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Existence of Solutions to a New Class of Fractional Differential Equations With Antiperiodic Boundary Conditions

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Abstract. This study introduces a novel class of fractional differential equations characterized by antiperiodic parametric boundary conditions of order $\mu \in (2,3]$. The parameters θ and ξ play a crucial role in shaping the boundary conditions by defining specific values and functional behavior. By employing fixed point theorems, we establish existence results for a fractional differential equation equipped with nonlocal antiperiodic boundary conditions involving a Caputo fractional derivative at one of the boundaries. Our investigation centers on a nonlocal point $0 \leq \theta < b$, in conjunction with a fixed endpoint at the interval's extremity $(0, b]$. This approach enables us to extend the interval of interest to $(-\infty, b]$. The findings presented in this study serve to expand and generalize the existing body of knowledge pertaining to nonlocal and classical fractional differential equations.

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Key Words and Phrases: Fractional derivatives, differential equations, existence of solutions, fixed-point techniques, evaluation metrics

1. Introduction

In this paper, we find the existence and uniqueness of a solution to the following fractional problem:

$$\begin{cases} {}^cD^{\mu}\omega(\rho) = \Omega(\rho, \omega(\rho)), \quad \rho \in [0,b], \quad 2 < \mu \leq 3, \ \theta \in [0,b), \\ \omega(\theta) = -\omega(b), \quad \omega^{'}(\theta) = -\omega^{'}(b), \ {}^cD^{\xi+1}\omega(\theta) = -{}^cD^{\xi+1}\omega(b), \ 0 < \xi < 1. \end{cases}, \tag{1}$$

where ${}^cD^{\mu}$ denotes the Caputo fractional derivative of order μ, $\Omega : [0,b] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function, $\omega(\rho)$ represents the solution of (1) for a variable $\rho \in [0,b]$, and

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$\omega(\theta) = -\omega(b)$, $\omega^{'}(\theta) = -\omega^{'}(b)$, ${}^{c}D^{\xi+1}\omega(\theta) = -{}^{c}D^{\xi+1}\omega(b)$, $0 < \xi < 1$ represents the nonlocal boundary conditions imposed on the solution $\omega(\rho)$. Problem (1) describes a physical phenomenon based on the nonlocal boundary conditions that restrict the solution's behavior at the domain boundaries.

To utilize these conditions, consider a fractional heat conduction problem. Then, we can understand these conditions as defining specific characteristics of the system at the boundaries. For example, $\omega(\theta) = -\omega(b)$ indicates a symmetry requirement in which the temperature at θ is equal in magnitude but opposite in sign to the temperature at b, suggesting a reflective boundary of some kind. Likewise, the other statements could impose further restrictions on the derivatives of ω at these boundaries.

The exploration of differential equations with a fractional order became extensively researched. Models with non-integer order can grant an astonishing description of memory and hereditary properties of many physical phenomena. Moreover, using fractional derivatives in non-integer order models of real systems can be more accurate and adequate than the integer order models. Fractional differential equations can provide a comprehensive scheme to examine complex and regular systems in numerous fields, such as physics, biology, economics, engineering, social sciences, and material science [17–19, 27]. For example

- In Physics, the nonlinear fractional–stochastic wave equation can be used to describe numerous nonlinear physical phenomena with gas bubbles in liquids. Moreover, fractional kinetic equations describe a system's evolution over time and fundamental in modeling anomalous diffusion [25].

- In Biology, models of a fractional order can be used to investigate the spread of a disease in communities. In particular, an accurate and efficient description of the COVID-19 pandemic and its growing variant can be obtained by using a noninteger order of COVID-19 models [26].

- In Economics, growth model of fractional order with time delay can effectively describe the economic growth by adding a time lag to the capital stock [23].

- In Engineering, fractional order sliding mode is used to progress the control performance by reducing the error of steady-state and saturate. Also, fractional-order fast adaptive sliding mode control ensure prompt convergence of the human Knee Joint Orthosis state and its finite-time stability to the intended trajectory [22].

Boundary conditions have a significant impact on comprehending physical systems and solving differential equations. Periodic and antiperiodic boundary conditions are crucial in solving fractional differential equations. In quantum mechanics, antiperiodic boundary problems will give a meaningful description of the system and how it behave in a certain domain. In spin systems, problems with antiperiodics can assist in examining the characteristics of spins located at the extremities of limited chains. In general, fractional differential equations with antiperiodic boundary conditions offer a flexible mathematical tool for comprehending and modeling different engineering and physical systems that display cyclic or alternating behaviors at their boundaries.

Fractional derivatives have been defined in various ways by mathematicians, including Grünwald–Letnikov, Liouville, Hadamard, Marchaud, Riesz, and Caputo derivatives. These definitions have been used to investigate solutions and stability of systems, as well as to define and characterize spaces [12, 30]. In this work, we focus on the Caputo fractional derivative because of its similarity to ordinary differential equations and its effectiveness in handling antiperiodic boundary problems.

Tremendous papers discuss the existence and uniqueness of a solution or solving different types of the boundary problems of fractional differential equations. For interesting results see [5, 8, 10, 13, 14, 16, 24, 28]. In [4], the author investigated and confirmed the existence of solutions to the following problem:

$$\begin{cases} {}^{c}D^{\mu}\omega(\rho) = \Omega(\rho, \omega(\rho)), \quad \rho \in [0, b], \quad 2 < \mu \leq 3, \\ \omega(0) = -\omega(b), \omega^{'}(0) = -\omega^{'}(b), \ \omega^{''}(0) = -\omega^{''}(b), \end{cases} \tag{2}$$

where $\Omega : [0, b] \times \mathbb{R} \longrightarrow \mathbb{R}$, by using fixed point theorem of Krasnoselskii's.

An interesting result was proved in [2], in which the domain $[0, b]$ was not considered for both its boundaries for the fractional differential equation of order $\mu \in (1, 2]$. They selected a nonlocal intermediate θ point and one of the fixed endpoints of the interval $(0, b]$ for:

$$\begin{cases} {}^{c}D^{\mu}\omega(\rho) = \Omega(\rho, \omega(\rho)), \quad \rho \in [0, b], \quad 1 < \mu \leq 2, \ 0 < \theta < b, \\ \omega(\theta) = -\omega(b), \quad \omega^{'}(\theta) = -\omega^{'}(b), \end{cases} \tag{3}$$

where $\Omega : [0, b] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a given function. The existence of a solution for this type of problems is based on standard fixed point theorems. In [6], the author studied the existence of solutions for the following NFDE:

$$\begin{cases} {}^{c}D^{\mu}\omega(\rho) = \Omega(\rho, \omega(\rho)), \quad \rho \in [0, b], \quad 1 < \mu \leq 2, \ 0 < \theta < b, \\ a_0\omega(\theta) = -b_0\omega(b), \ a_1\omega^{'}(\theta) = -b_1\omega^{'}(b), \end{cases} \tag{4}$$

where $a_i, b_i \in \mathbb{R}^+$, for $i = 0, 1$ and a continuous function $\Omega : [0, b] \times \mathbb{R} \longrightarrow \mathbb{R}$, by using the Krasnoselskii fixed-point theorem and the contraction principle. In recent work [15], positive solutions of fractional order Riemann Liouville and Caputo type Langevin equations was investigated. The results was obtained by upper and lower solution techniques along with fixed point theorems.

The paper is organized into four main sections, each serving a specific purpose in presenting and analyzing the research. The introduction provides a general overview of the topic, serving as an entry point into the study. The preliminary section delves deeper into the theoretical framework, methodological approach, and essential background information, providing context for the subsequent analysis. The third section presents and demonstrates important theorems that establish the existence of solutions to Equation (1). Finally, the conclusion summarizes the key findings and contributions of the research.

2. Basic facts

In this manuscript, we consider $A = C([0,b],\mathbb{R})$ is the Banach space of all continuous functions on the interval $[0,b]$ equiped with the norm $||\omega|| = \sup_{\rho\in[0,b]}|\omega(\rho)|$, and $L_1([0,b),\mathbb{R}^+)$ is the space of all integrable functions on the interval $[0,b]$.

