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•-Paradistributive Latticoids

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Abstract. In this paper, we introduce the notion of a new class of Paradistributive latticoids termed •-PDLs and investigate its properties. In addition, we prove that a Paradistributive latticoid(PDL) V is a •-PDL iff V/θ is a Boolean algebra with a minimal element, where $\theta = \{(\wp, \hbar) \in V \times V \mid [\wp]^\bullet = [\hbar]^\bullet\}$ is a congruence relation on V. Further, we explore the concept of dense elements in a PDL and characterize •-PDL in terms of dense elements. Also, we introduce the notion of a disjunctive PDL and we give equivalent conditions for a •-PDL to be a disjunctive PDL.

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Key Words and Phrases: •-PDL, Congruence relation, Boolean algebra, Dense element, Disjunctive PDL

1. Introduction

A variety of generalisations have emerged as a result of Booles' axiomatization of two valued propositional calculus as a Boolean algebra, both ring theoretically and lattice theoretically. Tarski, Moisil and others studied filters in lattices, and many of their findings are found in Birkhoff's lattice theory[1] . The thought of an Almost Distributive Lattice (ADL) was developed by Swamy and Rao[2] as a common abstraction of multiple existing ring theoretic generalisations of a Boolean algebra on one hand and the class of distributive lattices on the other. They developed ideals and filters in an ADL and showed that the set of all filters in an ADL forms a distributive lattice. They also devised a set of

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identities to ensure that the lattice of all filters in an ADL becomes a complete lattice. G.Nanaji Rao and Habtamu Tiruneh Alemu[3, 4] developed the hypothesis of Almost Lattice (AL) as a common abstraction of all lattice-theoretic generalisations of Boolean algebra, such as distributive lattices, almost distributive lattices, and lattices. Distributive pseudo-complemented lattices[5] form an extensively studied class of distributive lattices. Examples are the lattice of all open sets of a topological space, the lattice of all ideals of a distributive lattice with zero and the lattice of all congruences of an arbitrary lattice. Lattice which are just pseudo-complemented have been studied in detail, however, the most interesting results require at least the assumption of modularity, sometimes distributivity. With this motivation, Bandaru et al.[6] introduced the notion of paradistributive latticoid and studied its important properties. Also, authors given its subdirect representation. Later, Ajjarapu et al.[7] introduced the concept of parapseudo-complementation on a paradistributive latticoid and its characterisations are given. Later, Bandaru et al. [8] introduced the concept of a normal paradistributive latticoid and characterized in terms of the prime filters and minimal prime filters. Also, Ajjarapu et al. [9] studied topological properties of (minimal) prime filters in a paradistributive latticoid.

The main purpose of this paper is to introduce the concept of •-PDL. In Section 1, basic introduction is given, in continuation to this in Section 2, preliminaries related to this topic are mentioned. Further, in Section 3, we discuss the basic definition of •-PDL and study various results associated to the concept, connecting the results to Boolean algebra. Later, in Section 4, we discuss the definition of dense elements, and prove certain characterization theorems for dense elements. We conclude this section by introducing disjunctive PDL and provide results related to disjunctive and Boolean algebra.

2. Preliminaries

First we recall the necessary definitions and results from [6].

Definition 1 ([6]). *An algebra $(V, \vee, \wedge, 1)$ of type (2,2,0) is called a Paradistributive Latticoid, abbreviated as PDL, if it assures the subsequent axioms:*
$(LD\vee)$ $\partial_1 \vee (\partial_2 \wedge \partial_3) = (\partial_1 \vee \partial_2) \wedge (\partial_1 \vee \partial_3)$,
$(RD\vee)$ $(\partial_1 \wedge \partial_2) \vee \partial_3 = (\partial_1 \vee \partial_3) \wedge (\partial_2 \vee \partial_3)$,
(L_1) $(\partial_1 \vee \partial_2) \wedge \partial_2 = \partial_2$,
(L_2) $(\partial_1 \vee \partial_2) \wedge \partial_1 = \partial_1$,
(L_3) $\partial_1 \vee (\partial_1 \wedge \partial_2) = \partial_1$,
(I_1) $\partial_1 \vee 1 = 1$,
for any $\partial_1, \partial_2, \partial_3 \in V$.

For any $\partial_1, \partial_2 \in V$, we say that ∂_1 is less than or equal to ∂_2 and write $\partial_1 \leq \partial_2$ if $\partial_1 \wedge \partial_2 = \partial_1$ or equivalently $\partial_1 \vee \partial_2 = \partial_2$ and it can be easily observed that $\leq$ is a partial order on V. We can observe that, the element 1, in Definition 1, is the greatest element with respect to the partial ordering $\leq$.

Example 1 ([6]). *Let V be a non-empty set. Fix some element $g \in V$. Then, for any*

$\wp, \hbar \in V$ define $\vee$ and $\wedge$ on V by

$$\wp \vee \hbar = \begin{cases} \wp & \hbar \neq g \\ g & \hbar = g \end{cases}$$

and

$$\wp \wedge \hbar = \begin{cases} \hbar & \hbar \neq g \\ \wp & \hbar = g \end{cases}$$

Then $(V, \vee, \wedge, g)$ is a disconnected PDL with g as its greatest element.

According to Lemma 7, Theorem 1, Lemma 8, Theorem 4, Corollary 8, Lemma 9 and Lemma 10 of [6], the following Lemma holds.

Lemma 1 ([6]). *Let $(V, \vee, \wedge, 1)$ be a PDL. Then for any $\partial_1, \partial_2, \partial_3, \partial_4 \in V$, we have the following:*
(1) $1 \wedge \partial_1 = \partial_1$,
(2) $\partial_1 \wedge 1 = \partial_1$,
(3) $1 \vee \partial_1 = 1$,
(4) $(\partial_1 \vee \partial_2) \wedge \partial_3 = (\partial_1 \wedge \partial_3) \vee (\partial_2 \wedge \partial_3)$,
(5) $\partial_1 \vee (\partial_2 \wedge \partial_3) = \partial_1 \vee (\partial_3 \wedge \partial_2)$,
(6) *the operation $\vee$ is associative in V i.e., $\partial_1 \vee (\partial_2 \vee \partial_3) = (\partial_1 \vee \partial_2) \vee \partial_3$,*
(7) *the set $V_{\mu_1} = \{\partial_1 \in V \mid \mu_1 \leq \partial_1\} = \{\mu_1 \vee \partial_1 \mid \partial_1 \in V\}$ is a distributive lattice under induced operations $\vee$ and $\wedge$ with μ_1 as its least element,*
(8) $\partial_4 \vee \{\partial_1 \wedge (\partial_2 \wedge \partial_3)\} = \partial_4 \vee \{(\partial_1 \wedge \partial_2) \wedge \partial_3\}$,
(9) $\partial_1 \vee (\partial_2 \vee \partial_3) = \partial_1 \vee (\partial_3 \vee \partial_2)$,
(10) $\partial_1 \vee \partial_2 = 1$ *if and only if* $\partial_2 \vee \partial_1 = 1$,
(11) $\partial_1 \wedge \partial_2 = \partial_2 \wedge \partial_1$ *whenever* $\partial_1 \vee \partial_2 = 1$.

