EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 1, Article Number 5689
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Singular Value Inequalities for Concave and Convex Functions of Matrix Sums and Products

Ahmad Al-Natoor[1,*], Fadi Alrimawi[2]

[1] *Department of Mathematics, Isra University, Amman, Jordan*
[2] *Department of Basic Sciences, Al-Ahliyya Amman University, Amman, Jordan*

Abstract. In this paper, we present several singular value inequalities for special types of functions of matrix sums and products. Some of special cases of our results give a generalization of some recent inequalities.

2020 Mathematics Subject Classifications: 15A18, 15A42, 47A30, 93C05
Key Words and Phrases: Singular value, spectral norm, positive semidefinite matrix, concave function, convex function, inequality, control theory

1. Introduction

Let $\mathbb{M}_n(\mathbb{C})$ be the $\mathcal{C}^*$-algebra of all $n \times n$ complex matrices. A matrix $X \in \mathbb{M}_n(\mathbb{C})$ is said to be positive semidefinite if $x^*Ax \geq 0$ for all $x \in \mathbb{C}^n$. The singular values of $X \in \mathbb{M}_n(\mathbb{C})$, denoted by $s_1(X) \geq s_2(X) \geq ... s_n(X) \geq 0$ are the eigenvalues of $|X|$. In this paper, when we write s_j we mean $s_1(X), s_2(X), ..., s_n(X)$, i.e., $j = 1, 2, ..., n$.

The spectral norm of $X \in \mathbb{M}_n(\mathbb{C})$ is defined by $\|X\| = \max_{\|x\|=1} \|Ax\|$.

For $X, Y \in \mathbb{M}_n(\mathbb{C})$, let $X \oplus Y$ be the direct sum of X and Y, that is, the matrix given by $X \oplus Y = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}$. It is known that $\|X \oplus Y\| = \max(\|X\|, \|Y\|)$.

It is known [14] that if $X, Y \in \mathbb{M}_n(\mathbb{C})$, then

$$s_j(X + Y) \leq 2s_j(X \oplus Y), \tag{1}$$

A generalization of inequality (1) has been given in [8] by

$$s_j(XZ + ZY) \leq 2\|Z\| s_j(X \oplus Y). \tag{2}$$

The authors in [7] have proved several singular value inequalities. One of these inequalities asserts that if $X, Y \in \mathbb{M}_n(\mathbb{C})$, then

$$s_j(XY - YX) \leq \|Y\| s_i(X \oplus X) + \frac{1}{2} s_{j-i+1}((XY - YX) \oplus (XY - YX))$$

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i1.5689

Email addresses: ahmad.alnatoor@iu.edu.jo (A. Al-Natoor), f.rimawi@ammanu.edu.jo (F. Alrimawi)

for $1 \leq i \leq j \leq n$. In particular, if $j = i$, then

$$s_j (XY - YX) \leq \|Y\| s_j (X \oplus X) + \frac{1}{2} \|XY - YX\| . \tag{3}$$

Singular values of a matrix play a critical role in various applications, including data compression, noise reduction, and the resolution of ill-posed problems (see, e.g., [1], [2], and [12]), particularly in fields such as electrical and mechanical engineering, where these concepts are used to optimize signal processing and system performance. The spectral norm, defined as the largest singular value, is fundamental for assessing the stability of the matrix and quantifying its maximum impact as a linear transformation, which is crucial because it ensures that small perturbations in the input data do not lead to disproportionately large errors in the output, making computations reliable and consistent in practical applications.

In this paper, we give several singular value inequalities. Among other results, we give a related inequality to inequality (2) and we give a generalization of inequality (3). For recent articles related to matrix and singular value inequalities, we refer the reader to [4], [3], [5], [10] and [11].

2. Main results

To start our analysis, we need the following lemmas. The first lemma is a consequence of the spectral theorem for matrices (see, e.g., [13, p. 5]), the second lemma was given in [7], while the third lemma can be found in [13, p. 75].

