v \neq w$. Since $vw \in E(G)$ if and only if $\overline{v}\,\overline{w} \notin E(\overline{G})$, it follows that $d_{G\overline{G}}(v, \overline{w}) = 2$, contrary to the assumption that S is a hop independent set in $G\overline{G}$. Thus, $\overline{w} = \overline{v}$. Suppose there exists $x \in (S \cap V(G)) \setminus \{v\}$. Then $d_{G\overline{G}}(x, \overline{v}) = 2$ which is not possible. Thus, $S \cap V(G) = \{v\}$. Similarly, $S \cap V(\overline{G}) = \{\overline{v}\}$. This shows that (iii) holds.

The converse is clear. □

The next result is immediate from Theorem 4.

Corollary 4. *Let G be a graph. Then $\alpha_h(G\overline{G}) = \max\{2, \alpha_h(G), \alpha_h(\overline{G}\}$.*

7. Disjunction of Two Graphs

Theorem 5. *Let G and H be non-trivial connected graphs. Then $C = \cup_{x\in S}(\{x\} \times T_x)$ is a hop independent set in $G \vee H$ if and only if following conditions hold.*

(i) *S is a hop independent set in G.*

(ii) *T_x is a clique in H for every $x \in S$.*

(iii) *If $x, y \in S$ and $d_G(x, y) \geq 3$, then $T_x \cap T_y = \varnothing$ and $d_H(T_x, T_y) \geq 3$.*

(iv) *$\cup_{x\in S_0} T_x$ is a hop independent set in H for every independet set $S_0 \subseteq S$.*

Proof. Suppose C is a hop independent set in $G \vee H$. Suppose S is not hop independent in G. Then there exist $u, v \in S$ such that $d_G(u, v) = 2$. Let $w \in N_G(u) \cap N_G(v)$, $a \in T_u$ and $b \in T_v$. If $a = b$, then $[(u, a), (w, a), (v, a)]$ is a (u, a)-(v, b) geodesic in $G \vee H$. If $a \neq b$, then $[(u, a), (w, b), (v, b)]$ is a (u, a)-(v, b) geodesic in $G \vee H$. Both cases are contrary to the assumption that C is a hop independent set in $G \vee H$. Hence, S is hop independent in G, showing that (i) holds. Now, let $x \in S$ and let $p, q \in T_x$ where $p \neq q$. Since C is a hop independent set in $G \vee H$, $d_{G\vee H}((x, p), (x, q)) \neq 2$. From the fact that $1 \leq d_{G\vee H}((x, p), (x, q)) \leq 2$ and because $(x, p) \neq (x, q)$, we must have $d_{G\vee H}((x, p), (x, q)) = 1$. This implies that $pq \in E(G)$. Therefore, T_x is a clique in H, showing that (ii) holds.

Next, let $x, y \in S$ with $d_G(x,y) \geq 3$. Suppose $T_x \cap T_y \neq \varnothing$, say $t \in T_x \cap T_y$. Let $s \in N_H(t)$. Then $[(x,t),(y,s),(y,t)]$ is an (x,t)-(y,t) geodesic in $G \vee H$, contrary to the assumption that C is hop independent in $G \vee H$. Thus, $T_x \cap T_y = \varnothing$. Let $a_1 \in T_x$ and $a_2 \in T_y$. Suppose $d_H(a_1,a_2) = 2$ and let $q \in N_H(a_1,a_2)$. Then $[(x,a_1),(y,q),(y,a_2)]$ is an (x,a_1)-(y,a_2) geodesic in $G \vee H$, a contradiction. This implies that $d_H(a_1,a_2) \neq 2$. Since a_1 and a_2 were arbitrarily chosen, it follows that $d_H(T_x,T_y) \geq 3$, showing that (iii) holds. Finally, suppose that S_0 is an independent subset of S. Suppose $\cup_{x\in S_0} T_x$ is not hop independent in H. Then there exist $c, d \in \cup_{x\in S_0} T_x$ with $d_H(c,d) = 2$. By (ii), it follows that $c \in T_z$ and $d \in T_w$, where $z \neq w$ and $z, w \in S_0$. Let $r \in N_H(c) \cap N_H(d)$. Then $[(z,c),(w,r),(w,d)]$ is a (z,c)-(w,d) geodesic in $G \vee H$ which is not possible. Therefore, $\cup_{x\in S_0} T_x$ is hop independent in H, showing that (iv) holds.