Theorem 1 ([6]). *An algebra $(V, \vee, \wedge, 1)$ of type $(2, 2, 0)$ is a PDL if and only if it satisfies the following:*
$(LD\vee)$ $\partial_1 \vee (\partial_2 \wedge \partial_3) = (\partial_1 \vee \partial_2) \wedge (\partial_1 \vee \partial_3)$,
$(RD\vee)$ $(\partial_1 \wedge \partial_2) \vee \partial_3 = (\partial_1 \vee \partial_3) \wedge (\partial_2 \vee \partial_3)$,
$(RD\wedge)$ $(\partial_1 \vee \partial_2) \wedge \partial_3 = (\partial_1 \wedge \partial_3) \vee (\partial_2 \wedge \partial_3)$,
(L_1) $(\partial_1 \vee \partial_2) \wedge \partial_2 = \partial_2$,
(L_3) $\partial_1 \vee (\partial_1 \wedge \partial_2) = \partial_1$,
(I_1) $\partial_1 \vee 1 = 1$,
(I_2) $1 \wedge \partial_1 = \partial_1$,
for all $\partial_1, \partial_2, \partial_3 \in V$.

Definition 2 ([6]). *A Paradistributive Latticoid $(V, \vee, \wedge, 1)$ is said to be associative if it satisfies the following condition*

$$\partial_1 \wedge (\partial_2 \wedge \partial_3) = (\partial_1 \wedge \partial_2) \wedge \partial_3$$

for all $\partial_1, \partial_2, \partial_3 \in V$.

Definition 3 ([6]). *Let V be a PDL. Then, an element $\mu_1 \in V$ is said to be a minimal element if for any $u \in V$, $u \leq \mu_1 \Rightarrow u = \mu_1$.*

Lemma 2 ([6]). *Let V be a PDL. Then, for any $\mu_1 \in V$, the following are equivalent:*
(1) *μ_1 is minimal,*
(2) *$\partial_1 \wedge \mu_1 = \mu_1$ for all $\partial_1 \in V$,*
(3) *$\partial_1 \vee \mu_1 = \partial_1$ for all $\partial_1 \in V$.*

Definition 4 ([6]). *A non-empty subset F of a PDL V is said to be a filter if it satisfies the following:*

$$\partial_1, \partial_2 \in F \Rightarrow \partial_1 \wedge \partial_2 \in F,$$
$$\partial_1 \in F, \mu_1 \in V \Rightarrow \mu_1 \vee \partial_1 \in F.$$

Theorem 2 ([6]). *Let S be a non-empty subset of V. Then*

$$[S) \;=\; \{\partial_1 \vee (\bigwedge_{i=1}^{n} s_i) \mid s_i \in S, \partial_1 \in V,\ n \text{ is a positive integer }\}$$

is the smallest filter of V containing S.

Note that if $S = \{\partial_1\}$, then we write $[S) = [\partial_1)$, the principal ideal of V generated by '∂_1'. Hence $[\partial_1) = \{\wp \vee \partial_1 \mid \wp \in V\}$.
According to Corollary 8 and Lemma 12 of [6], the following Lemma holds.

Lemma 3 ([6]). *Let V be a PDL and F be a filter of V. Then for any $\partial_1, \partial_2 \in V$, we have the following:*
(1) *$\partial_1 \in [\partial_2)$ if and only if $\partial_1 = \partial_1 \vee \partial_2$ for all $\partial_1, \partial_2 \in V$,*
(2) *$\partial_1 \vee \partial_2 \in F$ if and only if $\partial_2 \vee \partial_1 \in F$,*
(3) *$[\partial_1 \vee \partial_2) = [\partial_2 \vee \partial_1)$,*
(4) *$[\partial_1 \wedge \partial_2) = [\partial_2 \wedge \partial_1) = [\partial_1) \vee [\partial_2)$.*

Theorem 3 ([6]). *The collection $F(V)$ of all filters of a PDL V forms a distributive lattice under set inclusion, in which, the glb and lub of any two filters F and G are given by $F \wedge G = F \cap G$ and $F \vee G = \{\partial_1 \wedge \partial_2 \mid \partial_1 \in F$ and $\partial_2 \in G\}$, respectively.*

Definition 5 ([6]). *A non-empty subset I of a PDL V is said to be an ideal if it satisfies the following:*

$$\partial_1, \partial_2 \in I \Rightarrow \partial_1 \vee \partial_2 \in I,$$
$$\partial_1 \in I, \mu_1 \in V \Rightarrow \partial_1 \wedge \mu_1 \in I.$$

Theorem 4 ([6]). *Let S be a non-empty subset of V. Then*

$$(S] \;=\; \{(\bigvee_{i=1}^{n} s_i) \wedge \partial_1 \mid s_i \in S, \partial_1 \in V,\ n \text{ is a positive integer }\}$$

is the smallest ideal of V containing S.

Note that if $S = \{\partial_1\}$, then we write $(S] = (\partial_1]$, the principal ideal of V generated by '∂_1'. Hence $(\partial_1] = \{\partial_1 \wedge \wp \mid \wp \in V\}$.
According to Corollary 5, Lemma 11 and Corollary 6 of [6], the following Lemma holds.

Lemma 4 ([6]). *Let V be a PDL and I be an ideal of V. Then, for any $\partial_1, \partial_2 \in V$, we have the following:*
(1) $\partial_1 \in (\partial_2]$ *if and only if* $\partial_1 = \partial_2 \wedge \partial_1$,
(2) $\partial_1 \wedge \partial_2 \in I$ *if and only if* $\partial_2 \wedge \partial_1 \in I$,
(3) $(\partial_1 \wedge \partial_2] = (\partial_2 \wedge \partial_1] = (\partial_1] \wedge (\partial_2]$.

Theorem 5 ([6]). *The collection $I(V)$ of all ideals of a PDL V forms a distributive lattice under set inclusion, in which, the glb and lub of any two ideals I and J are given by $I \wedge J = I \cap J$ and $I \vee J = \{\partial_1 \vee \partial_2 \mid \partial_1 \in I$ and $\partial_2 \in J\}$, respectively.*

A proper filter(ideal) P of V is said to be a prime filter(ideal) if for any $\wp, \hbar \in V$, $\wp \vee \hbar \in P(\wp \wedge \hbar \in P) \Rightarrow \wp \in P$ or $\hbar \in P$. A proper filter(ideal) M of V is said to be maximal if it is not properly contained in any proper filter(ideal) of V. A prime filter P of V is said to be minimal, if it is minimal among all the prime filters of V. A prime filter P is said to be a minimal prime filter belonging to a filter I, if it is minimal among all the prime filters of V containing I. A prime filer P of V is a minimal prime filter if and only if for each $\wp \in P$, there exists $\hbar \notin P$ such that $\wp \vee \hbar = 1$.

