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Development of the Nyström Method for Weakly Singular Functional Integral Equations

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Abstract. In this research, we apply the standard product integration method (Nyström method) for solving the delay nonlinear weakly singular Volterra integral equations. Typically, in weakly singular integral equations, the singularity of the kernel leads to the derivatives of the solution becoming singular at the boundary of the domain. The Chelyshkov polynomials serving as orthogonal polynomials, find application in numerical integration. Here, we use roots of these polynomials to make Lagrange interpolating polynomial for approximating the kernel functions in weakly singular functional integral equation. The proposed method's convergence analysis is developed, and numerical examples demonstrate the method's reliability and efficiency.

2020 Mathematics Subject Classifications: 65R20, 65L20, 65L80, 34K28

Key Words and Phrases: Functional equations, Nyström method, Non-vanishing delays, Vanishing delays, Convergence

1. Introduction

The focal issue involves delay nonlinear weakly singular Volterra integral equations

$$\begin{cases} y(t) = f(t) + (V_\alpha y)(t) + (V_{\alpha,\theta} y)(t), & t \in (t_0, T], \\ y(t) = \phi(t), & t \in [\theta(t_0), t_0]. \end{cases} \tag{1}$$

where

$$(V_\alpha y)(t) = \int_{t_0}^{t} \mathcal{P}_{1,h}(t,w) k_1(t,w,y(w)) dw, \tag{2}$$

with $k_1 \in C(\mathcal{D} \times \mathbb{R})$, $\mathcal{D} = \{(t,w) : \ t_0 \le w \le t \le T\}$, $k_2 \in C(\mathcal{D}_\theta \times \mathbb{R})$, $\mathcal{D}_\theta = \{(t,w) : \ \theta(t_0) \le w \le \theta(t)\}$ and

$$(V_{\alpha,\theta} y)(t) = \int_{t_0}^{\theta(t)} \mathcal{P}_{2,h}(t,w) k_2(t,w,y(w)) dw, \tag{3}$$

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ϕ, f as given functions are at least continuous on their respective domains. Normal forms of weakly singular kernels $\mathcal{P}_{.,h}(t,w)$, are

$$\mathcal{P}_{.,h}(t,w) = \begin{cases} \frac{1}{|t-w|^{\mu}}, & 0<\mu<1, \quad h=1, \\ \\ \log|t-w|, & h=2. \end{cases} \tag{4}$$

Here, the delay function θ will be constrained by the following conditions:

(O1) $\theta(t) = t - \eta(t),\ \tau \in C^{\nu}(I), I = [t_0, T],$ for some $\nu \geq 0$,

(O2) $$\begin{cases} \text{For vanishing delay:} \quad \eta(t_0) = 0, \quad \text{and} \quad \eta(t) > t_0 > 0, \text{ for } t_0 < t \leq T, \\ \\ \text{For non-vanishing delay:} \quad \eta(t) \geq \eta_0 > 0, \qquad \text{for all } t \in I, \end{cases}$$

(O3) θ is strictly increasing on I.

Due to the assumption (O2) that the delay $\eta(t)$ does not become zero on I, having smooth data in (1) typically does not result in a solution y that is globally smooth. It is widely recognized that these equations often exhibit discontinuity in the solution or its derivatives at the initial point of the integration domain. This discontinuity propagates along the integration interval, giving rise to subsequent points referred to as singular points. Determining these singular points in advance is challenging, and the solution derivatives at these points tend to smooth out along the interval. Many existing numerical methods for such equations are highly sensitive to these singular points. Therefore, it is crucial for these methods to incorporate a process that detects and includes these points in the mesh to ensure the desired accuracy (For further details see [1].) By virtue of Volterra integral equations and also their functional counterparts along with delay terms whether vanishing or non-vanishing, a wide spectrum of science subjects, namely in biology, ecology, physics, and chemistry have been mathematically well-formulated in order to analyse and study underlying phenomena. Particularly, these classes of mathematical modellings can be found in fluid dynamics, viscoelasticity of materials, population growth dynamics, heat conduction, epidemiology, controlled liquidation in obsolete production units, and renovation in economic systems, see [2–6] and references therein. Some numerical methods have been proposed to obtain approximate solutions of weakly singular Volterra integral equation, such as spectral method [7–9], collocation method [10, 11], radial basis function method [12], Bernstein and Genocchi polynomial method [13, 14] and successive approximation method [15].

Delay IEs have been solved approximately by many authors, such as, spectral methods [16], two-point multistep block method with constant step-size [17], collocation and continuous implicit Runge-Kutta methods [18], spline collocation method [19, 20], piecewise collocation method [21], and linear multistep methods [22]. Also, one- step polynomial collocation method [1] has been applied to find numerical solution of delay IEs and delay weakly singular IEs. In [23], a category of Volterra delay integral equations (VDIEs) involving noncompact operators is estimated using collocation methods. The study explores the characteristics of the associated operators and delves into the discussions regarding the existence, uniqueness, and regularity of the exact solution. The paper establishes the existence and uniqueness of collocation solutions, specifically under two distinct graded meshes. Additionally, the convergence conditions and order of convergence are presented. To validate the theoretical orders of convergence, numerical examples are provided.

The paper is organized as follows: Section 2 is dedicated to proposing some preliminaries about the product rules using the roots of certain types of generalized Jacobi orthogonal polynomials and specific properties of orthogonal polynomials. In Section 3, we construct and analyze the product integration method to solve the equation (1) numerically and in section 4 convergence of numerical solutions is investigated. This is succeeded by the discussion of three test problems in Section 5 to validate the theoretical results. Finally, in Section 6, we conclude the paper and suggest potential future avenues for research, which are currently less explored.

2. Preliminaries

Consider the product rules

$$\int_0^1 \mathcal{P}_{\cdot,h}(t,w)G(w)dw \approx \sum_{i=1}^{m} w_{m,i}(t)G(t_{m,i}) = \mathcal{I}_N(G,t), \tag{5}$$

derived from the roots of particular classes of generalized Jacobi orthogonal polynomials. We can consider the error term of (5) as

$$\mathcal{R}_N(G;t) = \int_0^1 \mathcal{P}_{\cdot,h}(t,w)G(w)dw - \mathcal{I}_N(G,t). \tag{6}$$

From[24, 25], we have $\mathcal{R}_N(G,t) = O(N^{-m})$ when $G \in C^m[0,1]$. If G is a polynomial of degree N, then $\mathcal{R}_N(G,t) = 0$, so from [24], for all polynomial P_N of degree N, we have

$$\mathcal{R}_N(G,t) = \int_0^1 \mathcal{P}_{\cdot,h}(t,w)\big(G(w) - P_N(w)\big)dw - \int_0^1 \mathcal{P}_{\cdot,h}(t,w)\Omega_N(G - P_N, w)dw,$$

where $\Omega_N(G,t)$ represents the Lagrange interpolation polynomial that interpolates G at the points $\{t_i\}_{i=0}^N$, the expression is defined as follows:

$$\Omega_N(G,t) = \sum_{i=0}^{N} G(t_i)l_{N,j}(t),$$

where

$$l_{N,j}(t) = \prod_{\substack{i=0\\ i\neq j}}^{N} \frac{t-t_i}{t_j-t_i}, \quad j = 0,1,...,N. \tag{7}$$