For the converse, suppose that C satisfies properties (i), (ii), (iii), and (iv). Let $(v,p),(w,q) \in C$ such that $(v,p) \neq (w,q)$. Consider the following cases:

Case 1. $v = w$.
Then $p, q \in T_v$ and $p \neq q$. By property (ii), $d_H(p,q) = 1$. Hence, $d_{G\vee H}((v,p),(w,q)) = 1$.

Case 2. $v \neq w$.
Suppose first that $d_G(v,w) = 1$. Then $d_{G\vee H}((v,p),(w,q)) = 1$. Next, suppose that $d_G(v,w) > 1$. By property (i), we must have $d_G(v,w) \geq 3$. Now, by property (iii), $p \neq q$. If $d_H(p,q) = 1$, then $d_{G\vee H}((v,p),(w,q)) = 1$. Suppose $d_H(p,q) \neq 1$. Then, by property (iv) (using the fact that $S_0 = \{v,w\}$ is an independent set), $d_H(p,q) \geq 3$. Therefore, $d_{G\vee H}((v,p),(w,q)) \geq 3$.

Accordingly, C is a hop independent set in $G \vee H$. □

A sequence of cliques $\langle S_{q_1}, S_{q_2}, \cdots, S_{q_m}\rangle$ in an undirected graph G, where each $q_j = |S_{q_j}|$, is a decreasing d_3-*sequence* if $d_G(S_{q_i}, S_{q_j}) \geq 3$ for every pair of distinct indices $i, j \in \{1,2,\cdots,m\}$ and $q_1 \geq q_2 \geq \ldots \geq q_m$. It is a maximum decreasing d_3-sequence if for every decreasing d_3-sequence of cliques $\langle S_{t_1}, S_{t_2}, \cdots, S_{t_s}\rangle$ in G, it holds that $s \leq m$ and $t_j \leq q_j$ for each $j \in \{1,2,\ldots,s\}$.

Corollary 5. *Let G and H be non-trivial connected graphs and let $\langle S_{q_1}, S_{q_2}, \cdots, S_{q_m}\rangle$ and $\langle D_{p_1}, D_{p_2}, \cdots, D_{p_r}\rangle$ be maximum decreasing d_3-sequences of cliques in G and H, respectively. Then*

$$\alpha_h(G \vee H) = \sum_{k=1}^{\rho_G^H} q_k p_k,$$

where $\rho_G^H = \min\{m, r\}$.

Proof. Let $\langle S_{q_1}, S_{q_2}, \cdots, S_{q_m}\rangle$ and $\langle D_{p_1}, D_{p_2}, \cdots, D_{p_r}\rangle$ be maximum decreasing d_3-sequences of cliques in G and H, respectively. Then $S = \cup_{k=1}^{\rho_G^H} S_{q_k}$ is a hop dominating set in G. For each $x \in S$, set $T_x = D_{p_j}$ if $x \in S_{q_j}$, where $1 \leq j \leq \rho_G^H$. Then $C = \cup_{x\in S}(\{x\} \times T_x)$

is a hop independent set in $G \vee H$ by Theorem 5. This implies that

$$\alpha_h(G \vee H) \geq |C| = \sum_{x \in S} |T_x| = \sum_{k=1}^{\rho_G^H} \sum_{x \in S_{q_k}} |T_x| = \sum_{k=1}^{\rho_G^H} \sum_{x \in S_{q_k}} |D_{p_k}| = \sum_{k=1}^{\rho_G^H} q_k p_k.$$

On the other hand, suppose $C_0 = \cup_{x \in S_0} (\{x\} \times R_x)$ is an α_h-set in $G \vee H$. Then, by (i) and (ii) of Theorem 5, S_0 is a hop independent set in G and R_x is a clique in H for each $x \in S_0$. By Theorem 1, the components of S_0 are cliques in G. Let $G_{r_1}, G_{r_2}, \cdots,$ and G_{r_n} be the components of G, where $r_1 \geq r_2 \cdots \geq r_n$, where $r_j = |V(G_{r_j})|$ for each $j \in \{1, 2, \cdots, n\}$. By (iii) of Theorem 5, $\langle V(G_{r_1}), V(G_{r_2}), \cdots, V(G_{r_n}) \rangle$ is a decreasing d_3-sequence of cliques in G. Hence, $n \leq m$ and $V(G_{r_j}) \subseteq S_{q_j}$ for each $j \in \{1, 2, \cdots, n\}$. Since C_0 is an α_h-set in $G \vee H$, it follows that for each j with $1 \leq j \leq n$ and for each $x \in V(G_{r_j})$, $R_x = Q_{t_j}$ and $t_j \geq t_{j+1}$ for $j \in \{1, 2, \cdots, n-1\}$. Thus, $\langle Q_{t_1}, Q_{t_2}, \cdots, Q_{t_n} \rangle$ is a decreasing d_3-sequence of cliques in H. Hence, $n \leq r$ and $Q_{t_j} \subseteq D_{p_j}$ for each $j \in \{1, 2, \cdots, n\}$. It follows that $n \leq \rho_G^H$. Therefore,