Definition 1. *[1] For the function $\delta \in L_1([0,b],\mathbb{R}^+)$, the Caputo fractional derivative of order $\mu > 0$ is described as*

$$^cD^\mu\delta(\rho) = \frac{1}{\Gamma(\beta-\mu)}\int_0^\rho(\rho-\nu)^{\beta-\mu-1}\delta^{(\beta)}(\nu)d\nu, \quad \beta-1<\mu<\beta, \beta=[\mu]+1,$$

where $[\mu]$ is the integer part of μ.

Definition 2. *[1] For the function $\delta \in L_1([0,b],\mathbb{R}^+)$, the Riemann-Liouville fractional integral of order $\mu > 0$ is defined by*

$$I^\mu\delta(\rho) = \frac{1}{\Gamma(\mu)}\int_0^\rho(\rho-\nu)^{\mu-1}\delta(\nu)d\nu.$$

Lemma 1. *[1] Assume that $\mu > 0$, then, the solution of equation $^cD^\mu\omega(\rho) = 0$ is given by*

$$\omega(\rho) = \sum_{k=1}^{[\mu]+1}\tau_k\rho^{k-1}, \tag{5}$$

where $\tau_k \in \mathbb{R}$, for $k = 1, 2, \cdots, [\mu]+1$.

Lemma 2. *Assume that $\gamma \in C[0,b]$. The solution of the fractional differential equation*

$$\begin{cases} ^cD^\mu\omega(\rho) = \gamma(\rho), \quad \rho\in[0,b], \quad 2<\mu\leq 3, \ 0<\theta<b, \\ \omega(\theta) = -\omega(b), \ \omega^{'}(\theta) = -\omega^{'}(b), \ ^cD^{\xi+1}\omega(\theta) = -^cD^{\xi+1}\omega(b), \ 0<\xi<1, \end{cases} \tag{6}$$

is given by

$$\begin{aligned} \omega(\rho) \quad = \quad & \int_0^\rho \frac{(\rho-\nu)^{\mu-1}}{\Gamma(\mu)}\gamma(\nu)d\nu - \frac{1}{2}\left(\int_0^\theta \frac{(\theta-\nu)^{\mu-1}}{\Gamma(\mu)}\gamma(\nu)d\nu + \int_0^b \frac{(b-\nu)^{\mu-1}}{\Gamma(\mu)}\gamma(\nu)d\nu\right) \\ & + \frac{\Gamma(2-\xi)[(\theta+b)-2\rho]}{2(\theta^{1-\xi}+b^{1-\xi})}\left(\int_0^\theta \frac{(\theta-\nu)^{\mu-\xi-1}}{\Gamma(\mu-\xi)}\gamma(\nu)d\nu + \int_0^b \frac{(b-\nu)^{\mu-\xi-1}}{\Gamma(\mu-\xi)}\gamma(\nu)d\nu\right) \\ & + \frac{\Gamma(2-\xi)}{4\Gamma(3-\xi)(\theta^{1-\xi}+b^{1-\xi})^2}\begin{pmatrix} 4\rho(\theta^{2-\xi}+b^{2-\xi})\Gamma(2-\xi) \\ -2\rho^2\Gamma(3-\xi)(\theta^{1-\xi}+b^{1-\xi}) \\ -2(\theta+b)(\theta^{2-\xi}+b^{2-\xi})\Gamma(2-\xi) \\ +(\theta^2+b^2)\Gamma(3-\xi)(\theta^{1-\xi}+b^{1-\xi}) \end{pmatrix} \\ & \times\left(\int_0^\theta \frac{(\theta-\nu)^{\mu-\xi-2}}{\Gamma(\mu-\xi-1)}\gamma(\nu)d\nu + \int_0^b \frac{(b-\nu)^{\mu-\xi-2}}{\Gamma(\mu-\xi-1)}\gamma(\nu)d\nu\right). \end{aligned} \tag{7}$$

Proof. By Lemma 1 there are constants $\tau_j \in \mathbb{R}$, for $j = 1,2,3$ such that

$$\omega(\rho) = I^{\mu}\ \delta(\rho) - \tau_1 - \tau_2\rho - \tau_3\rho^2 = \int_0^{\rho} \frac{(\rho-\nu)^{\mu-1}}{\Gamma(\mu)}\delta(\nu)d\nu - \tau_1 - \tau_2\rho - \tau_3\rho^2. \tag{8}$$

Applying the conditions (6), we have

$$\begin{aligned}\tau_1 &= \frac{1}{2}\left(\int_0^{\theta} \frac{(\theta-\nu)^{\mu-1}}{\Gamma(\mu)}\delta(\nu)d\nu + \int_0^{b} \frac{(b-\nu)^{\mu-1}}{\Gamma(\mu)}\delta(\nu)d\nu\right)\\ &\quad - \frac{(\theta+b)\Gamma(2-\xi)}{2(\theta^{1-\xi}+b^{1-\xi})}\left(\int_0^{\theta} \frac{(\theta-\nu)^{\mu-\xi-1}}{\Gamma(\mu-\xi)}\delta(\nu)d\nu + \int_0^{b} \frac{(b-\nu)^{\mu-\xi-1}}{\Gamma(\mu-\xi)}\delta(\nu)d\nu\right)\\ &\quad + \frac{\Gamma(2-\xi)}{2(\theta^{1-\xi}+b^{1-\xi})}\left(\frac{(\theta+b)(\theta^{2-\xi}+b^{2-\xi})(\Gamma(2-\xi))}{\Gamma(3-\xi)(\theta^{1-\xi}+b^{1-\xi})} - \frac{(\theta^2+b^2)}{2}\right)\\ &\quad \times\left(\int_0^{\theta} \frac{(\theta-\nu)^{\mu-\xi-2}}{\Gamma(\mu-\xi-1)}\delta(\nu)d\nu + \int_0^{b} \frac{(b-\nu)^{\mu-\xi-2}}{\Gamma(\mu-\xi-1)}\delta(\nu)d\nu\right),\end{aligned}$$

$$\begin{aligned}\tau_2 &= \frac{\Gamma(2-\xi)}{\theta^{1-\xi}+b^{1-\xi}}\left(\int_0^{\theta} \frac{(\theta-\nu)^{\mu-\xi-1}}{\Gamma(\mu-\xi)}\delta(\nu)d\nu + \int_0^{b} \frac{(b-\nu)^{\mu-\xi-1}}{\Gamma(\mu-\xi)}\delta(\nu)d\nu\right)\\ &\quad - \frac{(\theta^{2-\xi}+b^{2-\xi})(\Gamma(2-\xi))^2}{\Gamma(3-\xi)(\theta^{1-\xi}+b^{1-\xi})^2}\left(\int_0^{\theta} \frac{(\theta-\nu)^{\mu-\xi-2}}{\Gamma(\mu-\xi-1)}\delta(\nu)d\nu + \int_0^{b} \frac{(b-\nu)^{\mu-\xi-2}}{\Gamma(\mu-\xi-1)}\delta(\nu)d\nu\right),\end{aligned}$$

and

$$\tau_3 = \frac{\Gamma(2-\xi)}{2(b^{1-\xi}+\theta^{1-\xi})}\left(\int_0^{\theta} \frac{(\theta-\nu)^{\mu-\xi-2}}{\Gamma(\mu-\xi-1)}\delta(\nu)d\nu + \int_0^{b} \frac{(b-\nu)^{\mu-\xi-2}}{\Gamma(\mu-\xi-1)}\delta(\nu)d\nu\right).$$

Substituting the values of τ_1, τ_2 and τ_3 in (8), we obtain the result.

Remark 1. *It's important to note that when a equals* 0*, the first three terms in Equation (7) represent the solution to the fractional problem of order* $\mu \in (1,2]$ *[3]. Additionally, two more terms are added to the solution when the order* μ *is increased to a number in the interval* $(2,3]$*, as demonstrated in Lemma 2.*

Remark 2. *The solution of the fractional problem*

$$\begin{cases} {}^{c}D^{\mu}\omega(\rho) = u(\rho), \quad \rho \in [0,b], \quad 2 < \mu \le 3,\\ \omega(0) = -\omega(b),\ \omega'(0) = -\rho'(b),\ \omega''(0) = -\omega''(\rho), \end{cases} \tag{9}$$

is equivalent to

$$\begin{aligned}\omega(\rho) &= \int_0^{\rho} \frac{(\rho-\nu)^{\mu-1}}{\Gamma(\mu)}u(\nu)d\nu - \frac{1}{2}\int_0^{b} \frac{(b-\nu)^{\mu-1}}{\Gamma(\mu)}u(\nu)d\nu\\ &\quad + \frac{b-2\rho}{4}\int_0^{b} \frac{(b-\nu)^{\mu-2}}{\Gamma(\mu-1)}u(\nu)d\nu + \frac{\rho(b-\rho)}{4}\int_0^{b} \frac{(b-\nu)^{\mu-3}}{\Gamma(\mu-2)}u(\nu)d\nu,\end{aligned} \tag{10}$$

which is given in [4].