With a careful selection of the polynomial sequence $\{\Omega_N\}$, it becomes possible to establish upper bounds as:

$$\mathcal{R}_{1,N}(G,t) = \int_0^1 \mid \mathcal{P}_{.,h}(t,w) \mid\mid G(w) - P_N(w) \mid dw,$$

$$\mathcal{R}_{2,N}(G,t) = \int_0^1 \mid \mathcal{P}_{.,h}(t,w) \mid\mid \Omega_N(G - P_N, w) \mid dw,$$

where $\mathcal{P}_{.,h}(t,w)$ is defined in (4).

Theorem 2.1. *(From [8]) Assume that $G(t) = (1-t)^\mu$ where $\mu > -1$ is not an integer and $\tau > -1$, with $\mu + \tau > -1$, then*

$$\int_0^1 \mid G(t) - \Omega_N(G,t) \mid\mid t-w \mid^\tau dt \leq C \begin{cases} N^{-2-2\mu-2\tau} \log N, & \mid w \mid \leq 1, \ \tau < 0, \\ N^{-2-2\mu} \log N, & \mid w \mid \leq 1, \ \tau \geq 0, \end{cases} \tag{8}$$

such that C represents a constant that remains unaffected by both w and N.

Corollary 2.2. *(From [8]) Assume that $G(t) = (1-t)^\mu$ where $\mu > 0$, is not an integer, then we have*

$$\int_0^1 \mid G(t) - \Omega_N(G,t) \mid\mid \log \mid t-w \mid\mid dt \leq C \begin{cases} N^{-2-2\mu} \log^2 N, & \mid w \mid \leq 1, \\ \\ N^{-2-2\mu} \log N, & 0 \leq w < 1, \end{cases} \tag{9}$$

where C represents a constant that remains unaffected by both w and N.

2.1. Orthogonal polynomials

Chelyshkov has introduced polynomials in [26], specifically designed to be orthogonal over the interval $[0,1]$ which the Rodrigues type representation as

$$\mathcal{P}_{n,l}(t) = \frac{1}{(n-l)!} \frac{1}{t^{l+1}} \frac{d^{n-l}}{dt^{n-l}} (t^{n+l+1}(1-t)^{n-l}), \quad l = 0, 1, \ldots, n,$$

and are explicitly characterized by

$$\mathcal{P}_{n,l}(t) = \sum_{j=0}^{n-l} (-1)^j \binom{n-l}{j} \binom{n+l+1+j}{n-l} t^{l+j}, \quad l = 0, 1, ..., n. \tag{10}$$

Given a fixed value for n, the orthogonality property within the interval $[0,1]$ establishes an immediate link between the polynomials $\mathcal{P}_{n,l}(t)$ and a set of Jacobi polynomials $P_m^{(\alpha,\beta)}(t)$ [26] as:

$$\mathcal{P}_{n,l}(t) = (-1)^{n-l} t^l P_{n-l}^{(0,2l+1)}(2t-1), \quad l = 0, 1, \ldots, n. \tag{11}$$

From [27], Jacobi polynomials are the polynomial eigenfunctions of the singular Sturm-Liouville problem. An explicit formula is given by

$$P_m^{(\alpha,\beta)}(t) = \frac{1}{2^m}\sum_{j=0}^{m}\binom{m+\alpha}{j}\binom{m+\beta}{m-j}(t-1)^{m-j}(t+1)^j.$$

An important consequence of the symmetry of the weight function $w(t) = (1-t)^\alpha(1+t)^\beta$, and the orthogonality of the Jacobi polynomials, is the symmetry relation

$$P_m^{(\alpha,\beta)}(t) = (-1)^m P_m^{(\alpha,\beta)}(-t).$$

The ultraspherical polynomials are simply Jacobi polynomials with $\alpha = \beta$, and normalized differently:

$$P_m^{(\alpha)}(t) = \frac{\Gamma(\alpha+1)\Gamma(m+2\alpha+1)}{\Gamma(2\alpha+1)\Gamma(m+\alpha+1)}P_m^{(\alpha,\alpha)}(t),$$

where $\Gamma(\cdot)$ is the gamma function. The relation between Legendre and Chebyshev polynomials with the ultraspherical polynomials is

$$P_m(t) = P_m^{(0)}(t), \quad T_m(t) = \frac{P_m^{(-\frac{1}{2},-\frac{1}{2})}(t)}{P_m^{(-\frac{1}{2},-\frac{1}{2})}(1)}.$$

Also for considering the other orthogonal polynomials, we can refer to [28, 29]. According to [26], for each value of n, the polynomial $\mathcal{P}_{n,0}(t)$ has precisely n distinct roots within the interval $(0,1)$ and this set of polynomials exhibits all the typical characteristics found in other widely recognized orthogonal polynomial families such as Legendre or Chebyshev polynomials. Yon can see that distribution of roots of these polynomials in Fig. 1 and Fig. 2.

3. Nyström method

Here, we outline the Nyström method employed for the numerical solution of equation (1). In equation (1), for the sake of convenience and without any loss of generality, we assume certain conditions

$$\begin{cases} y(t) = f(t) + (V_\alpha y)(t) + (V_{\alpha,\theta}y)(t), & t \in (0,1], \\ y(t) = \phi(t), & t \in [\theta(0), 0]. \end{cases} \tag{12}$$

By chosen $(N+1)$ distinct points $\{t_i\}_{i=1}^N \bigcup \{t_0 = 0\}$, in the interval $I_h \subseteq [0,1]$ and collocate the equation (12) at the underlying local mesh $\{t_i\}_{i=0}^N$, we have

$$y(t_i) = \quad f(t_i) + (V_\alpha y)(t_i) + (V_{\alpha,\theta}y)(t_i), \quad i = 0,1,...,N, \tag{13}$$

where

$$(V_\alpha y)(t_i) = \quad \int_0^{t_i} \mathcal{P}_{1,h}(t_i,w)k_1(t_i,w,y(w))dw, \tag{14}$$