$$\alpha_h(G \vee H) = |C_0| = \sum_{x \in S_0} |R_x| = \sum_{k=1}^{n} r_k t_k \leq \sum_{k=1}^{\rho_G^H} q_k p_k.$$

This proves the desired equality. □

Corollary 6. *Let G and H be non-trivial connected graphs. If G or in H has a maximum decreasing d_3-sequence consisting of a single clique as a term, then $\alpha_h(G \vee H) = \omega(G)\omega(H)$.*

Proof. Suppose, without loss of generality, that $\langle S_q \rangle$ is a maximum d_3-sequence of a single clique in G. Then $\rho_G^H = 1$ and S_q is a maximum clique in G. Let D be a maximum clique in H. Then $\alpha_h(G \vee H) = |S_q||D| = \omega(G)\omega(H)$ by Corollary 5. □

Example 1. *For any non-trivial connected graph H, $\alpha_h(K_n \vee H) = n\omega(H)$ and $\alpha_h(P_3 \vee H) = 2\omega(H)$.*

8. Strong Product of Two Graphs

Theorem 6. *Let G and H be non-trivial connected graphs. Then $C = \cup_{x \in S}(\{x\} \times T_x)$ where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for every $x \in S$, is a hop independent set in $G \boxtimes H$ if and only if following conditions hold.*

(i) *T_x is a hop independent set in H for every $x \in S$.*

(ii) *$T_x \cup T_y$ is hop independent in H for every pair of vertices $x, y \in S$ with $d_G(x, y) \leq 2$.*

(iii) *$T_x \cap T_y = \varnothing$ and $d_H(T_x, T_y) \geq 3$ for every pair of vertices $x, y \in S$ with $d_G(x, y) = 2$.*

Proof. Suppose C is a hop independent set in $G \boxtimes H$. Let $x \in S$ and $a, b \in T_x$ such that $a \neq b$. If $ab \in E(G)$, then $(x, a)(x, b) \in E(G \boxtimes H)$. Suppose $ab \notin E(G)$. Then $(x, a)(x, b) \notin E(G \boxtimes H)$. Since C is hop independent in $G \boxtimes H$, $d_{G \boxtimes H}((x, a), (x, b)) \geq 3$. This implies that $d_H(a, b) \geq 3$, showing that T_x is hop independent in H. This shows that (i) holds. Next, let $x, y \in S$ with $d_G(x, y) \leq 2$ and suppose that $T_x \cup T_y$ is not hop independent in H. By (i), it follows that there exist $p \in T_x$ and $q \in T_y$ such that $d_H(p, q) = 2$. Let $t \in N_H(p) \cap N_H(q)$. Suppose first that $d_G(x, y) = 1$. Then $[(x, p)(y, t), (y, q)]$ is an (x, p)-(y, q) geodesic. Suppose $d_G(x, y) = 2$ and let $z \in N_G(x) \cap N_G(y)$. Then $[(x, p)(z, t), (y, q)]$ is an (x, p)-(y, q) geodesic. In both cases, we have $d_{G \boxtimes H}((x, p), (y, q)) = 2$, contrary to our assumption that C is hop independent. Thus, $T_x \cup T_y$ is hop independent in H, showing that (ii) holds. Finally, let $x, y \in S$ such that $d_G(x, y) = 2$. Let $z \in N_G(x) \cap N_G(y)$. Suppose there exists $t \in T_x \cap T_y$. Then $d_{G \boxtimes H}((x, t), (y, t)) = 2$ which is not possible. Hence, $T_x \cap T_y = \varnothing$. Suppose $d_H(T_x, T_y) = 1$. Then this would imply that there exist $c \in T_x$ and $d \in T_y$ with $d_H(c, d) = 1$. Consequently, $[(x, c), (z, d), (y, d)]$ is an (x, c)-(y, d) geodesic in $G \boxtimes H$. If $d_H(T_x, T_y) = 2$, then there exist $g \in T_x$ and $h \in T_y$ with $d_H(g, h) = 2$. Let $l \in N_H(g) \cap N_H(h)$. Then $[(x, g), (z, l), (y, h)]$ is an (x, c)-(y, d) geodesic in $G \boxtimes H$. In any case, we get a contradiction. Therefore, (iii) holds.