Lemma 1. *Let $X \in \mathbb{M}_n(\mathbb{C})$ and let f be nonnegative increasing function on $[0, \infty)$. Then*

$$f(s_j(X)) = s_j(f(|X|)),$$

where $|X|$ is the absolute value of the matrix X.

Lemma 2. *Let $X, Y \in \mathbb{M}_n(\mathbb{C})$. Then*

$$s_j (X + Y) \leq s_j (X \oplus Y) + \frac{1}{2} \|X + Y\| .$$

Lemma 3. *Let $X, Y, Z \in \mathbb{M}_n(\mathbb{C})$. Then*

$$s_j(XZY) \leq \|X\| \|Y\| s_j(Z).$$

Theorem 1. *Let $A, B, X, Y \in \mathbb{M}_n(\mathbb{C})$. Then*

(a)

$$\begin{aligned} s_j (f (|XAY + YBX|)) \quad &\leq \quad f (\|X\|) f (\|Y\|) s_j(f (|A|) \oplus f (|B|)) \\ &\quad + f \left(\frac{1}{2}\right) f (\|XAY + YBX\|) , \end{aligned} \tag{4}$$

where f is a nonnegative increasing submultiplicative concave function on $[0, \infty)$ with $f(0) = 0$.

(b)

$$\begin{aligned} s_j\left(f\left(\left|XAY+YBX\right|\right)\right) &\leq \frac{f(2)}{2} f\left(\|X\|\right) f\left(\|Y\|\right) s_j(f\left(|A|\right)\oplus f\left(|B|\right)) \\ &+\frac{1}{2} f\left(\|XAY+YBX\|\right), \end{aligned} \tag{5}$$

where f is a nonnegative increasing submultiplicative convex function on $[0,\infty)$.

Proof. We have

$$\begin{aligned} & s_j\left(f\left(\left|XAY+YBX\right|\right)\right) \\ &= f\left(s_j\left(XAY+YBX\right)\right) \text{ (by Lemma 1)} \\ &\leq f\left(s_j\left(XAY\oplus YBX\right)+\frac{1}{2}\left\|XAY+YBX\right\|\right) \qquad (6) \\ &\qquad \text{(by Lemma 2)} \\ &\leq f\left(s_j\left(XAY\oplus (YBX)^*\right)\right)+f\left(\frac{1}{2}\left\|XAY+YBX\right\|\right) \\ &\qquad \text{(since } f \text{ is concave and } f(0)=0) \\ &\leq f\left(s_j\left(XAY\oplus (YBX)^*\right)\right)+f\left(\frac{1}{2}\right) f\left(\|XAY+YBX\|\right) \\ &\qquad \text{(since } f \text{ is submultiplicative)} \\ &= f\left(s_j\left(\begin{bmatrix} X & 0 \\ 0 & X^* \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & B^* \end{bmatrix}\begin{bmatrix} Y & 0 \\ 0 & Y^* \end{bmatrix}\right)\right) \\ &\quad +f\left(\frac{1}{2}\right) f\left(\|XAY+YBX\|\right) \qquad (7) \\ &\leq f\left(\left\|\begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix}\right\|\left\|\begin{bmatrix} Y & 0 \\ 0 & Y \end{bmatrix}\right\| s_j\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right)\right) \\ &\quad +f\left(\frac{1}{2}\right) f\left(\|XAY+YBX\|\right) \text{ (by Lemma 3)} \\ &= f\left(\|X\|\,\|Y\|\, s_j\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right)\right)+f\left(\frac{1}{2}\right) f\left(\|XAY+YBX\|\right) \\ &\leq f(\|X\|)f(\|Y\|)f(s_j(A\oplus B))+f\left(\frac{1}{2}\right) f\left(\|XAY+YBX\|\right) \\ &= f(\|X\|)f(\|Y\|)s_j(f\left(|A|\right)\oplus f\left(|B|\right))+f\left(\frac{1}{2}\right) f\left(\|XAY+YBX\|\right), \end{aligned}$$