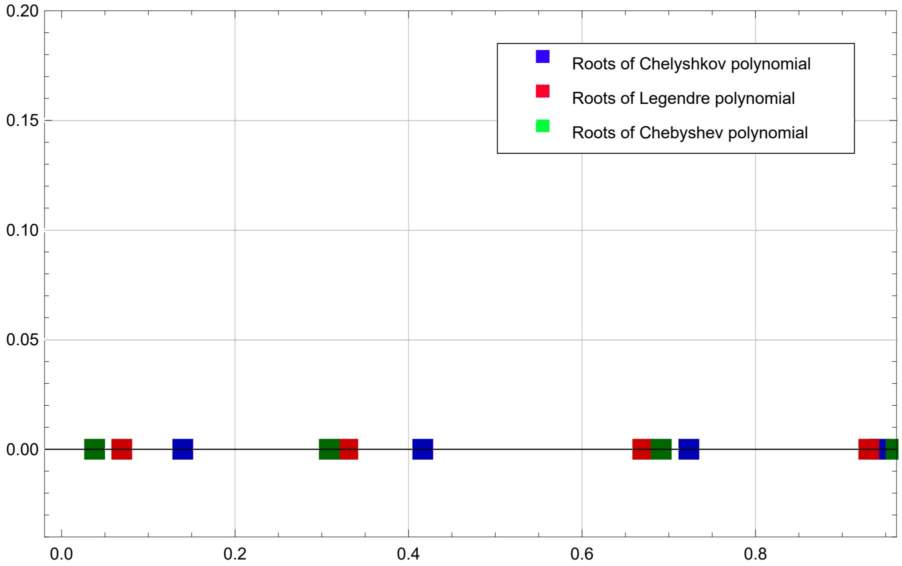

Figure 1: Plot of distribution of roots of Chelyshkov, Legendre and Chebyshev polynomials with $m = 4$.

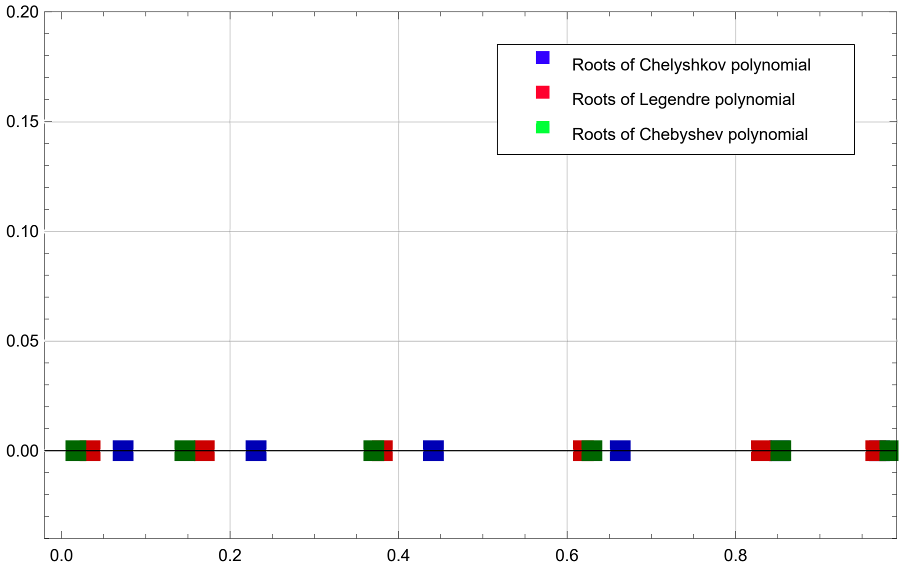

Figure 2: Plot of distribution of roots of Chelyshkov, Legendre and Chebyshev polynomials with $m = 6$.

$$(V_{\alpha,\theta}y)(t_i) \quad = \int_0^{\theta(t_i)} \mathcal{P}_{2,h}(t_i, w) k_2(t_i, w, y(w)) dw. \tag{15}$$

Subsequently, we employ the Lagrange interpolation polynomial

$$\Omega_N(k_h, w) = \sum_{j=0}^{N} l_{N,j}(w) k_h(t_i, t_j, y(t_j)), \quad h = 1, 2, \tag{16}$$

to approximate $k_h(t_i, w, y(w))$ and get

$$\begin{aligned} y_N(t_i) = \quad & f(t_i) + \sum_{j=0}^{N} [W_{1,i,j}\ k_1(t_i, t_j, y_N(t_j)) \\ & + W_{2,i,j}\ k_2(t_i, t_j, y_N(t_j))] , \end{aligned} \tag{17}$$

with

$$\begin{aligned} W_{1,i,j} = \quad & \int_0^{t_i} \mathcal{P}_{1,h}(t_i, w) l_{N,j}(w) dw, \\ W_{2,i,j} = \quad & \int_0^{\theta(t_i)} \mathcal{P}_{2,h}(t_i, w) l_{N,j}(w) dw, \quad i, j = 0, 1, 2, \cdots, N. \end{aligned} \tag{18}$$

Note that the relationship given in equation (17) constitutes a nonlinear system of equations with dimensions $(N + 1) \times (N + 1)$. This system possesses a unique solution, as demonstrated in sources such as [30] and [31]. The resolution of this nonlinear system yields the values of $y_N(t_i)$ for $i = 0, 1, 2, ..., N$, representing the solutions of (12) at the points $\{t_i\}_{i=0}^N$.

Remark 3.1. *Using the relation* (16)*, approximate the integrals in* (12)*, obtaining a new equation:*

$$y_N(t) = f(t) + \sum_{j=0}^{N} \Big(W_{1,t,j}\ k_1(t, t_j, y_N(t_j)) + W_{2,t,j}\ k_2(t, t_j, y_N(t_j)) \Big), \tag{19}$$

where for $j = 0, 1, 2, \ldots, N$*,*

$$W_{1,t,j} = \int_0^{t} \mathcal{P}_{1,h}(t, w) l_{N,j}(w) dw, \quad W_{2,t,j} = \int_0^{\theta(t)} \mathcal{P}_{2,h}(t, w) l_{N,j}(w) dw,$$

We write this as an exact equation with a new unknown function $y_N(t)$*. To find the solution at the node points, let* $y(t)$ *run through the quadrature node points* t_i*. This yields the nonlinear system* (17) *of order* $N+1$*. Each solution* $y_N(t)$ *of* (19) *furnishes a solution to* (17)*: merely evaluate* $y_N(t)$ *at the node points. The converse is also true. To each solution* $(y_N(t_0), \ldots, y_N(t_N))^T$ *of* (17)*, there is a unique solution of* (19)*. Therefore, by inserting* $(y_N(t_0), \ldots, y_N(t_N))^T$ *in* (19)*, we can obtain the approximate solution in each desired point* t*. In fact, the relation* (19) *is the Nyström interpolation formula.*

With these symbols in place, we can encapsulate the steps in the following algorithm:

Algorithm 1.

Input:
N;
begin
For l=0,1,2. . . ,N:
  Compute t_l as simple roots of $N+1$ st-degree orthogonal polynomial in $[0,1]$;
  For i=0,1. . . ,N:
    For j=0,1,. . . ,N:
      For r=1,2:
        Compute $W_{r,i,j}$ from (18);
        Solve nonlinear system (17) and Compute $y_N(t_i)$;
end.