Lemma 5 ([7]). *Let V be a PDL and $A \subseteq V$. Then*
(1). $A^{\bullet} = \{\tau \in V \mid \tau \vee \wp = 1$ for all $\wp \in A\}$ is a filter of V.
(2). *for any $\wp, \hbar \in V$, $[\wp \wedge \hbar]^{\bullet} = [\wp]^{\bullet} \cap [\hbar]^{\bullet}$, where*

$$[\wp \wedge \hbar]^{\bullet} = \{\tau \in V \mid \tau \vee (\wp \wedge \hbar) = 1\}.$$

(3). *for any $\wp, \hbar \in V$, $[\wp \vee \hbar]^{\bullet\bullet} = [\wp]^{\bullet\bullet} \cap [\hbar]^{\bullet\bullet}$, where*

$$[\wp \wedge \hbar]^{\bullet\bullet} = \{\tau \in V \mid \tau \vee \jmath = 1 \text{ for all } \wp \in [\wp \wedge \hbar]^{\bullet}\}.$$

Definition 6 ([7]). *Let $(V, \vee, \wedge, 1)$ be a Paradistributive Latticoid (PDL) and consider a unary operation denoted as $\wp \mapsto \wp^{\blacklozenge}$ on V. This operation is called a parapseudo-complementation on V if it satisfies the following conditions:*
(PPC_1) *If $\wp \vee \hbar = 1$, then $\wp \vee \hbar^{\blacklozenge} = \wp$.*
(PPC_2) $\wp \vee \wp^{\blacklozenge} = 1$.
(PPC_3) $(\wp \wedge \hbar)^{\blacklozenge} = \wp^{\blacklozenge} \vee \hbar^{\blacklozenge}$.

Definition 7 ([7]). *By a homomorphism of a PDL $(V, \vee, \wedge, 1)$ into a PDL $(V', \vee', \wedge', 1')$, we mean, a mapping $f : V \to V'$ satisfying the following:*
(1) $f(\mu_1 \vee \mu_2) = f(\mu_1) \vee' f(\mu_2)$,
(2) $f(\mu_1 \wedge \mu_2) = f(\mu_1) \wedge' f(\mu_2)$,
(3) $f(1) = f(1')$.

3. •-PDL

In this section, we introduce a new type of PDLs called •-PDLs and demonstrate that if V is parapseudo-complemented PDL, then for any $\wp \in V$, $[\wp]^{\bullet} = [\wp^{\blacklozenge})$ and $[\wp]^{\bullet\bullet} = [\wp^{\blacklozenge}]^{\bullet}$.

With this motivation, we define a $\bullet$ - PDL to be a PDL wherein for any $\wp \in V$, $[\wp]^{\bullet^{\bullet}} = [\wp^{'}]^{\bullet}$ for some $\wp^{'} \in V$. This class of $\bullet$-PDLs contains the class of parapseudo-complemented PDLs. Further, we define a congruence relation $\theta = \{(\wp, \hbar) \mid [\wp]^{\bullet} = [\hbar]^{\bullet}\}$ and prove that V is a $\bullet$-PDL if and only if V/θ is a Boolean algebra. Throughout this paper, V means $(V, \vee, \wedge, 1)$.

Lemma 6. *If V is a parapseudo-complemented PDL, then $[\wp]^{\bullet} = [\wp^{\blacklozenge})$ and $[\wp]^{\bullet^{\bullet}} = [\wp^{\blacklozenge}]^{\bullet} = [\wp^{\bullet^{\bullet}}]$ where $\wp \in V$.*

Proof. $\mu_1 \in [\wp]^{\bullet} \Leftrightarrow \mu_1 \vee \wp = 1 \Leftrightarrow \mu_1 \vee \wp^{\blacklozenge} = \mu_1 \Leftrightarrow \mu_1 \in [\wp^{\blacklozenge})$. Therefore, $[\wp]^{\bullet} = [\wp^{\blacklozenge})$.
Let $\mu_1 \in [\wp]^{\bullet^{\bullet}}$. Then $\mu_1 \vee \mu_2 = 1$ for all $\mu_2 \in [\wp]^{\bullet}$. Clearly $\wp^{\blacklozenge} \in [\wp]^{\bullet}$ (Since $\wp^{\blacklozenge} \vee \wp = 1$). Therefore $\mu_1 \vee \wp^{\blacklozenge} = 1$ implies $\mu_1 \in [\wp^{\blacklozenge}]^{\bullet}$.
Conversely, let $\mu_1 \in [\wp^{\blacklozenge}]^{\bullet}$. Then $\mu_1 \vee \wp^{\blacklozenge} = 1$. Let $\mu_2 \in [\wp]^{\bullet}$. Then $\mu_2 \vee \wp = 1$ and hence $\mu_2 \vee \wp^{\blacklozenge} = \mu_2$.

$$\begin{aligned} \mu_1 \vee \mu_2 &= \mu_1 \vee \mu_2 \vee \wp^{\blacklozenge} \\ &= \mu_1 \vee \wp^{\blacklozenge} \vee \mu_2 \\ &= 1 \vee \mu_2 \\ &= 1 \end{aligned}$$

Therefore $\mu_1 \in [\wp]^{\bullet^{\bullet}}$. Hence $[\wp^{\blacklozenge}]^{\bullet} = [\wp]^{\bullet^{\bullet}} = [\wp^{\blacklozenge^{\blacklozenge}})$.

Definition 8. *A PDL $(V, \vee, \wedge, 1)$ is called a $\bullet$-PDL if, for each $\wp \in V$, $[\wp]^{\bullet^{\bullet}} = [\wp^{'}]^{\bullet}$ where $\wp^{'} \in V$.*

Note that every parapseudo-complemented PDL is a $\bullet$-PDL. In the following we give an example of a $\bullet$-PDL.

Example 2. *Let $A = \{1, \wp_1, \wp_2\}$ and $B = \{1, \hbar\}$ are two disconnected PDLs. Let $V = A \times B = \{(1,1), (1,\hbar), (\wp_1, 1), (\wp_1, \hbar), (\wp_2, 1), (\wp_2, \hbar)\}$. Define $\vee$ and $\wedge$ on V under point-wise:*

$\vee$	$(1,1)$	$(1,\hbar)$	$(\wp_1,1)$	$(\wp_1,\hbar)$	$(\wp_2,1)$	$(\wp_2,\hbar)$
$(1,1)$	$(1,1)$	$(1,1)$	$(1,1)$	$(1,1)$	$(1,1)$	$(1,1)$
$(1,\hbar)$	$(1,1)$	$(1,\hbar)$	$(1,1)$	$(1,\hbar)$	$(1,1)$	$(1,\hbar)$
$(\wp_1,1)$	$(1,1)$	$(1,1)$	$(\wp_1,1)$	$(\wp_1,1)$	$(\wp_1,1)$	$(\wp_1,1)$
$(\wp_1,\hbar)$	$(1,1)$	$(1,\hbar)$	$(\wp_1,1)$	$(\wp_1,\hbar)$	$(\wp_1,1)$	$(\wp_1,\hbar)$
$(\wp_2,1)$	$(1,1)$	$(1,1)$	$(\wp_2,1)$	$(\wp_2,1)$	$(\wp_2,1)$	$(\wp_2,1)$
$(\wp_2,\hbar)$	$(1,1)$	$(1,\hbar)$	$(\wp_2,1)$	$(\wp_2,\hbar)$	$(\wp_2,1)$	$(\wp_2,\hbar)$