which proves part (a). For part (b), we start from inequality (6), so we have

$$s_j\left(f\left(\left|XAY+YBX\right|\right)\right)$$

$$\begin{aligned}
&\le f\left(s_j\left(XAY\oplus YBX\right)+\frac{1}{2}\left\|XAY+YBX\right\|\right)\\
&\le \frac{1}{2}f\left(2s_j\left(XAY\oplus\left(YBX\right)^*\right)\right)+\frac{1}{2}f\left(\left\|XAY+YBX\right\|\right)\quad\text{(since f is convex)}\\
&= \frac{1}{2}f\left(2s_j\left(XAY\oplus\left(YBX\right)^*\right)\right)+\frac{1}{2}f\left(\left\|XAY+YBX\right\|\right)\\
&= \frac{1}{2}f\left(2s_j\left(\begin{bmatrix}X&0\\0&X^*\end{bmatrix}\begin{bmatrix}A&0\\0&B^*\end{bmatrix}\begin{bmatrix}Y&0\\0&Y^*\end{bmatrix}\right)\right)\\
&\quad+\frac{1}{2}f\left(\left\|XAY+YBX\right\|\right)\\
&\le \frac{1}{2}f\left(2\left\|\begin{bmatrix}X&0\\0&X\end{bmatrix}\right\|\left\|\begin{bmatrix}Y&0\\0&Y\end{bmatrix}\right\|s_j\left(\begin{bmatrix}A&0\\0&B\end{bmatrix}\right)\right)\\
&\quad+\frac{1}{2}f\left(\left\|XAY+YBX\right\|\right)\ \text{(by Lemma 3)}\\
&= \frac{1}{2}f\left(2\left\|X\right\|\left\|Y\right\|s_j\left(\begin{bmatrix}A&0\\0&B\end{bmatrix}\right)\right)\\
&\quad+\frac{1}{2}f\left(\left\|XAY+YBX\right\|\right)\\
&\le \frac{f(2)}{2}f(\|X\|)f(\|Y\|)f(s_j(A\oplus B))+\frac{1}{2}f\left(\left\|XAY+YBX\right\|\right)\\
&= \frac{f(2)}{2}f(\|X\|)f(\|Y\|)s_j(f\left(|A|\right)\oplus f\left(|B|\right))+\frac{1}{2}f\left(\left\|XAY+YBX\right\|\right),
\end{aligned}$$

which completes the proof.

Taking $f(t)=t, t\in[0,\infty)$ in inequalities (4) and (5), we have

$$s_j\left(XAY+YBX\right)\le\left\|X\right\|\left\|Y\right\|s_j\left(A\oplus B\right)+\frac{1}{2}\left\|XAY+YBX\right\|. \tag{8}$$

Letting $X=I$ in inequality (8), we obtain

$$s_j\left(AY+YB\right)\le\left\|Y\right\|s_j\left(A\oplus B\right)+\frac{1}{2}\left\|AY+YB\right\|. \tag{9}$$

Replacing A by X, Y by Z, and B by Y in inequality (9), we have

$$s_j\left(XZ+ZY\right)\le\left\|Z\right\|s_j\left(X\oplus Y\right)+\frac{1}{2}\left\|XZ+ZY\right\|. \tag{10}$$

Combining inequalities (2) and (10), we have

$$s_j\left(XZ+ZY\right)\le\min\left\{2\left\|Z\right\|s_j(X\oplus Y),\left\|Z\right\|s_j\left(X\oplus Y\right)+\frac{1}{2}\left\|XZ+ZY\right\|\right\},$$

which is a refinement of inequality (2).

Inequality (8) generalizes inequality (3). In fact, replacing B by $-B$ in inequality (8), we have

$$s_j(XAY - YBX) \leq \|X\| \|Y\| s_j(A \oplus B) + \frac{1}{2} \|XAY - YBX\|, \tag{11}$$

which is a generalization of inequality (3). To see this, let $X = I$ and then replace A and B by X in inequality (11), we have

$$s_j(XY - YX) \leq \|Y\| s_j(X \oplus X) + \frac{1}{2} \|XY - YX\|,$$

which is inequality (3).