---

4. Convergence Theorem

Here, we investigate the convergence analysis of the equation

$$y(t) = f(t) + (V_\alpha y)(t) + (V_{\alpha,\theta} y)(t), \qquad t \in [0,1]. \tag{20}$$

where

$$\begin{aligned} (V_\alpha y)(t) = &\int_0^t \mathcal{P}_1(t,s)k_1(t,s)y(s)ds, \\ (V_{\alpha,\theta} y)(t) = &\int_0^{\theta(t)} \mathcal{P}_2(t,s)k_2(t,s)y(s)ds. \end{aligned} \tag{21}$$

Assuming that the function $f(t)$ belongs to the space $C([0,1])$ and the kernels $\mathcal{P}_h$, where $h=1,2$, exhibit weak singularity in the forms (4), the equation (20) possesses a distinct solution y in the interval $C[0,1]$. It is anticipated that this solution may have unbounded derivatives at the endpoints.

If we utilize the method (17) on the test problem (20) for a given mesh $\{t_i\}_{i=1}^N \bigcup \{0\}$, the resulting approximate solution $y_N(t)$ takes the following form

$$y_N(t) = f(t) + \sum_{j=0}^{N} \left(\omega_{1,j}(\mathcal{P}_{1,h}, t) + \omega_{2,j}(\mathcal{P}_{2,h}, \theta(t))\right) y_N(t_j), \tag{22}$$

where

$$\begin{aligned} \omega_{1,j}(\mathcal{P}_{1,h}, t) = &\int_0^t \mathcal{P}_{1,h}(t_i, w) l_{N,j}(w) dw, \\ \omega_{2,j}(\mathcal{P}_{2,h}, \theta(t)) = &\int_0^{\theta(t)} \mathcal{P}_{2,h}(t_i, w) l_{N,j}(w) dw. \end{aligned} \tag{23}$$

To assess the uniform convergence of $y_N(t)$ to $y(t)$ as solution of (20), we rewrite

$$\begin{aligned} y(t) - y_N(t) = & \sum_{j=0}^{N} \omega_{1,j}(\mathcal{P}_{1,h}, t)\big(y(t_j) - y_N(t_j)\big) \\ & + \sum_{j=0}^{N} \omega_{2,j}(\mathcal{P}_{2,h}, \theta(t))\big(y(t_j) - y_N(t_j)\big) \\ & + \mathcal{T}_{1,N}(\mathcal{P}_{1,h}, y; t) + \mathcal{T}_{2,N}(\mathcal{P}_{2,h}, y; \theta(t)), \end{aligned} \tag{24}$$

where

$$\begin{aligned} \mathcal{T}_{1,N}(\mathcal{P}_{1,h}, y; t) = & \int_0^t \mathcal{P}_{1,h}(t, w) y(w) dw - \sum_{j=0}^{N} \omega_{1,j}(\mathcal{P}_{1,h}, t) y(t_j), \\ \mathcal{T}_{2,N}(\mathcal{P}_{2,h}, y; \theta(t)) = & \int_0^{\theta(t)} \mathcal{P}_{2,h}(t, w) y(w) dw - \sum_{j=0}^{N} \omega_{2,j}(\mathcal{P}_{2,h}, \theta(t)) y(t_j). \end{aligned} \tag{25}$$

If we set

$$\begin{aligned} \omega_j(\mathcal{P}_{1,h}, \mathcal{P}_{2,h}; t, \theta(t)) = & \ \omega_{1,j}(\mathcal{P}_{1,h}, t) + \omega_{2,j}(\mathcal{P}_{2,h}, \theta(t)), \\ \mathcal{T}_N(\mathcal{P}_{1,h}, \mathcal{P}_{2,h}, y; t, \theta(t)) = & \ \mathcal{T}_{1,N}(\mathcal{P}_{1,h}, y; t) + \mathcal{T}_{2,N}(\mathcal{P}_{2,h}, y; \theta(t)), \end{aligned} \tag{26}$$

then equation (24) reduces to the following equation

$$\begin{aligned} y(t) - y_N(t) = & \sum_{j=0}^{N} \omega_j(\mathcal{P}_{1,h}, \mathcal{P}_{2,h}; t, \theta(t))\big(y(t_j) - y_N(t_j)\big) \\ & + \mathcal{T}_N(\mathcal{P}_{1,h}, \mathcal{P}_{2,h}, y; t, \theta(t)). \end{aligned} \tag{27}$$

Now, define linear operator $\mathcal{A}_N$ as:

$$\begin{cases} \mathcal{A}_N : C[0,1] \longrightarrow C[0,1] \\ \mathcal{A}_N(f(t)) = \displaystyle\sum_{j=0}^{N} \omega_j(\mathcal{P}_{1,h}, \mathcal{P}_{2,h}; t, \theta(t)) f(t_j), \quad f \in C[0,1], \end{cases} \tag{28}$$

then

$$\begin{aligned} \| y(t) - y_N(t) \|_\infty = & \ \| \mathcal{A}_N\big(y(t) - y_N(t)\big) + \mathcal{T}_N(\mathcal{P}_{1,h}, \mathcal{P}_{2,h}, y; t, \theta(t)) \|_\infty \\ \leq & \ \| \mathcal{A}_N \|_\infty \| y(t) - y_N(t) \|_\infty + \| \mathcal{T}_N(\mathcal{P}_{1,h}, \mathcal{P}_{2,h}, y; t, \theta(t)) \|_\infty, \end{aligned} \tag{29}$$

and by considering (27), we have

$$\parallel y(t) - y_N(t) \parallel_\infty \leq \parallel (I - \mathcal{A}_N)^{-1} \parallel_\infty \parallel \mathcal{T}_N \parallel_\infty . \tag{30}$$

Our ultimate objective is to establish an upper bound for (30). We begin by preparing a set of auxiliary theorems and lemmas concerning kernels of types (4).