$\wedge$	$(1,1)$	$(1,\hbar)$	$(\wp_1,1)$	$(\wp_1,\hbar)$	$(\wp_2,1)$	$\wp_2,\hbar)$
$(1,1)$	$(1,1)$	$(1,\hbar)$	$(\wp_1,1)$	$(\wp_2,\hbar)$	$(\wp_2,1)$	$(\wp_2,\hbar)$
$(1,\hbar)$	$(1,\hbar)$	$(1,\hbar)$	$(\wp_1,\hbar)$	$(\wp_1,\hbar)$	$(\wp_2,\hbar)$	$(\wp_2,\hbar)$
$(\wp_1,1)$	$(\wp_1,1)$	$(\wp_1,\hbar)$	$(\wp_1,1)$	$(\wp_1,\hbar)$	$(\wp_2,1)$	$(\wp_2,\hbar)$
$(\wp_1,\hbar)$	$(\wp_1,\hbar)$	$(\wp_1,\hbar)$	$(\wp_1,\hbar)$	$(\wp_1,\hbar)$	$(\wp_2,\hbar)$	$(\wp_2,\hbar)$
$(\wp_2,1)$	$(\wp_2,1)$	$(\wp_2,\hbar)$	$(\wp_1,1)$	$(\wp_1,\hbar)$	$(\wp_2,1)$	$(\wp_2,\hbar)$
$(\wp_2,\hbar)$	$(\wp_2,\hbar)$	$(\wp_2,\hbar)$	$(\wp_1,\hbar)$	$(\wp_1,\hbar)$	$(\wp_2,\hbar)$	$(\wp_2,\hbar)$

Then $(V, \vee, \wedge, 1^{'})$ *is a PDL where* $(1,1) = 1^{'}$*. Now, we verify that this is a* $\bullet$*-PDL.*

(i) $[(1,1)]^{\bullet} = V \Rightarrow [(1,1)]^{\bullet^{\bullet}} = V^{\bullet} = \{(1,1)\} = [(\wp_2, \hbar)]^{\bullet}$.

(ii) $[(1,\hbar)]^{\bullet} = \{(1,1),(\wp_1,1),(\wp_2,1)\} \Rightarrow [(1,\hbar)]^{\bullet^{\bullet}} = \{(1,1),(\wp_1,1),(\wp_2,1)\}^{\bullet} = \{(1,1),(1,\hbar)\} = [(\wp_1,1)]^{\bullet}$.

(iii) $[(\wp_1,1)]^{\bullet} = \{(1,1),(1,\hbar)\} \Rightarrow [(\wp_1,1)]^{\bullet^{\bullet}} = \{(1,1),(1,\hbar)\}^{\bullet} = \{(1,1),(\wp_1,1),(\wp_2,1)\} = [(1,\hbar)]^{\bullet}$.

(iv) $[(\wp_1,\hbar)]^{\bullet} = \{(1,1)\} \Rightarrow [(\wp_1,\hbar)]^{\bullet^{\bullet}} = V = [(1,1)]^{\bullet}$.

(v) $[(\wp_2,1)]^{\bullet} = \{(1,1),(1,\hbar)\} \Rightarrow [(\wp_2,1)]^{\bullet^{\bullet}} = \{(1,1),(1,\hbar)\}^{\bullet} = \{(1,1),(\wp_1,1),(\wp_2,1)\} = [(1,\hbar)]^{\bullet}$.

(vi) $[(\wp_2,\hbar)]^{\bullet} = \{(1,1)\} \Rightarrow [(\wp_2,\hbar)]^{\bullet^{\bullet}} = V = [(1,1)]^{\bullet}$.

Thus $(V, \vee, \wedge, 1')$ *is a* $\bullet$*-PDL.*

Lemma 7. *Let* V *be a* $\bullet$*-PDL and* $\wp \in V$*. Then* $[\wp]^{\bullet} \cap [\wp]^{\bullet^{\bullet}} = \{1\}$*.*

Proof. Let $\wp \in V$. Then $[\wp]^{\bullet^{\bullet}} = [\hbar]^{\bullet}$ for some $\hbar \in V$. Let $\mu_1 \in [\wp]^{\bullet}$ and $\mu_1 \in [\wp]^{\bullet^{\bullet}}$. Since $\mu_1 \vee \mu_1 = 1$, we have $\mu_1 = 1$. Therefore $[\wp]^{\bullet} \cap [\wp]^{\bullet^{\bullet}} = \{1\}$.

Lemma 8. *Let* V *be a PDL. Then the relation* $\theta = \{(\wp, \hbar) \in V \times V \mid [\wp]^{\bullet} = [\hbar]^{\bullet}\}$ *is a congruence relation on* V*.*

Proof. Clearly θ is an equivalence relation on V. Let $(\wp, \hbar) \in \theta$ and $\mu_1 \in V$. Then $[\wp]^{\bullet} = [\hbar]^{\bullet}$. Now $[\wp \wedge \mu_1]^{\bullet} = [\wp]^{\bullet} \cap [\mu_1]^{\bullet} = [\hbar]^{\bullet} \cap [\mu_1]^{\bullet} = [\hbar \wedge \mu_1]^{\bullet}$. Therefore $(\wp \wedge \mu_1, \hbar \wedge \mu_1) \in \theta$. Let $\tau \in [\wp \vee \mu_1]^{\bullet}$. Then $\tau \vee \wp \vee \mu_1 = 1$ and hence $\tau \vee \mu_1 \vee \wp = 1$. Therefore $\tau \vee \mu_1 \in [\wp]^{\bullet}$ and hence $\tau \vee \mu_1 \in [\hbar]^{\bullet}$. So that $\tau \vee \mu_1 \vee \hbar = 1$ which implies that $\tau \vee \hbar \vee \mu_1 = 1$. Hence $\tau \in [\hbar \vee \mu_1]^{\bullet}$, we get $[\wp \vee \mu_1]^{\bullet} \subseteq [\hbar \vee \mu_1]^{\bullet}$. Similarly, we get that $[\hbar \vee \mu_1]^{\bullet} \subseteq [\wp \vee \mu_1]^{\bullet}$. Thus $[\wp \vee \mu_1]^{\bullet} = [\hbar \vee \mu_1]^{\bullet}$. Therefore $(\wp \vee \mu_1, \hbar \vee \mu_1) \in \theta$. Hence θ is a congruence relation on V.

Note that, by the above Lemma 8, V/θ is PDL under the induced operations $\vee$ and $\wedge$ in V and is defined as follows: For any $\wp/\theta, \hbar/\theta \in V/\theta$,
(i) $\wp/\theta \vee \hbar/\theta = (\wp \vee \hbar)/\theta$.
(ii) $\wp/\theta \wedge \hbar/\theta = (\wp \wedge \hbar)/\theta$.

Theorem 6. *Let* V *be a PDL with a minimal element* m*. Then* V/θ *is distributive lattice with the greatest element as* $1/\theta$*, and* m/θ *as the least element.*

Proof. Let $\wp/\theta, \hbar/\theta \in V/\theta$. Since $[\wp \vee \hbar]^{\bullet} = [\hbar \vee \wp]^{\bullet}$, we have $(\wp \vee \hbar, \hbar \vee \wp) \in \theta$. Therefore $(\wp \vee \hbar)/\theta = (\hbar \vee \wp)/\theta$ and hence $\wp/\theta \vee \hbar/\theta = \hbar/\theta \vee \wp/\theta$. Thus V/θ is a distributive lattice. Clearly $1/\theta$ is a greatest element. Suppose m is a minimal element of V. Then for any $\wp \in V$, $m/\theta \wedge \wp/\theta = \wp/\theta \wedge m/\theta = (\wp \wedge m)/\theta = m/\theta$. Therefore m/θ is the least element of V/θ.

Note that $[1]^{\bullet} = \{\wp \in V \mid \wp \vee 1 = 1\} = V$.

Lemma 9. *Let V be a PDL. Then $[\wp]^{\bullet} = [1]^{\bullet} \Leftrightarrow \wp = 1$.*

Proof. Suppose $\wp = 1$. Then $[\wp]^{\bullet} = [1]^{\bullet}$. Conversely, assume that $\wp \in V$ and $[\wp]^{\bullet} = [1]^{\bullet}$. Then $\wp \vee 1 = 1$ implies that $\wp \in [1]^{\bullet} = [\wp]^{\bullet}$. Hence $\wp = \wp \vee \wp = 1$.