Observing that the solutions (7) and (10), we notice that there are additional terms in 7. Moreover, when $\xi \longrightarrow 1^-$, it will be the solution of the nonlocal antiperiodic boundary condition of order $\mu \in (2,3]$.

If we set $\theta = 0$ then take $\xi \longrightarrow 1^-$, we obtain

$$\begin{aligned}\omega(\rho) &= \int_0^\rho \frac{(\rho-\nu)^{\mu-1}}{\Gamma(\mu)} u(\nu)d\nu - \frac{1}{2}\int_0^b \frac{(b-\nu)^{\nu-1}}{\Gamma(\nu)} u(\nu)d\nu \\ &+ \frac{b-2\rho}{2}\int_0^b \frac{(b-\nu)^{\mu-2}}{\Gamma(\mu-1)} u(\nu)d\nu + \frac{2\rho(2b-\rho)-b^2}{4}\int_0^b \frac{(b-s)^{\mu-3}}{\Gamma(\mu-2)} u(\nu)d\nu]. \quad (11)\end{aligned}$$

The solution of (10) and (11) will be different and contains additional different terms. Therefore, the boundary conditions give rise to a new class of problems. However, if we take first $\xi \longrightarrow 1^-$ and then set every $a = 0$, the solution will stay the same, due to the shifting in the position at the left end of the interval $[0, b]$. Other results with different boundary conditions can be found in [7, 9, 11, 20, 21].

Theorem 1. *[29] Assume that $\mathcal{P}$ is an open bounded subset of a Banach space Y with $0 \in \mathcal{P}$ and the operator $\mathcal{L} : \overline{\mathcal{P}} \longrightarrow Y$ is a completely continuous with $||\mathcal{L}\omega|| \leq ||\omega||$ for every $\omega \in \partial\mathcal{P}$. Then, there is a fixed point of the operator $\mathcal{L}$ in $\partial\mathcal{L}$.*

Theorem 2. *[29] Suppose that A is a non-empty closed and convex subset of a Banach space $\mathcal{B}$, W_1, and W_2 are operators such that*

- *$W_1\omega_1 + W_2\omega_2 \in A$, whenever $\omega_1, \omega_2 \in A$;*
- *W_1 is compact and continuous;*
- *W_2 is a contraction mapping.*

Then there is $\hat{\omega} \in A$ satisfies the equation $\hat{\omega} = W_1\hat{\omega} + W_2\hat{\omega}$.

3. Main results

We begin this part with the definition of the operator $\mathcal{L} : A \longrightarrow A$ by

$$\begin{aligned}(\mathcal{L}\omega)\rho &= \int_0^\rho \frac{(\rho-\nu)^{\mu-1}}{\Gamma(\mu)}\Omega(\nu,\omega(\nu))d\nu \\ &-\frac{1}{2}\left(\int_0^\theta \frac{(\theta-\nu)^{\mu-1}}{\Gamma(\mu)}\Omega(\nu,\omega(\nu))d\nu + \int_0^b \frac{(b-\nu)^{\mu-1}}{\Gamma(\mu)}\Omega(\nu,\omega(\nu))d\nu\right) \\ &+\frac{\Gamma(2-\xi)[(\theta+b)-2\rho]}{2(\rho^{1-\xi}+b^{1-\xi})}\left(\int_0^\theta \frac{(\theta-\nu)^{\mu-\xi-1}}{\Gamma(\mu-\xi)}\Omega(\nu,\omega(\nu))d\nu\right. \\ &\left.+\int_0^b \frac{(b-\nu)^{\mu-\xi-1}}{\Gamma(\mu-\xi)}\Omega(\nu,\omega(\nu))d\nu\right)\end{aligned}$$

$$+\frac{\Gamma(2-\xi)}{4\Gamma(3-\xi)(\theta^{1-\xi}+b^{1-\xi})^2}\begin{pmatrix} 4\rho(\theta^{2-\xi}+b^{2-\xi})\Gamma(2-\xi) \\ -4\rho^2\Gamma(3-\xi)(\theta^{1-\xi}+b^{1-\xi}) \\ -2(\theta+b)(\theta^{2-\xi}+b^{2-\xi})\Gamma(2-\xi) \\ +(\theta^2+b^2)\Gamma(3-\xi)(\theta^{1-\xi}+b^{1-\xi}) \end{pmatrix}$$
$$\times\left(\int_0^\theta \frac{(\theta-\nu)^{\mu-\xi-2}}{\Gamma(\mu-\xi-1)}\Omega(\nu,\omega(\nu))d\nu+\int_0^b \frac{(b-\nu)^{\mu-\xi-2}}{\Gamma(\mu-\xi-1)}\Omega(\nu,\omega(\nu))d\nu\right). \quad (12)$$

where $A=C([0,b],\mathbb{R})$ is a Banach space.

Remark 3. *The existence of the fixed point of the operator $\mathcal{L}$ ($\mathcal{L}\omega=\omega$) is equivalent to the existence of the solution to the problem (1).*

For simplicity, we consider the following notations:

$$\begin{aligned} M_1 &= \max_{\rho\in[0,b]||\Omega(\rho,0)||=n<\infty}\left\{\frac{3b^\mu+\theta^\mu}{\Gamma(\mu+1)}+\frac{\Gamma(2-\xi)\,|(\theta+b)-2\rho|\,(\theta^{\mu-\xi-1}+b^{\mu-\xi-1})}{(\theta^{1-\xi}+b^{1-\xi})\Gamma(\mu-\xi+1)}\right. \\ &+\frac{\Gamma(2-\xi)}{2\Gamma(3-\xi)(\theta^{1-\xi}+b^{1-\xi})^2\Gamma(\mu-\xi)} \\ &\times\left|4\rho(\theta^{2-\xi}+b^{2-\xi})\Gamma(2-\xi)-4\rho^2\Gamma(3-\xi)(\theta^{1-\xi}+b^{1-\xi})\right. \\ &\left.\left.-2(\theta+b)(\theta^{2-\xi}+b^{2-\xi})\Gamma(2-\xi)+(\theta^2+b^2)\Gamma(3-\xi)(\theta^{1-\xi}+b^{1-\xi})\right|\right\}. \quad (13) \end{aligned}$$

$$\begin{aligned} M_2 &= \max_{\rho\in[0,b]||\Omega(\rho,0)||=m<\infty}\left\{\frac{b^\mu+\theta^\mu}{\Gamma(\mu+1)}+\frac{\Gamma(2-\xi)\,|(\theta+b)-2\rho|\,(\theta^{\mu-\xi-1}+b^{\mu-\theta-1})}{(\theta^{1-\xi}+b^{1-\xi})\Gamma(\mu-\xi+1)}\right. \\ &+\frac{\Gamma(2-\xi)}{2\Gamma(3-\xi)(\theta^{1-\xi}+b^{1-\xi})^2\Gamma(\mu-\xi)} \\ &\times\left|4\rho(\theta^{2-\xi}+b^{2-\xi})\Gamma(2-\xi)-4\rho^2\Gamma(3-\xi)(\theta^{1-\xi}+b^{1-\xi})\right. \\ &\left.\left.-2(\theta+b)(\theta^{2-\xi}+b^{2-\xi})\Gamma(2-\xi)+(\theta^2+b^2)\Gamma(3-\xi)(\theta^{1-\xi}+b^{1-\xi})\right|\right\}. \quad (14) \end{aligned}$$

Lemma 3. *The operator $\mathcal{L}:A\longrightarrow A$ is completely continuous.*