Theorem 1. *Consider $\{t_i\}_{i=0}^N$ as the zeros of the $(N+1)$-th-degree member of a set of polynomials that are orthogonal on $I_h \subseteq [0,1]$ with respect to the weight function*

$$W(t) = g(t)(1-t)^{\bar{\mu}}(1+t)^{\bar{\nu}}, \quad -1 < \bar{\mu} \leq \frac{3}{2}, \quad \bar{\nu} \geq -\frac{1}{2}, \tag{31}$$

where $g(t)$ is a positive and continuous function within the interval I_h, and the modulus of continuity u for g fulfills the condition $\int_0^1 u(g,s)\frac{ds}{s} < \infty$. Also, let $\Omega_N(y,w)$ represent the interpolating polynomial of degree at most N that matches the function y at $\{t_i\}_{i=1}^N \cup \{t_0 = 0\}$. Additionally, assume that $\mathcal{P}_{1,h}(t,w)$ and $\mathcal{P}_{2,h}(t,w)$ are kernels of type (4). Then, for any function $y \in C(I_h)$, we have

$$\lim_{N\to\infty} \| \int_0^t \mathcal{P}_{1,h}(t,w)\big(y(w) - \Omega_N(y,w)\big)dw + \int_0^{\theta(t)} \mathcal{P}_{2,h}(t,w)\big(y(w) - \Omega_N(y,w)\big)dw \| = 0. \tag{32}$$

Especially, for $0 < \mu < 1$, the limits are as follows

$$\|\mathcal{T}_N(|t-w|^{-\mu}, |t-w|^{-\mu}, y; t, \theta(t))\|_\infty = O\{h_1\}, \tag{33}$$

$$\|\mathcal{T}_N(\log|t-w|, \log|t-w|, y; t, \theta(t))\|_\infty = O\{h_2\}, \tag{34}$$

$$\|\mathcal{T}_N(|t-w|^{-\mu}, \log|t-w|, y; t, \theta(t))\|_\infty = O\{h_3\}, \tag{35}$$

where $h_1 = (N+1)^{2\mu-2\kappa-2}\log(N+1)$, $h_2 = (N+1)^{-2-2\kappa}\log^2(N+1)$ and $h_3 = \min\{h_1, h_2\}$.

Proof. By using the equation (26), we have

$$\begin{aligned} \mid \mathcal{T}_N(\mathcal{P}_{1,h}, \mathcal{P}_{2,h}, y; t, \theta(t)) \mid = & \mid \int_0^t \mathcal{P}_{1,h}(t,w)y(w)dw - \sum_{j=0}^{N} \omega_{1j}(\mathcal{P}_{1,h}, t)y(t_j) \\ & + \int_0^{\theta(t)} \mathcal{P}_{2,h}(t,w)y(w)dw - \sum_{j=0}^{N} \omega_{2j}(\mathcal{P}_{2,h}, \theta(t))y(t_j) \mid \\ \leq & \int_0^t \mid \mathcal{P}_{1,h}(t,w) \mid\mid y(w) - \Omega_N(y,w) \mid dw \\ & + \int_0^{\theta(t)} \mid \mathcal{P}_{2,h}(t,w) \mid\mid y(w) - \Omega_N(y,w) \mid dw \\ \leq & \int_0^1 \mid \mathcal{P}_{1,h}(t,w) \mid\mid y(w) - \Omega_N(y,w) \mid dw \\ & + \int_0^1 \mid \mathcal{P}_{2,h}(t,w) \mid\mid y(w) - \Omega_N(y,w) \mid dw. \end{aligned} \tag{36}$$

The proof is readily derived from an outcome of Theorem 2.2. The inequalities (33) and (35) provide a measure of the convergence rate.

Now, let's examine the characteristics of the initial term $\| (I - \mathcal{A}_N)^{-1} \|_\infty$

Theorem 2. *For a given set of nodes $\{t_i\}_{i=0}^N$ defined as Theorem 1 with the restriction $-\frac{1}{2} < \bar{\mu}, \bar{\nu} < \frac{3}{2}$, let $l_{N,j}(w)$ show the corresponding $j-th$ Lagrange polynomial. Fix a subinterval $[a,b] \subseteq [0,1]$. Then, there is a positive number C and a value of q_1 greater than 1, such that*

$$\sup \sum_{j=0}^{N} \mid \int_a^b (\mathcal{P}_{1,h}(b,w) + \mathcal{P}_{2,h}(b,w)) l_{N,j}(w) dw \mid \leq \quad C(\| \mathcal{P}_{1,h}(b,w) \|_{q_1} + \| \mathcal{P}_{2,h}(b,w) \|_{q_1}), \tag{37}$$

for all $\mathcal{P}_{.,h} \in L_{q_1}$ with $\| \mathcal{P}_{.,h} \|_{q_1} = (\int_0^1 \mid \mathcal{P}_{.,h}(b,w) \mid^{q_1} dw)^{\frac{1}{q_1}}$.

Proof. In the same manner in [32], we set $B = \{f \in C[0,1] : \|f\|_\infty = 1\}$. Then we have

$$\sum_{j=0}^{N} | \int_a^b [\mathcal{P}_{1,h}(b,w) + \mathcal{P}_{2,h}(b,w)] l_{N,j}(w) dw |$$

$$= \sup_{f\in B} | \int_a^b \sum_{j=0}^{N} (\mathcal{P}_{1,h}(b,w) + \mathcal{P}_{2,h}(b,w)) l_{N,j}(w) f(t_j) dw |$$

$$\leq \sup_{f\in B} \int_a^b | \sum_{j=0}^{N} l_{Nj}(w) f(t_j) | \{| \mathcal{P}_{1,h}(b,w) | + | \mathcal{P}_{2,h}(b,w) |\} dw$$

$$\leq \sup_{f\in B} \{[\int_a^b | \sum_{j=0}^{N} l_{Nj}(w) f(t_j) |^{q_2} dw]^{\frac{1}{q_2}} \{[\int_a^b | \mathcal{P}_{1,h}(b,w) |^{q_1} dw]^{\frac{1}{q_1}} + [\int_a^b | \mathcal{P}_{2,h}(b,w) |^{q_1} dw]^{\frac{1}{q_1}}\}\}$$

$$\leq \sup_{f\in B} \{[\int_0^1 | \sum_{j=0}^{N} l_{Nj}(w) f(t_j) |^{q_2} dw]^{\frac{1}{q_2}} \{[\int_a^b | \mathcal{P}_{1,h}(b,w) |^{q_1} dw]^{\frac{1}{q_1}} + [\int_a^b | \mathcal{P}_{2,h}(b,w) |^{q_1} dw]^{\frac{1}{q_1}}\}\}$$

$$\leq \sup_{f\in B} \{\| \sum_{j=0}^{N} l_{Nj}(w) f(t_j) \|_{q_2} [\| \mathcal{P}_{1,h}(b,w) \|_{q_1} + \| \mathcal{P}_{2,h}(b,w) \|_{q_1}]\}, \tag{38}$$

for all $\mathcal{P}_{.,h} \in L_{q_1},\ h = 1,2$ with $q_1, q_2 > 1$ such that $\frac{1}{q_1} + \frac{1}{q_2} = 1$. Now, from [25], given the conditions specified in this theorem, we obtain:

$$\sup_{f\in B} \| \sum_{j=0}^{N} l_{N,j}(w) f(t_j) \|_{q_2} \leq C \| f \|_\infty, \tag{39}$$

for any bounded function f and $0 < q_2 < \infty$. Hence, the bound (37) follows.