We need the following lemma [6] to give our second result.

Lemma 4. *Let $X, Y, Z \in \mathbb{M}_n(\mathbb{C})$ be such that Z is positive semidefinite. Then*

$$s_j(XZY^*) \leq \frac{1}{2} \left\| \frac{X^*X}{\|X\|^2} + \frac{Y^*Y}{\|Y\|^2} \right\| \|X\| \|Y\| s_j(Z).$$

Theorem 2. *Let $A, B, X, Y \in \mathbb{M}_n(\mathbb{C})$ be such that A and B are positive semidefinite. Then*

(a)

$$\begin{aligned} & s_j(f(|XAY + YBX|)) \\ \leq\ & f\left(\frac{1}{2}\right) \left\| f\left(\frac{|X|^2 \oplus |X^*|^2}{\|X\|^2} + \frac{|Y|^2 \oplus |Y^*|^2}{\|Y\|^2} \right) \right\| f(\|X\|) f(\|Y\|) s_j(f(A) \oplus f(B)) \\ & + f\left(\frac{1}{2}\right) f(\|XAY + YBX\|), \end{aligned} \tag{12}$$

where f is a nonnegative increasing submultiplicative concave function on $[0, \infty)$ with $f(0) = 0$.

(b)

$$\begin{aligned} & s_j(f(|XAY + YBX|)) \\ \leq\ & \frac{1}{2} \left\| f\left(\frac{|X|^2 \oplus |X^*|^2}{\|X\|^2} + \frac{|Y|^2 \oplus |Y^*|^2}{\|Y\|^2} \right) \right\| f(\|X\|) f(\|Y\|) s_j(f(A) \oplus f(B)) \\ & + \frac{1}{2} f(\|XAY + YBX\|), \end{aligned} \tag{13}$$

where f is a nonnegative increasing submultiplicative convex function on $[0, \infty)$.

Proof. By inequality (7), we have

$$
\begin{aligned}
& s_j\left(f\left(|XAY+YBX|\right)\right) \\
\leq\ & f\left(s_j\left(\begin{bmatrix} X & 0 \\ 0 & X^* \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & B^* \end{bmatrix}\begin{bmatrix} Y & 0 \\ 0 & Y^* \end{bmatrix}\right)\right) \\
& +f\left(\frac{1}{2}\right) f\left(\|XAY+YBX\|\right) \\
\leq\ & f\left(\frac{1}{2}\left\| \frac{\begin{bmatrix} X^* & 0 \\ 0 & X \end{bmatrix}\begin{bmatrix} X & 0 \\ 0 & X^* \end{bmatrix}}{\|X\|^2} + \frac{\begin{bmatrix} Y^* & 0 \\ 0 & Y \end{bmatrix}\begin{bmatrix} Y & 0 \\ 0 & Y^* \end{bmatrix}}{\|Y\|^2} \right\| \times \|X\|\,\|Y\|\, s_j\left(A\oplus B\right)\right) \\
& +f\left(\frac{1}{2}\right) f\left(\|XAY+YBX\|\right) \\
& \text{(by Lemma 4)} \\
=\ & f\left(\frac{1}{2}\left\|\frac{|X|^2\oplus|X^*|^2}{\|X\|^2}+\frac{|Y|^2\oplus|Y^*|^2}{\|Y\|^2}\right\| \|X\|\,\|Y\|\, s_j\left(A\oplus B\right)\right) \\
& +f\left(\frac{1}{2}\right) f\left(\|XAY+YBX\|\right) \\
\leq\ & f\left(\frac{1}{2}\right) f\left(\left\|\frac{|X|^2\oplus|X^*|^2}{\|X\|^2}+\frac{|Y|^2\oplus|Y^*|^2}{\|Y\|^2}\right\|\right) f\left(\|X\|\right) f\left(\|Y\|\right) f\left(s_j\left(A\oplus B\right)\right) \\
& +f\left(\frac{1}{2}\right) f\left(\|XAY+YBX\|\right) \\
& \text{(since } f \text{ is submultiplicative)} \\
=\ & f\left(\frac{1}{2}\right)\left\| f\left(\frac{|X|^2\oplus|X^*|^2}{\|X\|^2}+\frac{|Y|^2\oplus|Y^*|^2}{\|Y\|^2}\right)\right\| f\left(\|X\|\right) f\left(\|Y\|\right) s_j\left(f\left(A\right)\oplus f\left(B\right)\right) \\
& +f\left(\frac{1}{2}\right) f\left(\|XAY+YBX\|\right),
\end{aligned}
$$