Proof. Suppose $\mathcal{S}\subset A$ be bounded. Then there exists $K_1>0$ such that $|\Omega(\rho,\omega)|\leq K_1$, $\forall\rho\in[0,b]$ and $\omega\in\mathcal{S}$. Let $\mathcal{L}$ be the operator defined in (12), then, we have

$$\begin{aligned} |(\mathcal{L}\omega)\rho| &= \int_0^\rho \frac{(\rho-\nu)^{\mu-1}}{\Gamma(\mu)}|\Omega(\nu,\omega(\nu))|d\nu \\ &+\frac{1}{2}\left(\int_0^\theta \frac{(\theta-\nu)^{\mu-1}}{\Gamma(\mu)}|\Omega(\nu,\omega(\nu))|d\nu+\int_0^b \frac{(b-\nu)^{\mu-1}}{\Gamma(\mu)}|\Omega(\nu,\omega(\nu))|d\nu\right) \\ &+\frac{\Gamma(2-\xi)|(\theta+b)-2\rho|}{2(\theta^{1-\xi}+b^{1-\xi})}\left(\int_0^\theta \frac{(\theta-\nu)^{\mu-\xi-1}}{\Gamma(\mu-\xi)}|\Omega(\nu,\omega(\nu))|d\nu+\int_0^b \frac{(b-\nu)^{\mu-\xi-1}}{\Gamma(\mu-\xi)}|\Omega(\nu,\omega(\nu))|d\nu\right) \end{aligned}$$

$$
\begin{aligned}
& +\frac{\Gamma(2-\xi)}{4\Gamma(3-\xi)(\theta^{1-\xi}+b^{1-\xi})^2}\Big|4\rho(\theta^{2-\xi}+b^{2-\xi})\Gamma(2-\xi)-4\rho^2\Gamma(3-\xi)(\theta^{1-\xi}+b^{1-\xi}) \\
& -2(\theta+b)(\theta^{2-\xi}+b^{2-\xi})\Gamma(2-\xi)+(\theta^2+b^2)\Gamma(3-\xi)(\theta^{1-\xi}+b^{1-\xi})\Big| \\
& \times\left(\int_0^\theta \frac{(\theta-\nu)^{\mu-\xi-2}}{\Gamma(\mu-\xi-1)}|\Omega(\nu,\omega(\nu))|d\nu+\int_0^b \frac{(b-\nu)^{\mu-\xi-2}}{\Gamma(\mu-\xi-1)}|\Omega(\nu,\omega(\nu))|d\nu\right) \\
\leq\ & \frac{K_1}{2}\left[2\int_0^\rho \frac{(\rho-\nu)^{\mu-1}}{\Gamma(\mu)}d\nu+\int_0^\theta \frac{(\theta-\nu)^{\mu-1}}{\Gamma(\mu)}d\nu+\int_0^b \frac{(b-\nu)^{\mu-1}}{\Gamma(\mu)}d\nu\right. \\
& +\frac{\Gamma(2-\xi)|(\theta+b)-2\rho|}{(\theta^{1-\xi}+b^{1-\xi})}\left[\int_0^\theta \frac{(\theta-\nu)^{\mu-\xi-1}}{\Gamma(\mu-\xi)}d\nu+\int_0^b \frac{(b-\nu)^{\mu-\xi-1}}{\Gamma(\mu-\xi)}d\nu\right] \\
& +\frac{\Gamma(2-\xi)}{2\Gamma(3-\xi)(\theta^{1-\xi}+b^{1-\xi})^2}4\Big|\rho(\theta^{2-\xi}+b^{2-\xi})\Gamma(2-\xi)-4\rho^2\Gamma(3-\xi)(\theta^{1-\xi}+b^{1-\xi}) \\
& -2(\theta+b)(\theta^{2-\xi}+b^{2-\xi})\Gamma(2-\xi)+(\theta^2+b^2)\Gamma(3-\xi)(\theta^{1-\xi}+b^{1-\xi})\Big| \\
& \left.\times\left(\int_0^\theta \frac{(\theta-\nu)^{\mu-\xi-2}}{\Gamma(\mu-\xi-1)}d\nu+\int_0^b \frac{(b-\nu)^{\mu-\xi-2}}{\Gamma(\mu-\xi-1)}d\nu\right)\right] \\
\leq\ & \frac{K_1 M_1}{2}=K_2,
\end{aligned}
$$

which implies that $||(\mathcal{L}\omega)||\leq K_2$, Now,

$$
\begin{aligned}
|(\mathcal{L}\omega)^{'}(\rho)|\ \leq\ & \int_0^\rho \frac{(\rho-\nu)^{\mu-2}}{\Gamma(\mu-1)}|\Omega(\nu,\omega(\nu))|d\nu \\
& +\frac{\Gamma(2-\xi)}{(\theta^{1-\xi}+b^{1-\xi})}\left(\int_0^\theta \frac{(\theta-\nu)^{\mu-2}}{\Gamma(\mu-1)}|\Omega(\nu,\omega(\nu))|d\nu+\int_0^b \frac{(b-\nu)^{\mu-2}}{\Gamma(\mu-1)}|\Omega(\nu,\omega(\nu))|d\nu\right) \\
& +\frac{\Gamma(2-\xi)}{\Gamma(3-\xi)(\theta^{1-\xi}+b^{1-\xi})^2}\Big|(\theta^{2-\xi}+b^{2-\xi})\Gamma(2-\xi)-2\rho\Gamma(3-\xi)(\theta^{1-\xi}+b^{1-\xi})\Big| \\
& \times\left(\int_0^\theta \frac{(\theta-\nu)^{\mu-\xi-2}}{\Gamma(\mu-\xi-1)}|\Omega(\nu,\omega(\nu))|d\nu+\int_0^b \frac{(b-\nu)^{\mu-\xi-2}}{\Gamma(\mu-\xi-1)}|\Omega(\nu,\omega(\nu))|d\nu\right) \\
\leq\ & K_1\left(\max_{\rho\in[0,b]}\left\{\frac{2|\rho^{\mu-1}|+\frac{\Gamma(2-\xi)}{(a^{1-\xi}+b^{1-\xi})}(\theta^{\mu-1}+b^{\mu-1})}{2\Gamma(\mu)}\right.\right. \\
& \left.\left.+\frac{|\Gamma(2-\xi)|}{\Gamma(3-\xi)(\theta^{1-\xi}+b^{1-\xi})^2}|(\theta^{2-\xi}+b^{2-\xi})\Gamma(2-\xi)-2\rho\Gamma(3-\xi)(\theta^{1-\xi}+b^{1-\xi})|\right\}\right) \\
=\ & K_3.
\end{aligned}
$$

Hence, for any $\rho_1,\rho_2\in[0,b]$, we have

$$
|(\mathcal{L}\omega)(\rho_2)-(\mathcal{L}\omega)(\rho_1)|\leq\int_{\rho_1}^{\rho_2}|(\mathcal{L}\rho)^{'}(\nu)|d\nu\leq K_3(\rho_2-\rho_1).
$$

Therefore, $\mathcal{L}$ satisfies equuicontinuity on $[0,b]$. By Arzelá-Ascoli theorem, we conclude that $\mathcal{L}$ is completely continuous.

Theorem 3. *Assume that $\Omega : [0,b] \times \mathbb{R} \longrightarrow \mathbb{R}$ is jointly continuous function such that*

$$||\Omega(\rho,\omega_1) - \Omega(\rho,\omega_2)|| \leq L||\omega_1 - \omega_2||,$$

for all $\rho \in [0,b]$ and $\omega_1,\ \omega_2 \in \mathbb{R}$, where M_1 is defined in (13). The problem (1) has a solution, provided that $LM_1 < 1$.