Lemma 1. *If the kernels $\mathcal{P}_h$ satisfy*

$$\begin{cases} \mathcal{P}_{.,h} \in L_{q_1}, \quad q_1 > 1, \quad h = 1,2, \\ \lim_{t_1\to t} \| \mathcal{P}_{.,h}(t_1,w) - \mathcal{P}_{.,h}(t,w) \|_{q_1} = 0, \quad \forall t \in I_h. \end{cases} \tag{40}$$

Then

$$\lim_{(t_1,\theta(t_1))\to(t,\theta(t))} \sup_N \sum_{j=0}^{N} | \omega_j(\mathcal{P}_{1,h}, \mathcal{P}_{2,h}; t_1, \theta(t_1)) - \omega_j(\mathcal{P}_{1,h}, \mathcal{P}_{2,h}; t, \theta(t)) | = 0, \tag{41}$$

for all $t \in I_h$.

Proof. By using $\omega_j = \omega_{1j} + \omega_{2j}$ we get

$$\sup_N\{\sum_{j=0}^{N} \mid \omega_j(\mathcal{P}_{1,h}, \mathcal{P}_{2,h}; t_1, \theta(t_1)) - \omega_j(\mathcal{P}_{1,h}, \mathcal{P}_{2,h}; t, \theta(t)) \mid\}$$

$$= \sup_N\{\sum_{j=0}^{N} \mid [\omega_{1j}(\mathcal{P}_{1,h}, t_1) - \omega_{1j}(\mathcal{P}_{1,h}, t)] + [\omega_{2j}(\mathcal{P}_{2,h}, t_1) - \omega_{2j}(\mathcal{P}_{2,h}, t)] \mid\}$$

$$= \sup_N\{\sum_{j=0}^{N} \mid \int_0^{t_1} \mathcal{P}_{1,h}(t_1, w) l_{N,j}(w) dw - \int_0^{t} \mathcal{P}_{1,h}(t, w) l_{N,j}(w) dw$$

$$+ \int_0^{\theta(t_1)} \mathcal{P}_{2,h}(t_1, w) l_{N,j}(w) dw - \int_0^{\theta(t)} \mathcal{P}_{2,h}(t, w) l_{N,j}(w) dw \mid\}$$

$$= \sup_N\{\sum_{j=0}^{N} \mid \int_0^{t} [\mathcal{P}_{1,h}(t_1, w) - \mathcal{P}_{1,h}(t, w)] l_{N,j}(w) dw + \int_t^{t_1} \mathcal{P}_{1,h}(t_1, w) l_{N,j}(w) dw$$

$$+ \int_0^{\theta(t)} [\mathcal{P}_{2,h}(t_1, w) - \mathcal{P}_{2,h}(t, w)] l_{N,j}(w) dw + \int_{\theta(t)}^{\theta(t_1)} \mathcal{P}_{2,h}(t_1, w) l_{N,j}(w) dw \mid\}$$

$$\le \sup_N\{\sum_{j=0}^{N} \mid \int_0^{t_1} \mathcal{P}_{1,h}(t_1, w) l_{N,j}(w) dw \mid - \sum_{j=0}^{N} \mid \int_0^{t} \mathcal{P}_{1,h}(t, w) l_{N,j}(w) dw \mid$$

$$+ \sum_{j=0}^{N} \mid \int_0^{\theta(t_1)} \mathcal{P}_{2,h}(t_1, w) l_{N,j}(w) dw \mid - \sum_{j=0}^{N} \mid \int_0^{\theta(t)} \mathcal{P}_{2,h}(t, w) l_{N,j}(w) dw \mid\} \tag{42}$$

$$\le C\{[\int_0^{t_1} \mid \mathcal{P}_{1,h}(t_1, w) - \mathcal{P}_{1,h}(t, w) \mid^{q_1} dw]^{\frac{1}{q_1}} + [\int_t^{t_1} \mid \mathcal{P}_{1,h}(t_1, w) \mid^{q_1} dw]^{\frac{1}{q_1}}$$

$$+ [\int_0^{t_1} \mid \mathcal{P}_{2,h}(t_1, w) - \mathcal{P}_{2,h}(t, w) \mid^{q_1} dw]^{\frac{1}{q_1}} + [\int_{\theta(t)}^{\theta(t_1)} \mid \mathcal{P}_{2,h}(t_1, w) \mid^{q_1} dw]^{\frac{1}{q_1}}\}$$

$$\le C\{[\int_0^{1} \mid \mathcal{P}_{1,h}(t_1, w) - \mathcal{P}_{1,h}(t, w) \mid^{q_1} dw]^{\frac{1}{q_1}} + [\int_t^{t_1} \mid \mathcal{P}_{1,h}(t_1, w) \mid^{q_1} dw]^{\frac{1}{q_1}}$$

$$+ [\int_0^{1} \mid \mathcal{P}_{2,h}(t_1, w) - \mathcal{P}_{2,h}(t, w) \mid^{q_1} dw]^{\frac{1}{q_1}} + [\int_{\theta(t)}^{\theta(t_1)} \mid \mathcal{P}_{2,h}(t_1, w) \mid^{q_1} dw]^{\frac{1}{q_1}}\}$$

$$\le C\{\parallel \mathcal{P}_{1,h}(t_1, w) - \mathcal{P}_{1,h}(t, w) \parallel_{q_1} + [\int_{\theta(t)}^{\theta(t_1)} \mid \mathcal{P}_{2,h}(t_1, w) \mid^{q_1} dw]^{\frac{1}{q_1}}$$

$$+ \parallel \mathcal{P}_{2,h}(t_1, w) - \mathcal{P}_{2,h}(t, w) \parallel_{q_1} + [\int_{\theta(t)}^{\theta(t_1)} \mid \mathcal{P}_{2,h}(t_1, w) \mid^{q_1} dw]^{\frac{1}{q_1}}\}.$$

The Lemma now follow from these statements.

Theorem 3. *Consider the operator* $\mathcal{A}_N$ *defined as in (28), and let the nodes* $\{t_i\}_{i=0}^N$ *be selected as outlined in Theorem 1. If conditions* (32), (37), *and* (41) *are satisfied, then for all sufficiently large values of* N*, there exists a constant* $C > 0$ *that is independent of* N*, so that*

$$\| (I - \mathcal{A}_N)^{-1} \| \leq C. \tag{43}$$

Proof. The proof immediately stems from an implication of Theorem 2 in [32] and Lemma 1 in [33].

The outcomes of our efforts in this section lead to the following principal theorem:

Theorem 4. *Consider* $y(t)$ *and* $y_N(t)$ *as the exact and approximated solutions, respectively, to the equation* (20)*. These solutions are constructed based on a collection of distinct nodes* $\{t_i\}_{i=1}^N \cup \{t_0 = 0\}$*. If the nodes* $\{t_i\}$ *are the zeroes of the orthogonal polynomial in* I_h *and* $p_h(t, w)$ *is the kernel function of the form* (4)*, then* $y_N(t)$ *converges uniformly to* $y(t)$*. Furthermore, the convergence rate aligns with the product integration quadrature that we select to approximate the integral term* (20)*.*

5. Numerical results

In this section, we present the numerical outcomes of various test problems solved using the method proposed in this article. Different forms of kernels are taken into account for computational purposes in the following test problems. The discretization algorithm relies on nodes that coincide with the zeros of the specified orthogonal polynomials of the (N)-th degree, along with $t = 0$. Additionally, a product integration method described in Section 3 is employed. All calculations were conducted using Mathematica software.