which proves part (a). For part (b), we start from inequality (6), so we have

$$
\begin{aligned}
& s_j\left(f\left(|XAY+YBX|\right)\right) \\
\leq\ & f\left(s_j\left(XAY\oplus YBX\right)+\frac{1}{2}\|XAY+YBX\|\right) \\
=\ & f\left(s_j\left(XAY\oplus (YBX)^*\right)+\frac{1}{2}\|XAY+YBX\|\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{2} f\left(2 s_j\left(\begin{bmatrix} X & 0 \\ 0 & X^* \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\begin{bmatrix} Y & 0 \\ 0 & Y^* \end{bmatrix}\right)\right) \\
&\quad +\frac{1}{2} f\left(\|XAY+YBX\|\right) \\
&\leq \frac{1}{2} f\left(\begin{array}{c}\left\|\frac{\begin{bmatrix} X^* & 0 \\ 0 & X \end{bmatrix}\begin{bmatrix} X & 0 \\ 0 & X^* \end{bmatrix}}{\|X\|^2}+\frac{\begin{bmatrix} Y^* & 0 \\ 0 & Y \end{bmatrix}\begin{bmatrix} Y & 0 \\ 0 & Y^* \end{bmatrix}}{\|Y\|^2}\right\| \\ \times\|X\|\|Y\| s_j(A \oplus B)\end{array}\right) \\
&\quad +\frac{1}{2} f\left(\|XAY+YBX\|\right) \\
&\qquad \text{(by Lemma 4)} \\
&= \frac{1}{2} f\left(\left\|\frac{|X|^2 \oplus |X^*|^2}{\|X\|^2}+\frac{|Y|^2 \oplus |Y^*|^2}{\|Y\|^2}\right\|\|X\|\|Y\| s_j(A \oplus B)\right) \\
&\quad +\frac{1}{2} f\left(\|XAY+YBX\|\right) \\
&\leq \frac{1}{2} f\left(\left\|\frac{|X|^2 \oplus |X^*|^2}{\|X\|^2}+\frac{|Y|^2 \oplus |Y^*|^2}{\|Y\|^2}\right\|\right) f(\|X\|) f(\|Y\|) f\left(s_j(A \oplus B)\right) \\
&\quad +\frac{1}{2} f\left(\|XAY+YBX\|\right) \\
&\qquad \text{(since } f \text{ is submultiplicative)} \\
&= \frac{1}{2}\left\| f\left(\frac{|X|^2 \oplus |X^*|^2}{\|X\|^2}+\frac{|Y|^2 \oplus |Y^*|^2}{\|Y\|^2}\right)\right\| f(\|X\|) f(\|Y\|) s_j\left(f(A) \oplus f(B)\right) \\
&\quad +\frac{1}{2} f\left(\|XAY+YBX\|\right),
\end{aligned}
$$

this completes the proof.

Taking $f(t)=t, t \in [0,\infty)$ in inequalities (12) and (13), we have

$$
\begin{aligned}
s_j(XAY+YBX) \leq\ & \frac{1}{2}\left\|\frac{|X|^2 \oplus |X^*|^2}{\|X\|^2}+\frac{|Y|^2 \oplus |Y^*|^2}{\|Y\|^2}\right\|\|X\|\|Y\| s_j(A \oplus B) \\
& +\frac{1}{2}\|XAY+YBX\|.
\end{aligned}
$$

To state our next result, we need the following lemma [9].