Proof. Assume that $\max_{\rho\in[0,b]} |\Omega(\rho,0)|| = \eta < \infty$ and selecting $r \geq \frac{\kappa_1}{1-\kappa_2}$ where $\kappa_1 = \frac{\eta M_1}{2}$ and $\kappa_2 = \frac{LM_1}{2}$, we will demonstrate that $\mathcal{L}\mathcal{B}_r \subset \mathcal{B}_r$ where $\mathcal{B}_r = \{\omega \in C[0,b] : ||\omega|| \leq r\}$. Now, for $\omega \in \mathcal{B}_r$, we have

$$\begin{aligned}
||(\mathcal{L}\omega)(\rho)|| \leq\ & \int_0^\rho \frac{(\rho-\nu)^{\mu-1}}{\Gamma(\mu)} | \left(|\Omega(\nu,\omega(\nu)) - \Omega(\nu,0)|| + ||\Omega(\nu,0)||\right) d\nu \\
& + \frac{1}{2}\left(\int_0^\theta \frac{(\theta-\nu)^{\mu-1}}{\Gamma(\mu)} [||\Omega(\nu,\omega(\nu)) - \Omega(\nu,0)|| + ||\Omega(\nu,0)||d\nu] \right. \\
& \left. + \int_0^b \frac{(b-\nu)^{\mu-1}}{\Gamma(\mu)} \left(||\Omega(\nu,\omega(\nu)) - \Omega(\nu,0)|| + ||\Omega(\nu,0)||\right) d\nu \right) \\
& + \frac{\Gamma(2-\xi)|(\theta+b) - 2\rho|}{2(\theta^{1-\xi} + b^{1-\xi})} \left(\int_0^\theta \frac{(\theta-\nu)^{\mu-\xi-1}}{\Gamma(\mu-\xi)} \left(||\Omega(\nu,\omega(\nu)) - \Omega(\nu,0)|| + ||\Omega(\nu,0)||\right) d\nu \right. \\
& \left. + \int_0^b \frac{(b-\nu)^{\mu-\xi-1}}{\Gamma(\mu-\xi)} \left(||\Omega(\nu,\omega(\nu)) - \Omega(\nu,0)|| + ||\Omega(\nu,0)||\right) d\nu \right) \\
& + \frac{\Gamma(2-\xi)}{4\Gamma(3-\xi)(\theta^{1-\xi} + b^{1-\xi}))} \Big| 4\rho(\theta^{2-\xi} + b^{2-\xi})\Gamma(2-\xi) - 4\rho^2\Gamma(3-\xi)(\theta^{1-\xi} + b^{1-\xi}) \\
& - 2(\theta+b)(\theta^{2-\xi} + b^{2-\xi})\Gamma(2-\xi)) + (\theta^2 + b^2)\Gamma(3-\xi)(\theta^{1-\xi} + b^{1-\xi}) \Big| \\
& \times \left(\int_0^\theta \frac{(\theta-\nu)^{\mu-\xi-2}}{\Gamma(\mu-\xi-1)} ||\Omega(\nu,\omega(\nu)) - \Omega(\nu,0)|| + ||\Omega(\nu,0)|| \right) d\nu \\
& \left. + \int_0^b \frac{(b-\nu)^{\mu-\xi-2}}{\Gamma(\mu-\xi-1)} \left(||\Omega(\nu,\omega(\nu)) - \Omega(\nu,0)|| + ||\Omega(\nu,0)||\right) d\nu \right) \\
\leq\ & \frac{(Lr+\eta)}{2} \left(2\int_0^\rho \frac{(\rho-\nu)^{\mu-1}}{\Gamma(\mu)} d\nu + \int_0^\theta \frac{(\theta-\nu)^{\mu-1}}{\Gamma(\mu)} d\nu + \int_0^b \frac{(b-\nu)^{\mu-1}}{\Gamma(\mu)} d\nu \right) \\
& + \frac{\Gamma(2-\xi)|(\theta+b) - 2\rho|}{(\theta^{1-\xi} + b^{1-\xi})} \left(\int_0^\theta \frac{(\theta-\nu)^{\mu-\xi-1}}{\Gamma(\mu-\xi)} d\nu + \int_0^b \frac{(b-\nu)^{\mu-\xi-1}}{\Gamma(\mu-\xi)} d\nu \right) \\
& + \frac{\Gamma(2-\xi)}{2\Gamma(3-\xi)(\theta^{1-\xi} + b^{1-\xi}))} \Big| 4\rho(\theta^{2-\xi} + b^{2-\xi})\Gamma(2-\xi) - 4\rho^2\Gamma(3-\xi)(\theta^{1-\xi} + b^{1-\xi}) \\
& - 2(\theta+b)(\theta^{2-\xi} + b^{2-\xi})\Gamma(2-\xi)) + (\theta^2 + b^2)\Gamma(3-\xi)(\theta^{1-\xi} + b^{1-\xi}) \Big| \\
& \times \left(\int_0^\theta \frac{(\theta-\nu)^{\mu-\xi-2}}{\Gamma(\mu-\xi-1)} d\nu + \int_0^b \frac{(b-\nu)^{\mu-\xi-2}}{\Gamma(\mu-\xi-1)} d\nu \right) \\
\leq\ & \frac{(Lr+\eta)}{2} \left[\frac{2|\rho|^\mu + b^\mu + \theta^\mu}{\Gamma(\mu+1)} + \frac{\Gamma(2-\xi)|(\theta+b) - 2\rho|(\theta^{\mu-\xi} + b^{\mu-\xi})}{2\Gamma(\mu-\xi+1)} \right.
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\Gamma(2-\xi)(\theta^{\mu-\xi-1}+b^{\mu-\xi-1})}{2\Gamma(\mu-\xi)}\Big|4\rho(\theta^{2-\xi}+b^{2-\xi})\Gamma(2-\xi)-4\rho^2\Gamma(3-\xi)(\theta^{1-\xi}+b^{1-\xi})\\
&-\ 2(\theta+b)(\theta^{2-\xi}+b^{2-\xi})\Gamma(2-\xi))+(\theta^2+b^2)\Gamma(3-\xi)(\theta^{1-\xi}+b^{1-\xi})\Big|\Big]\\
\leq\ & (Lr+\eta)\frac{M_1}{2}\leq r.
\end{aligned}
$$

This proves that $\mathcal{L}\mathcal{B}_r \subset \mathcal{B}_r$. Now, we prove that the operator $\mathcal{L}$ is a contraction. For $\omega_1, \omega_2 \in \mathcal{B}_r$, we can write

$$
\begin{aligned}
||(\mathcal{L}\omega_1)(\rho)-(\mathcal{L}\omega_2)(\rho)|| \leq\ & \int_0^\rho \frac{(\rho-\nu)^{\mu-1}}{\Gamma(\mu)}||\Omega(\nu,\omega_1(\nu))-\Omega(\nu,\omega_2(\nu))||d\nu\\
&+\frac{1}{2}\left[\int_0^\theta \frac{(\theta-\nu)^{\mu-1}}{\Gamma(\mu)}||\Omega(\nu,y_1(\nu))-\Omega(\nu,\omega_2(\nu))||d\nu\right.\\
&\left.+\int_0^b \frac{(b-\nu)^{\mu-1}}{\Gamma(\mu)}||\Omega(\nu,\omega_1(\nu))-\Omega(\nu,\omega_2(\nu))||d\nu\right]\\
&+\frac{\Gamma(2-\xi)\,|[(\theta+b)-2\rho]|}{2(\theta^{1-\xi}+b^{1-\xi})}\left(\int_0^\theta \frac{(\theta-\nu)^{\mu-\xi-1}}{\Gamma(\mu-\xi)}||\Omega(\nu,\omega_1(\nu))-\Omega(\nu,\omega_2(\nu))||d\nu\right.\\
&\left.+\int_0^b \frac{(b-\nu)^{\mu-\xi-1}}{\Gamma(\mu-\xi)}||\Omega(\nu,\omega_1(\nu))-\Omega(\nu,\omega_2(\nu))||d\nu\right)\\
&+\frac{\Gamma(2-\xi)}{4\Gamma(3-\xi)(\theta^{1-\xi}+b^{1-\xi})^2}\Big[|4\rho(\theta^{2-\xi}+b^{2-\xi})\Gamma(2-\xi)-4\rho^2\Gamma(3-\xi)(\theta^{1-\xi}+b^{1-\xi})\\
&-2(\theta+b)(\theta^{2-\xi}+b^{2-\xi})\Gamma(2-\xi)+(\theta^2+b^2)\Gamma(3-\xi)(\theta^{1-\xi}+b^{1-\xi})\Big|\Big]\\
&\times\left[\int_0^\theta \frac{(\theta-\nu)^{\mu-\xi-2}}{\Gamma(\mu-\xi-1)}||\Omega(\nu,\omega_1(\nu))-\Omega(\nu,\omega_2(\nu))||d\nu\right.\\
&\left.+\int_0^b \frac{(b-\nu)^{\mu-\xi-2}}{\Gamma(\mu-\xi-1)}||\Omega(\nu,\omega_1(\nu))-\Omega(\nu,\omega_2(\nu))||d\nu\right]\\
\leq\ & L||\omega_1-\omega_2||\left[\int_0^\rho \frac{(\rho-\nu)^{\mu-1}}{\Gamma(\mu)}d\nu+\frac{1}{2}\left(\int_0^\theta \frac{(\theta-\nu)^{\mu-1}}{\Gamma(\mu)}d\nu+\int_0^b \frac{(b-\nu)^{\mu-1}}{\Gamma(\mu)}d\nu\right)\right.\\
&+\frac{\Gamma(2-\xi)\,|(\theta+b)-2\rho|}{2(\theta^{1-\xi}+b^{1-\xi})}\left(\int_0^\theta \frac{(\theta-\nu)^{\mu-\xi-1}}{\Gamma(\mu-\xi)}d\nu+\int_0^b \frac{(b-\nu)^{\mu-\xi-1}}{\Gamma(\mu-\xi)}d\nu\right)\\
&+\frac{\Gamma(2-\xi)}{4\Gamma(3-\xi)(\theta^{1-\xi}+b^{1-\xi})^2}\Big|4\rho(\theta^{2-\xi}+b^{2-\xi})\Gamma(2-\xi)-4\rho^2\Gamma(3-\xi)(\theta^{1-\xi}+b^{1-\xi})\\
&-2(\theta+b)(\theta^{2-\xi}+b^{2-\xi})\Gamma(2-\xi)+(\theta^2+b^2)\Gamma(3-\xi)(\theta^{1-\xi}+b^{1-\xi})\Big|\\
&\left.\times\left(\int_0^\theta \frac{(\theta-\nu)^{\mu-\xi-2}}{\Gamma(\mu-\xi-1)}d\nu+\int_0^b \frac{(b-\nu)^{\mu-\xi-2}}{\Gamma(\mu-\xi-1)}d\nu\right)\right]\\
\leq\ & LM_1||\omega_1-\omega_2||.
\end{aligned}
$$