Example 5.1. *The non-linear, weakly singular Volterra functional integral equation*

$$y(t) = f(t) + \int_0^t \ln|t-w|\,(tw^2 - y^2(w))dw + \int_0^{\frac{t}{10}} |t-w|^{-\frac{1}{2}}\, y^2(w)dw, \quad t \in [0,1]$$

with $f(t)$ *such that possesses the exact solution* $y(t) = t^{\frac{13}{2}}$*.*

Example 5.2. *The nonlinear weakly singular Volterra functional integral equation in* $[0, 1]$

$$y\,(t) = f\,(t) + \int_0^t |t-w|^{-\frac{1}{2}}\, y^2\,(w)\,dw + \int_0^{t-\frac{1}{10}} |t-w|^{-\frac{1}{2}}\, y^2\,(w)\,dw,$$

with $f(t)$ *such that possesses the analytical solution* $y(t) = e^{2t}$*.*

We consider the other example which the exact solution has the low regularity. The exact solution is $y(t) = \sqrt{t}$, then derivative of this solution is unbounded at $t = 0$ and reflects the general qualitative regularity behaviour of the solution near $t = 0^+$.

N	Chelyshkov Polynomials	Legendre Polynomials	Chebyshev Polynomials
4	2.81×10^{-2}	3.87×10^{-2}	7.31×10^{-2}
5	9.50×10^{-3}	1.48×10^{-2}	1.77×10^{-2}
6	2.48×10^{-3}	3.98×10^{-3}	5.10×10^{-3}
7	6.34×10^{-4}	1.00×10^{-3}	1.38×10^{-3}
8	1.25×10^{-4}	2.10×10^{-4}	2.57×10^{-4}
9	2.02×10^{-5}	3.44×10^{-5}	4.30×10^{-5}
10	2.40×10^{-6}	3.90×10^{-6}	5.06×10^{-6}

Table 1: Maximum errors by using different types of orthogonal polynomials in Example 5.1.

N	Chelyshkov Polynomials	Legendre Polynomials	Chebyshev Polynomials
4	1.70×10^{-2}	1.72×10^{-2}	2.13×10^{-2}
5	4.28×10^{-3}	4.42×10^{-3}	5.20×10^{-3}
6	8.35×10^{-4}	8.90×10^{-4}	9.93×10^{-4}
7	1.30×10^{-4}	1.45×10^{-4}	1.53×10^{-4}
8	1.61×10^{-5}	1.97×10^{-5}	1.98×10^{-5}
9	1.47×10^{-6}	2.31×10^{-6}	2.51×10^{-6}
10	1.64×10^{-7}	2.31×10^{-7}	6.71×10^{-7}

Table 2: Maximum errors by using different types of orthogonal polynomials in Example 5.2.

Example 5.3. *The non-linear, weakly singular Volterra functional integral equation*

$$y(t) = f(t) - \int_0^t |t-w|^{-\frac{1}{2}} y^3(w)dw + \int_0^{0.9t} |t-w|^{-\frac{1}{2}} y^2(w)dw, \quad t \in [0,1]$$

with $f(t)$ *such that possesses the exact solution* $y(t) = t^{\frac{1}{2}}$.

We give the numerical solution of the Examples, at the root of N-st-degree orthogonal polynomials. The maximum errors obtained using the presented method are compared with the exact solution in Tables 1, 2 and 3. In the case of examples, the obtained nonlinear systems are solved using Newton's method.

N	Chelyshkov Polynomials	Legendre Polynomials	Chebyshev Polynomials
4	6.16×10^{-4}	6.69×10^{-4}	1.16×10^{-3}
5	2.91×10^{-4}	3.44×10^{-4}	5.50×10^{-4}
6	1.54×10^{-4}	2.23×10^{-4}	3.19×10^{-4}
7	8.87×10^{-5}	1.44×10^{-4}	2.06×10^{-4}
8	5.45×10^{-5}	1.00×10^{-4}	1.40×10^{-4}
9	3.53×10^{-5}	6.93×10^{-5}	9.53×10^{-5}
10	2.38×10^{-5}	4.91×10^{-5}	4.67×10^{-5}

Table 3: Maximum errors by using different types of orthogonal polynomials in Example 5.3.

N	Example 5.1	Example 5.2	Example 5.3
4	6.03×10^{-4}	8.60×10^{-4}	2.72×10^{-5}
6	3.60×10^{-4}	9.54×10^{-6}	4.07×10^{-6}
8	4.79×10^{-6}	3.41×10^{-7}	8.67×10^{-7}
10	9.95×10^{-8}	1.81×10^{-7}	2.86×10^{-7}

Table 4: Absolute errors at the point $t_{N+1} = 1$ for different values of N.

In Tables 1, 2 and 3, we report maximum errors in Examples 5.1, 5.2 and 5.3. We consider Chelyshkov, Legendre and Chebyshev polynomials as orthogonal polynomials and observe that our proposed numerical method works well for theses different types of orthogonal polynomials. Also, according to the dispersion of the roots of these polynomials, it seems that the errors reported by Chelyshkov polynomials is slightly better than the other polynomials.

Remark 5.4. *To get the approximate solution at any given point η within the interval $I_h \subseteq [0, 1]$, our focus lies on a rule that relies on both β and $y(\eta)$*

$$\int_0^1 y(w)dw \approx \sum_{j=0}^{N} \beta_j y(t_j) + \beta y(\eta).$$

From [34] and [35], we know that this method is exact for all polynomials whose degree does not exceed $2N$. Nevertheless, when considering quadrature rules that incorporate the point η as one of their nodes, serving as a collocation point, we will be faced with a nonlinear system of equations of size $(N + 1) \times (N + 1)$. The solutions to this system provide the values at our grid points, particularly at $t_{N+1} = \eta$([8]).

In Table 4, we set $t_{N+1} = 1$ and report absolute value errors at this points for the examples 5.1, 5.2 and 5.3.

6. Conclusion

Using incorporating the Nyström method, the non linear weakly singular Volterra functional can be converted into a nonlinear system. This system can be solved by some classical techniques. The method's effectiveness and accuracy in solving nonlinear equations have been assessed through various problems. Another significant advantage of the proposed method is that the unknown coefficients can be determined quite easily using computer programs. In our future work, we will study the Nyström method to solve delay weakly singular integral-algebraic equations in the following form:

$$B(t)X(t) + \int_0^t \mathcal{P}_{1,h}(t,s)K_1(t,s)X(s)\mathrm{d}s + \int_0^{\theta(t)} \mathcal{P}_{2,h}(t,w)K_2(t,s)X(s)\mathrm{d}s = F(t), \quad t \in I,$$

subject to $\det(B(t)) = 0, \ \forall t \in I$.