Now we prove the following:

Theorem 7. *Let V be a PDL. If V/θ is a Boolean algebra then V is a $\bullet$-PDL.*

Proof. Let V/θ be Boolean algebra and $\wp \in V$. Then $\wp/\theta \in V/\theta$. Hence, there exists $\hbar/\theta \in V/\theta$ such that $\wp/\theta \vee \hbar/\theta = 1/\theta$ and $\wp/\theta \wedge \hbar/\theta = m/\theta$ where m/θ and $1/\theta$ are least and greatest elements in V/θ respectively. Now $\wp/\theta \wedge \hbar/\theta = (\wp \wedge \hbar)/\theta = m/\theta \Rightarrow (\wp \wedge \hbar, m) \in \theta$ so that $[\wp \wedge \hbar]^{\bullet} = [\wp]^{\bullet} \cap [\hbar]^{\bullet} = [m]^{\bullet}$ which is true for all $\wp \in V$. Hence $[m]^{\bullet} \subseteq [\wp]^{\bullet}$ for all $\wp \in V$. Let $\mu_1 \in [m]^{\bullet}$. Then $\mu_1 \in [\mu_1]^{\bullet}$. Therefore $\mu_1 = 1$. Hence $[m]^{\bullet} = \{1\}$. We get $[\wp]^{\bullet} \cap [\hbar]^{\bullet} = \{1\}$. Also, $\wp/\theta \vee \hbar/\theta = (\wp \vee \hbar)/\theta = 1/\theta$. So that $[\wp \vee \hbar]^{\bullet} = [1]^{\bullet} = V$. Therefore $\wp \vee \wp \vee \hbar = 1$ implies that $\wp \vee \hbar = 1$. Hence $\hbar \in [\wp]^{\bullet}$, we get $[\wp]^{\bullet\bullet} \subseteq [\hbar]^{\bullet}$. Let $\tau \in [\hbar]^{\bullet}$ and $\partial_4 \in [\wp]^{\bullet}$. Then $\tau \vee \hbar = 1$ and $\partial_4 \vee \wp = 1$. Now $\tau \vee \partial_4 \in [\wp]^{\bullet}$ and $\tau \vee \partial_4 \in [\hbar]^{\bullet}$. Therefore $\tau \vee \partial_4 \in [\wp]^{\bullet} \cap [\hbar]^{\bullet} = \{1\}$. Hence $\tau \vee \partial_4 = 1$. Thus $\tau \in [\wp]^{\bullet\bullet}$, we get $[\hbar]^{\bullet} \subseteq [\wp]^{\bullet\bullet}$. Therefore $[\wp]^{\bullet\bullet} = [\hbar]^{\bullet}$. Hence V is a $\bullet$-PDL.

Observe that the converse of the above theorem is true if V has a minimal element which we prove in the following:

Theorem 8. *Let V be a PDL with a minimal element m. If V is a $\bullet$-PDL, then V/θ is a Boolean algebra.*

Proof. Clearly $(V/\theta, \vee, \wedge)$ is a distributive lattice with the least element m/θ and the greatest element $1/\theta$. Now, we prove that V/θ is complemented. Let $\wp/\theta \in V/\theta$. Then there exists $\hbar \in V$ such that $[\wp]^{\bullet\bullet} = [\hbar]^{\bullet}$. Now, $[\wp \vee \hbar]^{\bullet\bullet} = [\wp]^{\bullet\bullet} \cap [\hbar]^{\bullet\bullet} = [\hbar]^{\bullet} \cap [\hbar]^{\bullet\bullet} = \{1\}$ which implies that $[\wp \vee \hbar]^{\bullet} = [1]^{\bullet}$ and hence $\wp/\theta \vee \hbar/\theta = 1/\theta$. Also, $[\wp \wedge \hbar]^{\bullet} = [\wp]^{\bullet} \cap [\hbar]^{\bullet} = [\wp]^{\bullet} \cap [\wp]^{\bullet\bullet} = \{1\} = [m]^{\bullet}$. Therefore $(\wp \wedge \hbar)/\theta = m/\theta \Rightarrow \wp/\theta \wedge \hbar/\theta = m/\theta$. Hence V/θ is a Boolean algebra.

4. Dense Elements in a PDL

In this section, we state the definition of a dense element in a PDL V and prove that the set of all dense elements is an ideal of V. We characterize $\bullet$-PDL V in terms of dense elements. Further, we introduce the concept of a disjunctive PDL as a PDL V in which for any $\wp, \hbar \in V$, $[\wp]^{\bullet} = [\hbar]^{\bullet}$ implies $\wp = \hbar$ and prove that, $\bullet$-PDL is disjunctive if and only if V is a Boolean algebra.

Definition 9. *An element $\wp$ of a PDL V is called a dense element if, $[\wp]^{\bullet} = \{1\}$.*

Lemma 10. *Every $\bullet$-PDL contains a dense element.*

Proof. Let V be a $\bullet$-PDL and $\wp \in V$. Then there exists $\hbar \in V$ such that $[\wp]^{\bullet\bullet} = [\hbar]^{\bullet}$. Now, $[\wp \wedge \hbar]^{\bullet} = [\wp]^{\bullet} \cap [\hbar]^{\bullet} = \{1\}$. Hence $\wp \wedge \hbar$ is a dense element.

Corollary 1. *In a $\bullet$-PDL, every element of $[1]^{\bullet}$ is a dense element.*

Lemma 11. *In a PDL, every minimal element is dense element.*

Proof. *Let m be a minimal element in V and $\wp \in [m]^{\bullet}$. Then $\wp \vee m = 1$ and hence $\wp = 1$. Therefore $[m]^{\bullet} = \{1\}$. Thus m is dense element.*

Lemma 12. *Let V be a PDL. Then, the set of all dense elements of V is an ideal of V.*

Proof. *Let $D = \{\mu_1 \in V \mid [\mu_1]^{\bullet} = \{1\}\}$ and $\wp, \hbar \in D$. Then $[\wp]^{\bullet} = \{1\}$ and $[\hbar]^{\bullet} = \{1\}$. Let $\tau \in [\wp \vee \hbar]^{\bullet}$. Then $\tau \vee \wp \vee \hbar = 1 \Rightarrow \tau \vee \wp \in [\hbar]^{\bullet} = \{1\} \Rightarrow \tau \vee \wp = 1 \Rightarrow \tau \in [\wp]^{\bullet} = \{1\} \Rightarrow \tau = 1$. Therefore, $[\wp \vee \hbar]^{\bullet} = \{1\}$. So that $\wp \vee \hbar \in D$. Now, let $\wp \in D$ and $\mu_1 \in V$. Then $[\wp]^{\bullet} = \{1\}$. If $\tau \in [\wp \wedge \mu_1]^{\bullet}$, then $\tau \vee (\wp \wedge \mu_1) = 1 \Rightarrow (\tau \vee \wp) \wedge (\tau \vee \mu_1) = 1 \Rightarrow \tau \vee \wp = 1$ and $\tau \vee \mu_1 = 1 \Rightarrow \tau = 1$. Therefore $[\wp \wedge \mu_1]^{\bullet} = \{1\}$. Hence $\wp \wedge \mu_1 \in D$. Thus D is an ideal of V.*

Lemma 13. *Let V be a PDL. Then, for any $\wp, \hbar \in V$,*
(1). $\wp \vee \hbar \in D$ if and only if $\hbar \vee \wp \in D$.
(2). $\wp \wedge \hbar \in D$ if and only if $\hbar \wedge \wp \in D$.