For the converse, suppose that C satisfies properties (i), (ii), (iii), and (iv). Let $(v, p), (w, q) \in C$ such that $(v, p) \neq (w, q)$. Consider the following cases:

Case 1. $v = w$.
Then $p, q \in T_v$ and $p \neq q$. By condition (i), $d_H(p, q) \neq 2$. Hence, $d_{G \boxtimes H}((v, p), (w, q)) \neq 2$.

Case 2. $v \neq w$.
Suppose first that $vw \in E(G)$. If $p = q$, then $d_{G \boxtimes H}((v, p), (w, q)) = 1$. Suppose $p \neq q$. If $pq \in E(H)$, then $d_{G \boxtimes H}((v, p), (w, q)) = 1$. Suppose $pq \notin E(H)$. By (ii), $T_v \cup T_w$ is a hop independent set. It follows that $d_H(p, q) \geq 3$. Thus, $d_{G \boxtimes H}((v, p), (w, q)) \geq 3$. Next, suppose that $vw \notin E(G)$. If $d_G(v, w) \geq 3$, then $d_{G \boxtimes H}((v, p), (w, q)) \geq 3$. If $d_G(v, w) = 2$, then $T_v \cap T_w = \varnothing$ by (iii). Hence, $p \neq q$ and $d_H(p, q) \geq 3$ by (ii). Therefore, $d_{G \boxtimes H}((v, p), (w, q)) \geq 3$.

Accordingly, C is a hop independent set in $G \boxtimes H$. □

Corollary 7. *Let G and H be non-trivial connected graphs and let $\langle S_{q_1}, S_{q_2}, \cdots, S_{q_m} \rangle$ and $\langle D_{p_1}, D_{p_2}, \cdots, D_{p_r} \rangle$ be maximum decreasing d_3-sequences of cliques in G and H, respectively. Then*

$$\alpha_h(G \boxtimes H) \geq \sum_{i=1}^{m} \sum_{j=1}^{r} q_i p_j.$$

Proof. Let $\langle S_{q_1}, S_{q_2}, \cdots, S_{q_m} \rangle$ and $\langle D_{p_1}, D_{p_2}, \cdots, D_{p_r} \rangle$ be maximum decreasing d_3-sequences of cliques in G and H, respectively. Let $S = \cup_{i=1}^{m} S_{q_i}$ and for each $x \in S$, set $R_x = \cup_{j=1}^{r} D_{p_j}$. Then each R_x is a hop independent set in H. Clearly, condition (ii) of Theorem 6 is satisfied. Moreover, because S is a hop independent set, condition (iii) of

Theorem 6 also holds. Thus, $C = \cup_{x\in S}(\{x\} \times T_x)$ is a hop independent set in $G \boxtimes H$ and

$$\begin{aligned} \alpha_h(G \boxtimes H) &\geq |C| \\ &= \sum_{x\in S} |R_x| \\ &= \sum_{i=1}^{m} \sum_{x\in S_{q_i}} |\cup_{j=1}^{r} D_{p_j}| \\ &= \sum_{i=1}^{m} q_i \sum_{j=1}^{r} p_j \\ &= \sum_{i=1}^{m} \sum_{j=1}^{r} q_i p_j. \end{aligned}$$

This proves the assertion.