References

[1] H. Brunner. *Collocation Methods for Volterra Integral and Related Functional Equations.* Cambridge University Press, Cambridge, 2004.

[2] H. Brunner and Y. Yatsenko. Spline collocation methods for non- linear volterra integral equations with unknown delay. *Journal of computational and applied mathematics*, 71(1):67–81, 1996.

[3] A. Cardone and D. Conte. Multistep collocation methods for volterra integro-differential equations. *Applied Mathematics and Computation*, 221:770–7856, 2013.

[4] W. Ming and C. Huang. Collocation methods for volterra functional integral equations with non-vanishing delays. *Applied mathematics and computation*, 269:198–214, 2017.

[5] C. Huang W. Ming and L. Zhao. Optimal super- convergence results for volterra functional integral equations with proportional vanishing delays. *Applied Mathematics and Computation*, 320:292–301, 2018.

[6] M. Wang. Multistep collocation method for fredholm integral equations of the second kind. *Applied Mathematics and Computation*, 420(126870):1–16, 2022.

[7] T. Tang C. Huang and Z. Zhang. Supergeometric convergence of spectral collocation methods for weakly singular Volterra and Fredholm integral equations with smooth solutions. *Journal of Computational Mathematics*, 29(6):698–719, 2011.

[8] M. Rasty and M. Hadizadeh. A product integration approach on new orthogonal polynomials for nonlinear weakly singular integral equations. *Acta Applicandae Mathematicae*, 109:861–873, 2010.

[9] M.A. Zaky and I. G. Ameen. A novel jacobi spectral method for multi-dimensional weakly singular nonlinear volterra integral equations with nonsmooth solutions. *Engineering with Computers-Germany*, 37:2623–2631, 2021.

[10] H. Cai and Y. Chen. A fractional order collocation method for second kind volterra integral equations with weakly singular kernels. *Journal of Scientific Computing*, 75:970–992, 2018.

[11] T. Herdman Y. Cao and Y. Xu. A hybrid collocation method for volterra integral equations with weakly singular kernels. *SIAM Journal on Numerical Analysis*, 41:364–381, 2003.

[12] P. Assari and M. Dehghan. A meshless method for the numerical solution of nonlinear weakly singular integral equations using radial basis functions. *The European Physical Journal Plus*, 132:1–23, 2017.

[13] M. A. Ebadi E. Hashemizadeh and S. Noeiaghdam. Matrix method by genocchi polynomials for solving nonlinear volterra integral equations with weakly singular kernels. *Symmetry*, 12:1–16, 2020.

[14] J. Huang Y. B. Pan and Y.Y Ma. Bernstein series solutions of multidimensional linear and nonlinear volterra integral equations with fractional order weakly singular kernels. *Applied Mathematics and Computation*, 374:149–161, 2019.

[15] A. M. Wazwaz. *Linear and Nonlinear Integral Equations.* Springer, Heidelberg, 2011.

[16] H. Brunner I. Ali and T. Tang. Spectral methods for pantograph-type differential and integral equations with multiple delays. *Frontiers of Mathematics in China*, 4:49–61, 2009.

[17] N. Senu N. A. Baharum, Z.A. Majid and H. Rosali. Numerical Approach for Delay Volterra Integro-Differential Equation. *Sains Malaysiana*, 51(12):4125–4144, 2002.

[18] H. Brunner. Collocation and continuous implicit runge-kutta methods for a class of delay volterra integral equations. *Journal of Computational and Applied Mathematics*, 53:61–72, 1994.

[19] E. Marchetti F. Calio and R. Pavani. About the deficient spline collocation method for particular differential and integral equations with delay. *Rendiconti del Seminario Matematico - Politecnico di Torino*, 61:287–300, 2003.

[20] R. Pavani F. Calio, E. Marchetti and G. Micula. About some volterra problems solved by a particular spline collocation. *Studia Universitatis Babes-Bolyai*, 48:45–52, 2003.

[21] F. Ghoreishi M. Khasi and M. Hadizadeh. Numerical analysis of a high order method for state-dependent delay integral equations. *Numerical Algorithms*, 66:177–201, 2013.

[22] S. Gan S. Wu. Errors of linear multistep methods for singularly perturbed volterra delay integro-differential equations. *Mathematics and Computers in Simulation*, 79:3148–3159, 2009.

[23] Y. Xiao H. Song and M. Chen. Collocation methods for third-kind volterra integral equations with proportional delays. *Applied Mathematics and Computation*, 388:1–11, 2020.

[24] G. Mastroianni G. Criscuolo and G. Monegato. Convergence properties of a class of product formulas for weakly singular integral equations. *Mathematics of Computation*, 55:213–230, 1990.

[25] P. Nevai. Mean convergence of lagrange interpolation. *Trans. Am. Math. Soc.*, 282:669–689, 1984.

[26] V.S. Chelyshkov. Alternative orthogonal polynomials and quadratures. *Electronic Transactions on Numerical Analysis*, 25(7):17–26, 2006.

[27] S. Gottlieb J. Hesthaven and D. Gottlieb. *Spectral Methods for Time-dependent Problems.* Cambridge University Press, Cambridge, 2007.

[28] C. Cesarano. Generalized Chebyshev polynomials. *Hacettepe Journal of Mathematics and Statistics*, 43(5):731–740, 2014.

[29] B. Germano C. Cesarano and P.E. Ricci. Laguerre-type Bessel functions. *Integral Transforms and Special Functions*, 16(4):315–322, 2005.

[30] H. Kaneko and Y. Xu. Gauss-type quadratures for weakly singular integrals and their application to fredholm integral equations of second kind. *Mathematics of Computation*, 62:739–753, 1994.

[31] L. Tao and H. Yong. Extrapolation method for solving weakly singular nonlinear volterra integral equations of second kind. *Journal of Mathematical Analysis and*

Applications, 324:225–237, 2006.

[32] A.P. Orsi. Product integration for volterra integral equations of the second kined with weakly singular kernels. *Mathematics of Computation*, 212:1201–1212, 1996.

[33] I.H. Sloan. Analysis of genral quadrature methods for integral equations with continuous or discontinuous terms. *Journal of the Institute of Mathematics and its Applications*, 26:175–186, 1980.

[34] V.I. Krylov. *Approximate Calculation of Integrals.* Macmillan Company, New York, 1962.

[35] P.K. Kythe and P. Puri. *Computational Methods for Linear Integral Equations.* Birkhuser, Boston, 2002.