Proof. (1). Let $\wp, \hbar \in V$ and $\wp \vee \hbar \in D$. Then $[\hbar \vee \wp]^{\bullet} = [\wp \vee \hbar]^{\bullet} = \{1\}$ and hence $\hbar \vee \wp \in D$. Conversely, assume that $\hbar \vee \wp \in D$. Then $[\wp \vee \hbar]^{\bullet} = [\hbar \vee \wp]^{\bullet} = \{1\}$ and hence $\wp \vee \hbar \in D$.
(2). Let $\wp, \hbar \in V$ and $\wp \wedge \hbar \in D$. Then $\hbar \wedge \wp = (\wp \wedge \hbar) \wedge (\hbar \wedge \wp) \in D$ since $\wp \wedge \hbar = (\wp \wedge \hbar) \vee (\hbar \wedge \wp)$ and D is an ideal of V. Conversely, assume that $\hbar \wedge \wp \in D$. Then $\wp \wedge \hbar = (\hbar \wedge \wp) \wedge (\wp \wedge \hbar) \in D$.

Lemma 14. *Let V be a PDL and $\wp, \hbar \in V$. If $\hbar \in [\wp]^{\bullet}$ then $[\wp]^{\bullet\bullet} \subseteq [\hbar]^{\bullet}$.*

Proof. Assume that $\hbar \in [\wp]^{\bullet}$ and $\mu_1 \in [\wp]^{\bullet\bullet}$. Then $\mu_1 \vee \mu_2 = 1$ for all $\mu_2 \in [\wp]^{\bullet}$. Hence $\mu_1 \vee \hbar = 1$. So that $\mu_1 \in [\hbar]^{\bullet}$. Therefore $[\wp]^{\bullet\bullet} \subseteq [\hbar]^{\bullet}$.

Theorem 9. *Let V be a PDL. Then V be a $\bullet$-PDL if and only if for any $\wp \in V$, there exists $\hbar \in V$ such that $\wp \vee \hbar = 1$ and $\wp \wedge \hbar$ is dense element.*

Proof. Suppose V is a $\bullet$-PDL and $\wp \in V$. Then there exists $\hbar \in V$ such that $[\wp]^{\bullet\bullet} = [\hbar]^{\bullet}$. Hence $\wp \wedge \hbar$ is a dense element. Also, $\wp \in [\wp]^{\bullet\bullet} = [\hbar]^{\bullet}$ implies that $\wp \vee \hbar = 1$. Conversely, assume that the condition holds. We prove that V is a $\bullet$-PDL. Let $\wp \in V$. Then there exists $\hbar \in V$ such that $\wp \vee \hbar = 1$ and $\wp \wedge \hbar$ is a dense element. Since $\wp \vee \hbar = 1$, we have $\hbar \in [\wp]^{\bullet}$. Therefore $[\wp]^{\bullet\bullet} \subseteq [\hbar]^{\bullet}$. Now, let $\tau \in [\hbar]^{\bullet}$ and $\mu_1 \in [\wp]^{\bullet}$. Then $\tau \vee \hbar = 1$ and $\mu_1 \vee \wp = 1$. Now $(\tau \vee \mu_1) \vee (\wp \wedge \hbar) = (\tau \vee \mu_1 \vee \wp) \wedge (\tau \vee \mu_1 \vee \hbar) = 1 \wedge 1 = 1$. Hence $\tau \vee \mu_1 \in [\wp \wedge \hbar]^{\bullet} = \{1\}$. Thus $\tau \vee \mu_1 = 1$, this is true for all $\mu_1 \in [\wp]^{\bullet}$. We get $\tau \in [\wp]^{\bullet\bullet}$. Hence $[\hbar]^{\bullet} \subseteq [\wp]^{\bullet\bullet}$. Therefore $[\wp]^{\bullet\bullet} = [\hbar]^{\bullet}$. Thus V is a $\bullet$-PDL.

Corollary 2. *Let V be a PDL. If for each $\wp \in V$, there is an element $\hbar \in V$ such that $\wp \vee \hbar = 1$ and $\wp \wedge \hbar$ is a minimal element, then V is a $\bullet$-PDL.*

Theorem 10. *Let V be $\bullet$-PDL. Then the following conditions are equivalent:*
(1). *For each $\wp \in V$, there is $\hbar \in V$ such that $\wp \vee \hbar = 1$ and $\wp \wedge \hbar$ is a minimal element.*
(2). *Every dense element of V is a minimal element.*

Proof. (1) $\Rightarrow$ (2): Assume (1). Let $\wp \in V$ be a dense element. Then by the assumption, there exists $\hbar \in V$ such that $\wp \vee \hbar = 1$ and $\wp \wedge \hbar$ is a minimal element. Now $\wp \vee \hbar = 1$ implies that $\hbar \in [\wp]^{\bullet} = \{1\}$. Therefore $\hbar = 1$. Now $\wp = \wp \wedge 1 = \wp \wedge \hbar$ is a minimal element in V.
(2) $\Rightarrow$ (1): Assume (2). Let $\wp \in V$. Then there exists $\hbar \in V$ such that $\wp \vee \hbar = 1$ and $\wp \wedge \hbar$ is dense element by Theorem 9. Then, by assumption, $\wp \wedge \hbar$ is minimal element.

Theorem 11. *Let V be a $\bullet$-PDL and D the set of all dense elements of V. Then $[\mu_4 \vee \mu_1]^{\bullet} = [\mu_4 \vee \mu_2]^{\bullet}$ for some $\mu_4 \in D$ if and only if $[\mu_1]^{\bullet} = [\mu_2]^{\bullet}$.*

Proof. Suppose $[\mu_4 \vee \mu_1]^{\bullet} = [\mu_4 \vee \mu_2]^{\bullet}$ for some $\mu_4 \in D$. Then $[\mu_4]^{\bullet} = \{1\}$, we get $[\mu_4]^{\bullet^{\bullet}} = \{1\}^{\bullet} = V$. Now $[\mu_1]^{\bullet^{\bullet}} = [\mu_4]^{\bullet^{\bullet}} \cap [\mu_1]^{\bullet^{\bullet}} = [\mu_4 \vee \mu_1]^{\bullet^{\bullet}} = [\mu_4 \vee \mu_2]^{\bullet^{\bullet}} = [\mu_4]^{\bullet^{\bullet}} \cap [\mu_2]^{\bullet^{\bullet}} = [\mu_2]^{\bullet^{\bullet}}$. Therefore $[\mu_1]^{\bullet^{\bullet}} = [\mu_2]^{\bullet^{\bullet}}$. Hence $[\mu_1]^{\bullet} = [\mu_2]^{\bullet}$. Conversely assume that $[\mu_1]^{\bullet} = [\mu_2]^{\bullet}$. Since V is a $\bullet$-PDL and $\mu_1 \in V$, there exists $\mu_1^{'} \in V$ such that $[\mu_1]^{\bullet^{\bullet}} = [\mu_1^{'}]^{\bullet}$. Also $[\mu_1]^{\bullet} = [\mu_2]^{\bullet}$ implies that $[\mu_1]^{\bullet} = [\mu_1 \wedge \mu_2]^{\bullet}$. Now if we write $\mu_4 = (\mu_1 \wedge \mu_2) \wedge \mu_1^{'}$ then,

$$\begin{aligned} [\mu_4]^{\bullet} &= [(\mu_1 \wedge \mu_2)]^{\bullet} \cap [\mu_1^{'}]^{\bullet} \\ &= [\mu_1]^{\bullet} \cap [\mu_1^{'}]^{\bullet} \\ &= [\mu_1]^{\bullet} \cap [\mu_1]^{\bullet^{\bullet}} \\ &= \{1\} \end{aligned}$$