Since $LM_1 < 1$, we conclude that $\mathcal{L}$ is a contraction operator. According to Lemma 3 and

Theorem 1, $\mathcal{L}$ has a fixed point, which is a solution to the problem (1).

Theorem 4. *Assume that* $\Omega : [0,b] \times \mathbb{R} \longrightarrow \mathbb{R}$ *is continuous such that*

$$||\Omega(\rho,\omega_1) - \Omega(\rho,\omega_2)|| \leq L||\omega_1 - \omega_2||,$$

for all $\rho \in [0,b]$ *and* $\omega_1, \omega_2 \in \mathbb{R}$*. If for all* $(\rho,\omega) \in [0,1] \times \mathbb{R}$ *and* $\psi \in L^1([0,b],\mathbb{R}^+)$,

$$|\Omega(\rho,\omega)| \leq \psi(\rho).$$

Then, the problem (1) has at least one solution on $[0,b]$*, provided that* $LM_2 < 1$*, where* M_2 *is defined in (14).*

Proof. Define the operators $\mathcal{L}_1$ and $\mathcal{L}_2$ by

$$(\mathcal{L}_1\omega)(\rho) = \int_0^\rho \frac{(\rho-\nu)^{\mu-1}}{\Gamma(\mu)}\Omega(\nu,\omega(\nu))d\nu,$$

and

$$\begin{aligned}
(\mathcal{L}_2\omega)(\rho) \;=\; & -\frac{1}{2}\left(\int_0^\theta \frac{(\theta-\nu)^{\mu-1}}{\Gamma(\mu)}\Omega(\nu,\omega(\nu))d\nu + \int_0^b \frac{(b-\nu)^{\mu-1}}{\Gamma(\mu)}\Omega(\nu,\omega(\nu))d\nu\right) \\
& + \frac{\Gamma(2-\xi)[(\theta+b)-2\rho]}{2(\theta^{1-\xi}+b^{1-\xi})} \\
& \times \left(\int_0^\theta \frac{(\theta-\nu)^{\mu-\xi-1}}{\Gamma(\mu-\xi)}\Omega(\nu,\omega(\nu))d\nu + \int_0^b \frac{(b-\nu)^{\mu-\xi-1}}{\Gamma(\mu-\xi)}\Omega(\nu,\omega(\nu))d\nu\right) \\
& + \frac{\Gamma(2-\xi)}{4\Gamma(3-\xi)(\theta^{1-\xi}+b^{1-\xi})^2}\Big[4\rho(\theta^{2-\xi}+b^{2-\xi})\Gamma(2-\xi) - 4\rho^2\Gamma(3-\xi)(\theta^{1-\xi}+b^{1-\xi}) \\
& -2(\theta+b)(\theta^{2-\xi}+b^{2-\xi})\Gamma(2-\xi) + (\theta^2+b^2)\Gamma(3-\xi)(\theta^{1-\xi}+b^{1-\xi})\Big] \\
& \times \left(\int_0^\theta \frac{(\theta-\nu)^{\mu-\xi-2}}{\Gamma(\mu-\xi-1)}\Omega(\nu,\omega(\nu))d\nu + \int_0^b \frac{(b-\nu)^{\mu-\xi-2}}{\Gamma(\mu-\xi-1)}\Omega(\nu,\omega(\nu))d\nu\right)
\end{aligned}$$

on $B_r = \{\omega \in \mathcal{L} : ||\omega|| \leq r\}$ where $r \geq \frac{||\mu||M_1}{2}$.
Clearly, for $\omega_1, \omega_2 \in B_r$,

$$||\mathcal{L}_1\omega_1 + \mathcal{L}_2\omega_2|| \leq \frac{||\psi||M_1}{2} \leq r,$$

which implies that $\mathcal{L}_1\omega_1 + \mathcal{L}_2\omega_2 \in B_r$. By the same steps of Theorem 4 and the assumption $LM_2 < 1$, we have $\mathcal{L}_2$ is a contraction mapping.

Further, the continuity of Ω implies that the continuity of $\mathcal{L}_1$. Since

$$||(\mathcal{L}_1\omega(\rho)|| \leq \frac{||\psi||}{\Gamma(\mu+1)}b^\mu,$$

then, $\mathcal{L}_1$ is uniformly bounded on B_r. In a view of the second assumption, for all $(\rho,\omega) \in [0,b] \times B_r$, we have

$$\begin{aligned} ||(\mathcal{L}_1\omega)(\rho_1) - (\mathcal{L}_1\omega)(\rho_2)|| &= \frac{1}{\Gamma(\mu)} \left\| \int_0^{\rho_1} [(\rho_1-\nu)^{\mu-1} - (\rho_2-\nu)^{\mu-1}]\Omega(\nu,\omega(\nu))d\nu \right. \\ &\quad \left. + \int_{\rho_1}^{\rho_2} (\rho_2-\nu)^{\mu-1}\Omega(\nu,\omega(\nu))d\nu| \right\| \\ &\leq \frac{\max\limits_{(\rho,\omega)\in[0,b]\times\mathcal{B}_r} |\Omega(\rho,\omega)|}{\Gamma(\mu+1)} |2(\rho_2-\rho_1)^\mu + \rho_1^\mu - \rho_2^\mu|. \end{aligned}$$

When $\rho_1 \to \rho_2$, then $||(\mathcal{L}_1\omega)(\rho_1) - (\mathcal{L}_1\omega)(\rho_2)|| \to 0$. By using Arzelá-Ascoli theorem, $\mathcal{L}_1$ is compact on B_r. Based on Theorem 2, there exists at least one solution to the problem (1).

Theorem 5. *Assume that $\Omega : [0,b] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function and there exists a constant $0 < \chi < \frac{1}{M}$ and $\delta > 0$ such that*

$$|\Omega(\rho,\omega)| \leq \chi|\omega| + \delta,$$

for all $\rho \in [0,b]$, and $\omega \in \mathbb{R}$. Then, the problem (1) has at least one solution, where

$$\begin{aligned} M &= \max_{\rho\in[0,b]||\Omega(\rho,0)||=n<\infty} \left\{ \frac{3b^\mu + \theta^\mu}{2\Gamma(\mu+1)} + \frac{\Gamma(2-\xi)|(\theta+b)-2\rho|(\theta^{\mu-\xi-1} + b^{\mu-\xi-1})}{2(\theta^{1-\xi}+b^{1-\xi})\Gamma(\mu-\xi+1)} \right. \\ &\quad + \frac{\Gamma(2-\xi)}{4\Gamma(3-\xi)(\theta^{1-\xi}+b^{1-\xi})^2\Gamma(\mu-\xi)} \Big| 4\rho(\theta^{2-\xi}+b^{2-\xi})\Gamma(2-\xi) - 4\rho^2\Gamma(3-\xi)(\theta^{1-\xi}+b^{1-\xi}) \\ &\quad \left. -2(\theta+b)(\theta^{2-\xi}+b^{2-\xi})\Gamma(2-\xi) + (\theta^2+b^2)\Gamma(3-\xi)(\theta^{1-\xi}+b^{1-\xi}) \Big| \right\}. \end{aligned}$$

Proof. Define the operator $\mathcal{L} : A \longrightarrow A$ as in (12), then $(\mathcal{L}\omega)\rho$ fulfills a fixed point problem $\omega = \mathcal{L}\omega$.