Lemma 5. *Let $A, B, X \in \mathbb{M}_n(\mathbb{C})$ be such that X is positive semidefinite. Then*

$$
s_j(AXB^*) \leq \frac{1}{2}\|X\| s_j(A^*A+B^*B).
$$

Theorem 3. *Let $A, B, X, Y \in \mathbb{M}_n(\mathbb{C})$ be such that A and B are positive semidefinite. Then*

(a)

$$s_j\left(f\left(|XAY+YBX|\right)\right)$$

$$\begin{aligned} \leq\ & f\left(\frac{1}{2}\right)\max\left(\|f(A)\|,\|f(B)\|\right)s_j\left(\left(f\left(X^*X+YY^*\right)\right)\oplus\left(f\left(XX^*+Y^*Y\right)\right)\right) \\ & +f\left(\frac{1}{2}\right)f\left(\|XAY+YBX\|\right), \end{aligned} \tag{14}$$

where f is a nonnegative increasing submultiplicative concave function on $[0,\infty)$ with $f(0)=0$.

(b)

$$s_j\left(f\left(|XAY+YBX|\right)\right)$$

$$\begin{aligned} \leq\ & \frac{1}{2}\max\left(\|f(A)\|,\|f(B)\|\right)s_j\left(\left(f\left(X^*X+YY^*\right)\right)\oplus\left(f\left(XX^*+Y^*Y\right)\right)\right) \\ & +\frac{1}{2}f\left(\|XAY+YBX\|\right), \end{aligned} \tag{15}$$

where f is a nonnegative increasing submultiplicative convex function on $[0,\infty)$.

Proof. By inequality (7), we have

$$s_j\left(f\left(|XAY+YBX|\right)\right)$$

$$\begin{aligned} \leq\ & f\left(s_j\left(\begin{bmatrix} X & 0 \\ 0 & X^* \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\begin{bmatrix} Y & 0 \\ 0 & Y^* \end{bmatrix}\right)\right) \\ & +f\left(\frac{1}{2}\right)f\left(\|XAY+YBX\|\right) \\ \leq\ & f\left(\frac{1}{2}\left\|\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right\|s_j\left(\begin{array}{c} \begin{bmatrix} X^* & 0 \\ 0 & X \end{bmatrix}\begin{bmatrix} X & 0 \\ 0 & X^* \end{bmatrix} \\ +\begin{bmatrix} Y & 0 \\ 0 & Y^* \end{bmatrix}\begin{bmatrix} Y^* & 0 \\ 0 & Y \end{bmatrix} \end{array}\right)\right) \\ & +f\left(\frac{1}{2}\right)f\left(\|XAY+YBX\|\right) \qquad \text{(by Lemma 5)} \\ \leq\ & f\left(\frac{1}{2}\right)f\left(\left\|\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right\|\right)f\left(s_j\left(\begin{bmatrix} X^*X+YY^* & 0 \\ 0 & XX^*+Y^*Y \end{bmatrix}\right)\right) \end{aligned}$$

$$
\begin{aligned}
& +f\left(\frac{1}{2}\right) f\left(\|XAY+YBX\|\right) \\
= \quad & f\left(\frac{1}{2}\right)\left\|\begin{bmatrix} f(A) & 0 \\ 0 & f(B) \end{bmatrix}\right\| s_j\left(f\left(\begin{bmatrix} X^*X+YY^* & 0 \\ 0 & XX^*+Y^*Y \end{bmatrix}\right)\right) \\
& +f\left(\frac{1}{2}\right) f\left(\|XAY+YBX\|\right) \\
= \quad & f\left(\frac{1}{2}\right)\max\left(\|f(A)\|,\|f(B)\|\right) s_j\left(\left(f\left(X^*X+YY^*\right)\right)\oplus\left(f\left(XX^*+Y^*Y\right)\right)\right) \\
& +f\left(\frac{1}{2}\right) f\left(\|XAY+YBX\|\right),
\end{aligned}
$$