Therefore $\mu_4 \in D$. Now $\mu_1 \vee \mu_4 = \mu_1 \vee ((\mu_1 \wedge \mu_2) \wedge \mu_1^{'}) = \mu_1 \wedge (\mu_1 \vee \mu_1^{'}) = \mu_1$.
Also $\mu_2 \vee \mu_4 = \mu_2 \vee ((\mu_1 \wedge \mu_2) \wedge \mu_1^{'}) = (\mu_2 \vee (\mu_1 \wedge \mu_2)) \wedge (\mu_2 \vee \mu_1^{'}) = \mu_2 \wedge (\mu_2 \vee \mu_1^{'}) = \mu_2$.
Thus $[\mu_1 \vee \mu_4]^{\bullet} = [\mu_2 \vee \mu_4]^{\bullet}$ which implies $[\mu_4 \vee \mu_1]^{\bullet} = [\mu_4 \vee \mu_2]^{\bullet}$.

Lemma 15. *Let V be a PDL. For any $\mu_1 \in V$, $(\mu_1 : D) = \{\wp \in V \mid \wp \wedge \mu_1 \in D\}$ is an ideal of V.*

Proof. Let $\wp, \hbar \in (\mu_1 : D)$. Then $\wp \wedge \mu_1 \in D$, $\hbar \wedge \mu_1 \in D$. Therefore $(\wp \wedge \mu_1) \vee (\hbar \wedge \mu_1) \in D$. So that $(\wp \vee \hbar) \wedge \mu_1 \in D$. Hence $\wp \vee \hbar \in (\mu_1 : D)$. Now, let $\wp \in (\mu_1 : D)$ and $\tau \in V$. Then, $\wp \wedge \mu_1 \in D$ and $[\wp \wedge \mu_1]^{\bullet} = \{1\}$. Hence

$$\begin{aligned} [(\wp \wedge \tau) \wedge \mu_1]^{\bullet} &= [\wp \wedge \tau]^{\bullet} \cap [\mu_1]^{\bullet} \\ &= [\wp]^{\bullet} \cap [\tau]^{\bullet} \cap [\mu_1]^{\bullet} \\ &= [\tau]^{\bullet} \cap [\wp]^{\bullet} \cap [\mu_1]^{\bullet} \\ &= [\tau]^{\bullet} \cap [\wp \wedge \mu_1]^{\bullet} \\ &= \{1\} \end{aligned}$$

Therefore $(\wp \wedge \tau) \wedge \mu_1 \in D \Rightarrow \wp \wedge \tau \in (\mu_1 : D)$. Thus $(\mu_1 : D)$ is an ideal of V.

Theorem 12. *Let V be a $\bullet$-PDL. Then $\theta = \{(\mu_1, \mu_2) \in V \times V \mid (\mu_1 : D) = (\mu_2 : D)\}$, where $\theta = \{(\wp, \hbar) \in V \times V \mid [\wp]^\bullet = [\hbar]^\bullet\}$.*

Proof. Let $(\mu_1, \mu_2) \in \theta$. Then $[\mu_1]^\bullet = [\mu_2]^\bullet$. Now, for any $\wp \in V$

$$\begin{aligned} \wp \in (\mu_1 : D) &\Leftrightarrow \wp \wedge \mu_1 \in D \\ &\Leftrightarrow [\wp \wedge \mu_1]^\bullet = \{1\} \\ &\Leftrightarrow [\wp]^\bullet \cap [\mu_1]^\bullet = \{1\} \\ &\Leftrightarrow [\wp]^\bullet \cap [\mu_2]^\bullet = \{1\} \\ &\Leftrightarrow [\wp \wedge \mu_2]^\bullet = \{1\} \\ &\Leftrightarrow \wp \wedge \mu_2 \in D \\ &\Leftrightarrow \wp \in (\mu_2 : D) \end{aligned}$$

Therefore $(\mu_1 : D) = (\mu_2 : D)$.
On the other hand, let $\mu_1, \mu_2 \in V$ and $(\mu_1 : D) = (\mu_2 : D)$. Since V is a $\bullet$-PDL and $\mu_1 \in V$, there exists $\mu_3 \in V$ such that $[\mu_1]^{\bullet\bullet} = [\mu_3]^\bullet$. So that $\mu_1 \wedge \mu_3 \in D$ and hence $\mu_3 \wedge \mu_1 \in D$. Therefore $\mu_3 \in (\mu_1 : D)$. Hence $\mu_3 \in (\mu_2 : D)$. We get $\mu_3 \wedge \mu_2 \in D$. Therefore $[\mu_3 \wedge \mu_2]^\bullet = \{1\}$ and hence $[\mu_3]^\bullet \cap [\mu_2]^\bullet = \{1\}$. Thus $[\mu_1]^{\bullet\bullet} \cap [\mu_2]^\bullet = 1$. Now, let $\wp \in [\mu_2]^\bullet$. Then $\mu_1 \vee \wp \in [\mu_2]^\bullet$. Also $\mu_1 \in [\mu_1]^{\bullet\bullet}$ and hence $\mu_1 \vee \wp \in [\mu_1]^{\bullet\bullet}$. Thus $\mu_1 \vee \wp = 1$, we get $\wp \in [\mu_1]^\bullet$. Thus $[\mu_2]^\bullet \subseteq [\mu_1]^\bullet$. Similarly, we get $[\mu_1]^\bullet \subseteq [\mu_2]^\bullet$. Hence $[\mu_1]^\bullet = [\mu_2]^\bullet$ which implies $(\mu_1, \mu_2) \in \theta$. Therefore $\theta = \{(\mu_1, \mu_2) \in V \times V \mid (\mu_1 : D) = (\mu_2 : D)\}$.

Lemma 16. *Let V be a PDL. If $\mu_1 \vee \mu_3 = 1 = \mu_2 \vee \mu_3$ and $\mu_1 \wedge \mu_2$ is a dense element. Then $\mu_3 = 1$.*

Proof. Let $\mu_1 \vee \mu_3 = 1 = \mu_2 \vee \mu_3$. Then $(\mu_1 \wedge \mu_2) \vee \mu_3 = 1$. Therefore $\mu_3 \in [\mu_1 \wedge \mu_2]^\bullet = \{1\}$ and hence $\mu_3 = 1$.