Define a ball $B_r \subset C[0,b]$ with a suitable radius $r > 0$, where $r > \frac{\delta M}{1-\chi M}$, such that $B_r = \{\omega \in C[0,b] : ||\omega|| < r\}$. We want to show that $\mathcal{L}\omega : \overline{B_r} \longrightarrow C[0,b]$ satisfies

$$\omega \neq \lambda\mathcal{L}\omega, \text{ for all } \omega \in \partial B_r \text{ and all } \lambda \in [0,b]. \quad (15)$$

For $\omega \in C(\mathbb{R})$, $\lambda \in [0,1]$, setting $H(\lambda,\omega) = \lambda\mathcal{L}\omega$. Consequently, the complete continuity of

$$h_\lambda(\omega) = \omega - H(\lambda,\omega) = \omega - \lambda\mathcal{L}\omega$$

can be deduced via the Arzelá-Ascoli theorem. If (15) is satisfied, and since

$$deg(h_\lambda, B_r, 0) = deg(h_1, B_r, 0) = deg(h_0, B_r, 0) = 1 \neq 0 \in B_r,$$

then by Leray-Schauder degree there is at least one $\omega \in B_r$ such that $h_1(\omega) = \omega - \lambda\mathcal{L}\omega = 0$. So for some $\lambda \in [0,1]$ and for all $\rho \in [0,b]$ we conclude that $\omega = \lambda\mathcal{L}\omega$. Now,

$$\begin{aligned}
|\omega(\rho)| &= |\lambda\mathcal{L}\omega(\rho)| \leq \int_0^\rho \frac{(\rho-\nu)^{\mu-1}}{\Gamma(\mu)}|\Omega(\nu,\omega(\nu))|d\nu \\
&\quad + \frac{1}{2}\left(\int_0^\theta \frac{(\theta-\nu)^{\mu-1}}{\Gamma(\mu)}|\Omega(\nu,\omega(\nu))d\nu| + \int_0^b \frac{(b-\nu)^{\mu-1}}{\Gamma(\mu)}|\Omega(\nu,\omega(\nu))|d\nu\right) \\
&\quad + \frac{\Gamma(2-\xi)|(\theta+b)-2\rho|}{2(\theta^{1-\xi}+b^{1-\xi})}\left(\int_0^\theta \frac{(\theta-\nu)^{\mu-\xi-1}}{\Gamma(\mu-\xi)}|\Omega(\nu,\omega(\nu))|d\nu + \int_0^b \frac{(b-\nu)^{\mu-\xi-1}}{\Gamma(\mu-\xi)}|\Omega(\nu,\omega(\nu))|d\nu\right) \\
&\quad \frac{\Gamma(2-\xi)}{4\Gamma(3-\xi)(\theta^{1-\xi}+b^{1-\xi})^2}\Big|4\rho(\theta^{2-\xi}+b^{2-\xi})\Gamma(2-\xi) - 4\rho^2\Gamma(3-\xi)(\theta^{1-\xi}+b^{1-\xi}) \\
&\quad - 2(\theta+b)(\theta^{2-\xi}+b^{2-\xi})\Gamma(2-\xi) + (\theta^2+b^2)\Gamma(3-\xi)(\theta^{1-\xi}+b^{1-\xi})\Big| \\
&\quad \times\left(\int_0^\theta \frac{(\theta-\nu)^{\mu-\xi-2}}{\Gamma(\mu-\xi-1)}|\Omega(\nu,\omega(\nu))|d\nu + \int_0^b \frac{(b-\nu)^{\mu-\xi-2}}{\Gamma(\mu-\xi-1)}|\Omega(\nu,\omega(\nu))|d\nu\right) \\
&\leq \int_0^\rho \frac{(\rho-\nu)^{\mu-1}}{\Gamma(\mu)}(\chi|\omega|+\delta)d\nu + \frac{1}{2}\left(\int_0^\theta \frac{(\theta-\nu)^{\mu-1}}{\Gamma(\mu)}(\chi|\omega|+\delta)d\nu + \int_0^b \frac{(b-\nu)^{\mu-1}}{\Gamma(\mu)}(\chi|\omega|+\delta)d\nu\right) \\
&\quad + \frac{\Gamma(2-\xi)\,|(\theta+b)-2\rho|}{2(\theta^{1-\xi}+b^{1-\xi})}\left(\int_0^\theta \frac{(\theta-\nu)^{\mu-\xi-1}}{\Gamma(\mu-\xi)}(\chi|\omega|+\delta)d\nu + \int_0^b \frac{(b-\nu)^{\mu-\xi-1}}{\Gamma(\mu-\xi)}(\chi|\omega|+\delta)d\nu\right) \\
&\quad + \frac{\Gamma(2-\xi)}{4\Gamma(3-\xi)(\theta^{1-\xi}+b^{1-\xi})^2}\Big(\Big|4\rho(\theta^{2-\xi}+b^{2-\xi})\Gamma(2-\xi) - 4\rho^2\Gamma(3-\xi)(\theta^{1-\xi}+b^{1-\xi}) \\
&\quad -2(\theta+b)(\theta^{2-\xi}+b^{2-\xi})\Gamma(2-\xi) + (\theta^2+b^2)\Gamma(3-\xi)(\theta^{1-\xi}+b^{1-\xi})\Big|\Big] \\
&\quad \times\left(\int_0^\theta \frac{(\theta-\nu)^{\mu-\xi-2}}{\Gamma(\mu-\xi-1)}(\chi|\omega|+\delta)d\nu + \int_0^b \frac{(b-\nu)^{\mu-\xi-2}}{\Gamma(\mu-\xi-1)}(\chi|\omega|+\delta)d\nu\right),
\end{aligned}$$

which implies that

$$|\omega| \leq (\chi||\omega||+\delta)M.$$

Hence,

$$||\omega|| \leq \frac{\delta M}{1-\chi M}.$$

This proves that the relation (15) is satisfied. By applying Theorem 1, we get the desired result.

4. Illustrative examples

This section is devoted to studying some illustrative examples that support the theoretical results.

Example 1. *Consider the following fractional problem*

$$\begin{cases} {}^cD^{\frac{5}{2}}\omega(\rho) = \frac{1}{(\rho-3)^4}\frac{|\omega|}{1+|\omega|}, \ \rho \in [0,2], \\ \omega(\theta) = -\omega(2), \ \omega'(\theta) = -\omega'(2), \ {}^cD^{\frac{3}{2}}\omega(\theta) = -{}^cD^{\frac{3}{2}}\omega(2), \quad \theta = 1 \end{cases} \tag{16}$$

Clearly, the problem (16) is a special case of the problem (1) with $\mu = \frac{5}{2} \in (2,3]$, $\xi = \frac{1}{2} \in (0,1)$, $b = 2 > 0$, $\theta \in [0,2)$, $\rho \in [0,2]$, *and* $\Omega(\rho, \omega(\rho)) = \frac{1}{(\rho-3)^4}\frac{|\omega|}{1+|\omega|}$.

Now, for each $\omega_1, \omega_2 \in \mathbb{R}$, *we have*

$$\begin{aligned} \|\Omega_1(\rho,\omega_1) - \Omega_1(\rho,\omega_2)\| &= \frac{1}{(\rho-3)^4}\left\|\frac{|\omega_1|}{1+|\omega_1|} - \frac{|\omega_2|}{1+|\omega_2|}\right\| \\ &\leq \frac{1}{81}\|\omega_1 - \omega_2\|. \end{aligned}$$

Hence, $L = \frac{1}{81}$. *By simple calculations, according to Eq. (13), if we choose* $\theta = 1 \in [0,2)$, *we get* $M_1 \approx 7.59$. *Thus,* $LM_1 = (\frac{1}{81})(7.59) \approx 0.0937 < 1$. *Therefore, all conditions of Theorem 3 are fulfilled. Then the problem (16) has a solution on* $[0,2]$.