which proves part (a). For part (b), we start from inequality (6), so we have

$s_j\left(f\left(|XAY+YBX|\right)\right)$

$$
\begin{aligned}
\leq \quad & f\left(s_j\left(XAY\oplus YBX\right)+\frac{1}{2}\|XAY+YBX\|\right) \\
= \quad & f\left(s_j\left(XAY\oplus (YBX)^*\right)+\frac{1}{2}\|XAY+YBX\|\right) \\
\leq \quad & \frac{1}{2}f\left(2s_j\left(\begin{bmatrix} X & 0 \\ 0 & X^* \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\begin{bmatrix} Y & 0 \\ 0 & Y^* \end{bmatrix}\right)\right) \\
& +\frac{1}{2}f\left(\|XAY+YBX\|\right) \\
\leq \quad & \frac{1}{2}f\left(\left\|\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right\| s_j\left(\begin{matrix} \begin{bmatrix} X^* & 0 \\ 0 & X \end{bmatrix}\begin{bmatrix} X & 0 \\ 0 & X^* \end{bmatrix} \\ +\begin{bmatrix} Y & 0 \\ 0 & Y^* \end{bmatrix}\begin{bmatrix} Y^* & 0 \\ 0 & Y \end{bmatrix} \end{matrix}\right)\right) \\
& +\frac{1}{2}f\left(\|XAY+YBX\|\right) \quad \text{(by Lemma 5)} \\
\leq \quad & \frac{1}{2}f\left(\left\|\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right\|\right) f\left(s_j\left(\begin{bmatrix} X^*X+YY^* & 0 \\ 0 & XX^*+Y^*Y \end{bmatrix}\right)\right) \\
& +\frac{1}{2}f\left(\|XAY+YBX\|\right) \\
= \quad & \frac{1}{2}\left\|\begin{bmatrix} f(A) & 0 \\ 0 & f(B) \end{bmatrix}\right\| s_j\left(\begin{bmatrix} f\left(X^*X+YY^*\right) & 0 \\ 0 & f\left(XX^*+Y^*Y\right) \end{bmatrix}\right) \\
& +\frac{1}{2}f\left(\|XAY+YBX\|\right) \\
= \quad & \frac{1}{2}\max\left(\|f(A)\|,\|f(B)\|\right) s_j\left(\left(f\left(X^*X+YY^*\right)\right)\oplus\left(f\left(XX^*+Y^*Y\right)\right)\right) \\
& +\frac{1}{2}f\left(\|XAY+YBX\|\right),
\end{aligned}
$$

this completes the proof.

Taking $f(t) = t, t \in [0, \infty)$ in inequalities (14) and (15), we have

$$\begin{aligned} s_j (XAY + YBX) &\leq \frac{1}{2}\max(\|A\|, \|B\|)\, s_j ((X^*X + YY^*) \oplus (XX^* + Y^*Y)) \\ &\quad + \frac{1}{2}\|XAY + YBX\|. \end{aligned} \tag{16}$$

Corollary 1. *Let $A, B, X, Y \in \mathbb{M}_n(\mathbb{C})$ be such that A and B are positive semidefinite. Then*

$$s_j (AY + YB) \leq \max(\|A\|, \|B\|)\, s_j (Y \oplus Y) + \frac{1}{2}\|AY + YB\|.$$

Proof. Letting $X = I$ in inequality (16), we have

$$\begin{aligned} s_j (AY + YB) &= \frac{1}{2}\max(\|A\|, \|B\|)\, s_j \left(\begin{bmatrix} I + YY^* & 0 \\ 0 & I + Y^*Y \end{bmatrix} \right) \\ &\quad + \frac{1}{2}\|AY + YB\| \\ &= \frac{1}{2}\max(\|A\|, \|B\|)\, s_j \left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} YY^* & 0 \\ 0 & Y^*Y \end{bmatrix} \right) \\ &\quad + \frac{1}{2}\|AY + YB\| \\ &= \frac{1}{2}\max(\|A\|, \|B\|) \left(1 + s_j \left(\begin{bmatrix} YY^* & 0 \\ 0 & Y^*Y \end{bmatrix} \right) \right) \\ &\quad + \frac{1}{2}\|AY + YB\| \\ &= \frac{1}{2}\max(\|A\|, \|B\|) (1 + s_j (YY^* \oplus Y^*Y)) + \frac{1}{2}\|AY + YB\| \\ &= \frac{1}{2}\max(\|A\|, \|B\|) \left(1 + s_j^2 (Y \oplus Y)\right) + \frac{1}{2}\|AY + YB\|. \end{aligned}$$