Lemma 17. *Let V be a PDL. If $[\mu_1]^{\bullet\bullet} = [\mu_3]^\bullet$ and $[\mu_2]^{\bullet\bullet} = [\mu_4]^\bullet$ for some $\mu_1, \mu_2, \mu_3, \mu_4 \in V$, then $\mu_1 \vee \mu_2 = 1$ if and only if $\mu_3 \wedge \mu_4$ is dense element.*

Proof. Assume $[\mu_1]^{\bullet\bullet} = [\mu_3]^\bullet$ and $[\mu_2]^{\bullet\bullet} = [\mu_4]^\bullet$ for some $\mu_1, \mu_2, \mu_3, \mu_4 \in V$. Suppose $\mu_1 \vee \mu_2 = 1$. Then

$$\begin{aligned} [\mu_3 \wedge \mu_4]^\bullet &= [\mu_3]^\bullet \cap [\mu_4]^\bullet \\ &= [\mu_1]^{\bullet\bullet} \cap [\mu_2]^{\bullet\bullet} \\ &= [\mu_1 \vee \mu_2]^{\bullet\bullet} \\ &= [1]^{\bullet\bullet} \\ &= \{1\} \end{aligned}$$

Therefore $\mu_3 \wedge \mu_4$ is dense element. Conversely, assume that $\mu_3 \wedge \mu_4$ is a dense element. Then, $[\mu_3 \wedge \mu_4]^\bullet = \{1\}$. Clearly $\mu_1 \in [\mu_3]^\bullet$ and $\mu_2 \in [\mu_4]^\bullet$. Then $\mu_1 \vee \mu_3 = 1$ and $\mu_2 \vee \mu_4 = 1$. Therefore $\mu_1 \vee \mu_2 \vee \mu_3 = \mu_1 \vee \mu_3 \vee \mu_2 = 1 \vee \mu_2 = 1$, also $\mu_1 \vee \mu_2 \vee \mu_4 = \mu_1 \vee 1 = 1$. Therefore $(\mu_1 \vee \mu_2) \vee \mu_3 = (\mu_1 \vee \mu_2) \vee \mu_4$. Hence $\mu_3 \vee (\mu_1 \vee \mu_2) = \mu_4 \vee (\mu_1 \vee \mu_2) = 1$ which implies that $\mu_1 \vee \mu_2 \in [\mu_3]^\bullet \cap [\mu_4]^\bullet = [\mu_3 \wedge \mu_4]^\bullet = \{1\}$. Therefore $\mu_1 \vee \mu_2 = 1$.

Theorem 13. *Let V be a $\bullet$-PDL and $\wp, \hbar \in V$. Then the following conditions are equivalent:*
(1). $[\wp]^\bullet = [\hbar]^\bullet$.
(2). *For any $\tau \in V$, $\wp \wedge \tau$ is dense if and only if $\hbar \wedge \tau$ is dense.*

Proof. (1) $\Rightarrow$ (2) is clear.
(2) $\Rightarrow$ (1): Assume (2). Let $\mu_1 \in [\wp]^\bullet$. Then $\mu_1 \vee \wp = 1$. Now, for $\mu_1, \wp, \hbar \in V$ there exists $\mu_3, u, v \in V$ such that $[\mu_1]^{\bullet^\bullet} = [\mu_3]^\bullet$, $[\wp]^{\bullet^\bullet} = [u]^\bullet$, $[\hbar]^{\bullet^\bullet} = [v]^\bullet$. Since $\mu_1 \vee \wp = 1$, we have $\mu_3 \wedge u$ is dense element. Also $\hbar \vee v = 1$ and $[\hbar \wedge v]^\bullet = [\hbar]^\bullet \cap [\hbar]^{\bullet^\bullet} = \{1\}$. Hence $\hbar \wedge v$ is dense element. But, by the assumption, $\wp \wedge v$ is dense, so it is left to prove that $\mu_3 \wedge v$ is a dense element. Let $\mu_2 \in [\mu_3 \wedge v]^\bullet = [\mu_3]^\bullet \cap [v]^\bullet$. Then $\mu_2 \vee v = 1$. Now, $\wp \vee \mu_2 \vee u = 1 = v \vee \mu_2 \vee u$ and $\wp \wedge v$ is dense element implies that $\mu_2 \vee u = 1$. Now, $\mu_2 \vee u = 1 = \mu_2 \vee \mu_3$ and $\mu_3 \wedge u$ is dense element implies that $\mu_2 = 1$. Therefore $\mu_3 \wedge v$ is dense element. Again $[\mu_1]^{\bullet^\bullet} = [\mu_3]^\bullet$ and $[\hbar]^{\bullet^\bullet} = [v]^\bullet$ and $\mu_3 \wedge v$ is dense element implies that $\mu_1 \vee \hbar = 1$. Therefore $\mu_1 \in [\hbar]^{\bullet^\bullet}$. Hence $[\wp]^\bullet \subseteq [\hbar]^\bullet$. Similarly, it can be proved that $[\hbar]^\bullet \subseteq [\wp]^\bullet$. Thus $[\wp]^\bullet = [\hbar]^\bullet$.

Definition 10. *A PDL $(V, \vee, \wedge, 1)$ is called a disjunctive PDL, if for each $\wp, \hbar \in V$, $[\wp]^\bullet = [\hbar]^\bullet \Rightarrow \wp = \hbar$.*

Theorem 14. *Let $(V, \vee, \wedge)$ be a $\bullet$-PDL. Then the following conditions are equivalent:*
(1). *$(V, \vee, \wedge, 1)$ is a Boolean algebra.*
(2). *V is a disjunctive PDL.*
(3). *V has exactly one dense element.*

Proof. (1) $\Rightarrow$ (2). Assume $(V, \vee, \wedge, 1)$ is a Boolean algebra. Let $\wp, \hbar \in V$ and $[\wp]^\bullet = [\hbar]^\bullet$. Then there exists $\wp' \in V$ such that $\wp' \vee \wp = 1$ and hence $\wp' \in [\wp]^\bullet = [\hbar]^\bullet$. Therefore $\wp' \vee \hbar = 1$. Hence $\wp \leq \hbar$. By symmetry, $\hbar \leq \wp$. Hence $\wp = \hbar$. Therefore V is a disjunctive PDL.
(2) $\Rightarrow$ (3). Assume V is a disjunctive PDL. Since V is $\bullet$-PDL, V has a dense element. Suppose V has two dense elements say $\wp$ and $\hbar$. Then $[\wp]^\bullet = \{1\} = [\hbar]^\bullet$. Hence $\wp = \hbar$.
(3) $\Rightarrow$ (1). Assume V has exactly one dense element. Since V is a $\bullet$-PDL, choose $1' \in V$ such that $[1]^{\bullet^\bullet} = [1']^\bullet$. That is $[1']^\bullet = \{1\}$. Hence, by (3), $1'$ is the only dense element in V. Let $\wp \in V$. Then there exists $\wp'$ in V such that $[\wp]^{\bullet^\bullet} = [\wp']^\bullet$. Hence $\wp \vee \wp' = 1$ and $\wp \wedge \wp'$ is dense element by Lemma 10. Therefore $\wp \wedge \wp' = 1'$ which is true for all $\wp \in V$ and we have $\wp' \wedge \wp = 1' \leq \wp$. Therefore V is a bounded distributive lattice in which every element has a complement. Thus $(V, \vee, \wedge, 1', 1)$ is a Boolean algebra.

5. Conclusions

In conclusions, this paper presents the concept of $\bullet$- PDL, dense elements on a PDL. By establishing necessary conditions, we introduced various results related to this structure. Furthermore, our study focussed on providing certain characterization theorems for $\bullet$-PDL and proved equivalence conditions for V to be $\bullet$- PDL if and only if every V/θ is

Boolean algebra. Lastly, concluded concepts with showing that the V is disjunctive if and only if V is Boolean algebra.

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