Remark 2. *The bound given in Corollary 7 is tight.*

Consider graphs $G = P_6 = [v_1, v_2, v_3, v_4, v_5, v_6]$, $H = P_6 = [p_1, p_2, p_3, p_4, p_5, p_6]$, and the strong product $P_6 \boxtimes P_6$ in Figure 4. The sequences $\langle\{v_1, v_2\}, \{v_5, v_6\}\rangle$ and $\langle\{p_1, p_2\}, \{p_5, p_6\}\rangle$ are maximum (decreasing) d_3-sequences of cliques in G and H, respectively. Clearly,

$$\begin{aligned} \alpha_h(G \boxtimes H) &= |\{v_1, v_2\}||\{p_1, p_2\}| + |\{v_1, v_2\}||\{p_5, p_6\}| + \\ &\quad\ |\{v_5, v_6\}||\{p_1, p_2\}| + |\{v_5, v_6\}||\{p_5, p_6\}| \\ &= 16. \end{aligned}$$

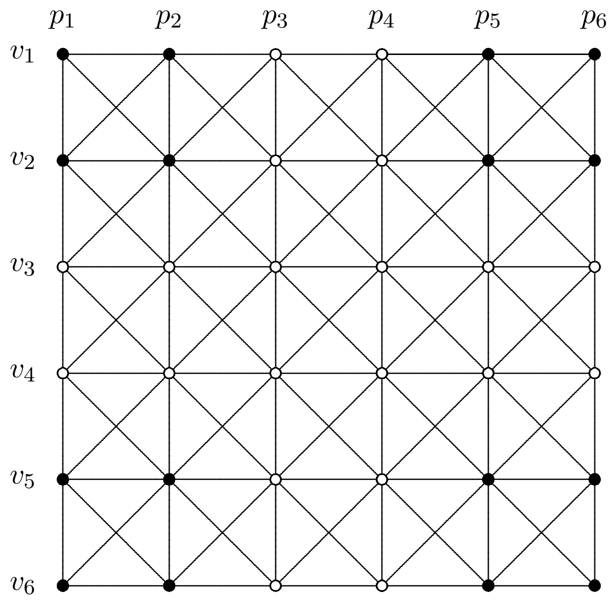

Figure 4: $P_6 \boxtimes P_6$

We now show that the hop independent set decision problem (HISP) is NP-complete. To this end, consider the following hop independent set decision problem (HISP):

Instance: Given a graph $G = (V(G), E(G))$ and a positive integer $k \leq |V(G)|$
Question: Does G contain a hop independent set of size k?

On the other hand, the clique decision problem (CP) is stated as follows:

Instance: Given a graph $G = (V(G), E(G))$ and a positive integer $k \leq |V(G)|$
Question: Does G contain a clique of size k?

Theorem 7 ([6])**.** *The clique problem is NP-complete.*

Theorem 8. *The hop independent set problem is NP-complete.*

Proof. Given a subset S of vertices of G, one can check in polynomial time if S is a hop independent set. Hence, the hop independent set problem is NP. We now use the clique problem (CP) to show that HISP is NP-Hard. To this end, let $G = (V(G), E(G)$ be a graph with $V(G) = \{v_1, v_2, ..., v_n\}$ and let k be a positive integer with $k \leq |V(G)|$. Let $H = G + K_1$, where $V(K_1) = \{v\}$. If S is a clique in G and $|S| = k$, then $S' = S \cup \{v\}$ is a hop independent set in H and $|S'| = k + 1$. Conversely, suppose S^* is a hop independent set in H with $|S^*| = k+1$. Suppose S^* is not a clique in H. Then, by Theorem 1, $\langle S^* \rangle$ has at least two complete components, say H_1 and H_2. Pick $x \in V(H_1)$ and $y \in V(H_2)$. Then

$d_H(x, y) = 2$, a contradiction. Thus, S^* is a clique in H. If $S^* \subseteq V(G)$, then $S^* \setminus \{x\}$, where $x \in S^*$, is a clique in G with size k. Suppose $v \in S^*$. Then $S^* \setminus \{v\}$ is a clique in G with size k. Therefore, G has a clique of size k if and only if H has a hop independent set of size $k + 1$. Accordingly, the hop independent set problem is NP-complete. □

9. Conclusion

The hop independence parameter has been explored for the shadow graph, complementary prism, edge corona, disjunction, and strong product of two graphs. It is conjectured that the hop independence number of the edge corona of two graphs may take only three possible values, namely; $\alpha_h(G)$, $\nu(G)\omega(G)$, and $\nu(G)\omega(G) + 2$. It is also conjectured that the lower bound in Corollary 7 is the exact value of the parameter. Using the fact that the clique decision problem (CP) is NP-complete, it was shown that the hop independent set problem (HISP) is also NP-complete.

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