Example 2. *Assume the boundary value problem below*

$$\begin{cases} {}^cD^{\frac{5}{2}}\omega(\rho) = \frac{1}{9}\omega(\rho)\cos(\rho), \ \rho \in [0,1], \\ \omega(\theta) = -\omega(1), \ \omega'(\theta) = -\omega'(1), \ {}^cD^{\frac{3}{2}}\omega(\theta) = -{}^cD^{\frac{3}{2}}\omega(1), \quad \theta = \frac{1}{2} \end{cases} \tag{17}$$

It is clear that, the problem (17) is a special form of the problem (1) with $\mu = \frac{5}{2} \in (2,3]$, $\xi = \frac{1}{2} \in (0,1)$, $b = 1 > 0$, $\theta \in [0,1)$, $\rho \in [0,1]$, *and* $\Omega(\rho, \omega(\rho)) = \frac{1}{9}\omega(\rho)\cos(\rho)$.

Now, for each $\omega_1, \omega_2 \in \mathbb{R}$, *we get*

$$\begin{aligned} \|\Omega_1(\rho,\omega_1) - \Omega_1(\rho,\omega_2)\| &= \frac{1}{9}|\cos(\rho)|\,\|\omega_1(\rho) - \omega_1(\rho)\| \\ &\leq \frac{1}{9}\|\omega_1 - \omega_2\|. \end{aligned}$$

Thus, $L = \frac{1}{9}$. *Further, for all* $(\rho, \omega) \in [0,1] \times \mathbb{R}^+$, *we have*

$$|\Omega(\rho,\omega)| = \frac{1}{9}|\omega(\rho)\cos(\rho)| \leq \frac{1}{9}\omega(\rho),$$

which implies that $\psi(\rho) = \frac{1}{9}\omega(\rho) \in L^1([0,1], \mathbb{R}^+)$. *Utilizing Eq. (14). Since* $\theta = \frac{1}{2} \in [0,1)$, *we get* $M_2 \approx 1.13$. *Thus,* $LM_2 = (\frac{1}{9})(1.13) \approx 0.1256 < 1$. *Therefore, all conditions of Theorem 4 are satisfied. Then the problem (17) has a unique solution on* $[0,1]$.

Example 3. *Assume the following fractional problem:*

$$\begin{cases} {}^cD^{\frac{7}{3}}\omega(\rho) = \frac{1}{15}\left(e^{-\rho} + \frac{\tan^{-1}(\omega(\rho))}{1+\omega^2(\rho)}\right), \ \rho \in [0,2], \\ \omega(\theta) = -\omega(2), \ \omega'(\theta) = -\omega'(2), \ {}^cD^{\frac{5}{4}}\omega(\theta) = -{}^cD^{\frac{5}{4}}\omega(2), \quad \theta = \frac{3}{2} \end{cases} \tag{18}$$

Obviousely, problems (17) and (1) are identical with $\mu = \frac{7}{2} \in (2,3]$, $\xi = \frac{1}{4} \in (0,1)$, $b = 2 > 0$, $\theta \in [0,2)$, $\rho \in [0,2]$, *and* $\Omega(\rho, \omega(\rho)) = \frac{1}{15}\left(e^{-\rho} + \frac{\tan^{-1}(\omega(\rho))}{1+\omega^2(\rho)}\right)$.

Now, for each $\omega_1, \omega_2 \in \mathbb{R}$, *we get*

$$\begin{aligned} \|\Omega_1(\rho,\omega_1) - \Omega_1(\rho,\omega_2)\| &= \frac{1}{15}\left\| e^{-\rho} + \frac{\tan^{-1}(\omega_1)}{1+\omega_1^2} - e^{-\rho} - \frac{\tan^{-1}(\omega_2)}{1+\omega_2^2} \right\| \\ &\leq \frac{1}{15}\|\omega_1 - \omega_2\|. \end{aligned}$$

Therefore, $L = \frac{1}{15}$. *Additionally, for all* $(\rho,\omega) \in [0,2] \times \mathbb{R}^+$, *we have*

$$|\Omega(\rho,\omega)| = \frac{1}{15}\left| e^{-\rho} + \frac{\tan^{-1}(\omega)}{1+\omega^2(\rho)} \right| \leq \frac{1}{15}\left(1 + \frac{\tan^{-1}(\omega)}{1+\omega^2(\rho)}\right),$$

which implies that $\psi(\rho) = \frac{1}{15}\left(1 + \frac{\tan^{-1}(\omega)}{1+\omega^2(\rho)}\right) \in L^1([0,2],\mathbb{R}^+)$. *Utilizing Eq. (14), if we select* $\theta = \frac{3}{2} \in [0,2)$, *we get* $M_2 \approx 4.18$ *and* $LM_2 = (\frac{1}{15})(4.18) \approx 0.2787 < 1$. *Hence, the requirements of Theorem 4 are satisfied. Then the problem (18) has a unique solution on* $[0,2]$. *Moreover, the same result can be obtained by Theorem 5 as follows: for all* $(\rho,\omega) \in [0,2] \times \mathbb{R}$, *we can write*

$$|\Omega(\rho,\omega(\rho))| = \frac{1}{15}\left| e^{-\rho} + \frac{\tan^{-1}(\omega(\rho))}{1+\omega^2(\rho)} \right| \leq e^{\rho} + \frac{1}{3}|\omega(\rho)|,$$

which implies that $\delta = e^{\rho} > 0$ *and* $\chi = \frac{1}{3}$. *By simple caculations, we have* $M = 1.28$. *Clearly,* $0 < \chi = \frac{1}{3} < \frac{1}{1.28} = \frac{1}{M}$.

Example 4. *Assume the following problem*

$$\begin{cases} {}^cD^{\frac{7}{3}}\omega(\rho) = \frac{\cos(2\pi\omega(\rho))}{12\pi} + \frac{|\sin(\omega(\rho))|}{1+|\sin(\omega(\rho))|}, \ \rho \in [0,1], \\ \omega(\theta) = -\omega(1), \ \omega'(\theta) = -\omega'(1), \ {}^cD^{\frac{4}{3}}\omega(\theta) = -{}^cD^{\frac{4}{3}}\omega(1), \quad \theta = 0 \end{cases} \tag{19}$$

The problems (17) and (1) are the same with $\mu = \frac{7}{2} \in (2,3]$, $\xi = \frac{1}{3} \in (0,1)$, $b = 2 > 0$, $\theta = 0 \in [0,2)$, $\rho \in [0,2]$, *and* $\Omega(\rho,\omega(\rho)) = \frac{\cos(2\pi\omega(\rho))}{12\pi} + \frac{|\sin(\omega(\rho))|}{1+|\sin(\omega(\rho))|}$. *According these values, we have* $M = 1.17$ *and*

$$|\Omega(\rho,\omega(\rho))| = \left| \frac{\cos(2\pi\omega(\rho))}{12\pi} + \frac{|\sin(\omega(\rho))|}{1+|\sin(\omega(\rho))|} \right| \leq 1 + \frac{1}{2}|\omega(\rho)|, \ \omega \in \mathbb{R}.$$

Thus, $\delta = 1$, $\chi = \frac{1}{2}$, *and the inequality* $0 < \chi < \frac{1}{M}$ *holds. Therefore, the assertions of Theorem 5 are fulfilled. Then, the problem (18) has at least one solution on* $[0,2]$.

5. Conclusion

The paper introduces a new approach to nonlinear fractional order nonlocal antiperiodic boundary conditions. The study reveals additional terms in the integral solutions that differ from classical antiperiodic boundary conditions. It is observed that under certain conditions, the resulting solution aligns with that of classical antiperiodic boundary conditions, extending the existing results, specifically as ξ approaches 1^- before θ approaches 0^+.However, a distinct solution for this problem type is encountered under different conditions. This offers a fresh perspective on the behavior of fractional differential equations under specific boundary conditions, potentially leading to novel solutions in various scientific fields. These new solutions could significantly impact resolving complex problems where traditional approaches fall short.

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Data Availability

The article contains all the necessary data that was utilized to back up the findings of this work.

Conflict of interest

The author does not have any conflict of interest.

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