Replacing Y by tY and taking the min over $t > 0$, we have

$$s_j (AY + YB) \leq \max(\|A\|, \|B\|)\, s_j (Y \oplus Y) + \frac{1}{2}\|AY + YB\|,$$

as required.

Conclusion. In this paper, we present several inequalities related to singular values for funcions of matrices and we give a general version of interesting recent results.

Acknowledgements

The authors are grateful to the reviewers for their careful reading and valuable suggestions.

References

[1] O. AbuAlghanam, O. Adwan, M. A. Al Shariah, and M. Qatawneh. Enhancing the speed of the learning vector quantization (lvq) algorithm by adding partial distance computation. *Cybernetics and Information Technologies*, 22:36–49, 2022.

[2] M. Abualhaj, A. A. Abu-Shareha, M. O. Hiari, Y. Alrabanah, M. Al-Zyoud, and M. A. Alsharaiah. A paradigm for dos attack disclosure using machine learning techniques. *International Journal of Advanced Computer Science and Applications*, 13:192–200, 2024.

[3] A. Al-Natoor. Norm inequalities for functions of matrices. *Heliyon*, 10, Issue 9:e30056.

[4] A. Al-Natoor. Norm inequalities for product of matrices. *Acta Sci. Math. (Szeged)*, https://doi.org/10.1007/s44146-024-00121-1, 2024.

[5] A. Al-Natoor and F. Alrimawi. Numerical radius inequalities involving 2×2 block matrices. *Eur. J. Pure Appl. Math.*, (Accepted).

[6] A. Al-Natoor, O. Hirzallah, and F. Kittaneh. Singular value inequalities for convex functions of positive semidefinite matrices. *Ann. Funct. Anal.*, 14:Papet No. 7:14 pp., 2023.

[7] A. Al-Natoor, O. Hirzallah, and F. Kittaneh. Singular value and unitarily invariant norm inequalities for matrices. *Adv. Oper. Theory*, 21:https://doi.org/10.1007/s43036–024–00319–8, 2024.

[8] A. Al-Natoor and F. Kittaneh. Singular value and norm inequalities for positive semidefinite matrices. *Linear Multilinear Algebra*, 70:4498–4507, 2022.

[9] H. Albadawi. Singular values and arithmetic-geometric mean inequalities for operators. *Ann. Funct. Anal.*, 3:10–18, 2024.

[10] F. Alrimawi, O. Hirzallah, and F. Kittaneh. Singular value inequalities involving convex and concave functions of positive semidefinite matrices. *Ann. Funct. Anal.*, 11:1257–1273, 2020.

[11] F. Alrimawi, H. Kawariq, and F.A. Abushaheen. Some matrix and norm inequalities for positive definite matrices. *Int. J. Math. Comput. Sci.*, 15:845–855, 2020.

[12] M.A. Alsharaiah, L.H. Baniata, O. Adwan, O. AbuAlghanam, A. A. Abu-Shareha, L. Alzboon, N. Mustafa, and M. Baniata. Neural network prediction model to explore complex nonlinear behavior in dynamic biological network. *Int. J. Interact. Mob. Technol.*, 16:32–51, 2022.

[13] R. Bhatia. *Matrix Analysis.* Springer-Verlag, New York, 1997.

[14] O. Hirzalleh and F. Kittaneh. Inequalities for sums and direct sums of hilbert space operator. *Linear Algebra and its Applications*, 424:71–82, 2007.