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Quasi Ruled Surfaces in Euclidean 3-space

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Abstract. This paper introduces three distinct types of ruled surfaces, namely, the quasi-tangent surfaces , the quasi-normal surfaces, and the quasi-binormal surfaces. These types are determined by the orientation of their direction curves tangent, normal, and binormal to the base curve, respectively. This paper does not only introduce these surfaces but also determines their fundamental properties, including the first, the second, and the third fundamental forms, as well as the Gaussian and the Mean curvatures. Also, the geodesic curvature, the normal curvature, and the geodesic torsion associated with the base curve for each type of surface are investigated. Furthermore, the conditions for the base curve to be as a geodesic, an asymptotic line, and a principal line for each type of surface are provided. Also, the conditions for these curves to be considered developable and minimal surfaces are introduced. Moreover, two illustrative examples are introduced to obtain our results.

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Key Words and Phrases: Ruled surface, Euclidean space, Quasi frame, developable surface, minimal surface

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1. Introduction

The study of ruled surfaces has garnered significant attention from researchers in recent decades due to their broad applications in various fields, including spatial mechanism design, computer-aided geometric design, architecture, civil engineering, and solid modeling [5–7, 23, 24, 31]. In differential geometry, ruled surfaces are generated by the motion of a straight line, called a ruling, along a base curve in space. These surfaces are central to many theoretical and practical advancements, including the study of developable ruled surfaces, which are characterized by zero Gaussian curvature and can be unfolded onto a plane without distortion [8, 26], and minimal ruled surfaces, which minimize surface area and are characterized by vanishing Mean curvature [29].

A significant body of work has explored the relationship between ruled surfaces and helical curves within the framework of the Frenet frame in three-dimensional Euclidean space [2, 3, 26, 32]. Historical contributions include the foundational work of Karger

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and Novak in 1978, introducing Frenet frames and invariants for ruled surfaces [27], and Pottmann et. al. exploration of rational ruled surfaces and their offsets in 1996 [34]. Later, Peternel et al. addressed computational aspects of ruled surfaces in 1999 [33], while recent research has extended these investigations to the differential geometry of ruled surfaces in Minkowski space [1, 4, 9, 28, 30, 35, 36, 39].

The quasi-frame has emerged as an alternative to the Frenet frame for studying curves and surfaces. Defined by a fixed projection vector and the angle between the principal normal and the quasi-normal vector field, the quasi-frame simplifies computations compared to the Frenet and Bishop frames. This simplicity has made the quasi-frame a valuable tool in exploring geometrical properties and applications in Euclidean, Minkowski, and Galilean spaces [13, 14, 18, 22, 25]. Furthermore, variants of quasi-frames, such as equiform and modified frames, have been utilized in diverse contexts [10–12, 15, 17, 19–21, 24, 37, 38].

This paper is organized as follows. In Section 2, we provide fundamental definitions and concepts used throughout the paper. Section 3 introduces three new types of ruled surfaces based on the quasi-frame: QRT-surfaces, QRN-surfaces, and QRB-surfaces. For each surface type, we discuss their fundamental properties and provide a detailed analysis. Finally, we present two illustrative examples in Section 4 to validate the theoretical results and demonstrate their practical relevance.

2. Preliminaries

Let E^3 be an Euclidean 3-space equipped with the metric $<,>$ given by

$$<,>= du^2 + dv^2 + dw^2,$$

where (u, v, w) is a coordinate system of E^3. For a space curve $\alpha(s) : (a, b) \in I \to \mathbf{R^3}$ represented by its arc-length s let $\{t_q(s), n_q(s), b_q(s)\}$ be the Quasi frame along $\alpha(s)$ in which $t_q(s)$, $n_q(s)$ and $b_q(s)$ are the Quasi-tangent, Quasi-normal and Quasi-binormal vectors, respectively, given in [22] by

$$t_q(s) = \frac{\alpha'(s)}{\| \alpha'(s) \|},\ n_q(s) = \frac{t_q \times \mathbf{M}}{\| t_q \times \mathbf{M} \|},\ b_q(s) = t_q \times n_q, \tag{2.1}$$

where $'$ is the derivative with respect to s and $\mathbf{M}$ is the projection vector which we could choose $\mathbf{M} = (0, 1, 0),\ (1, 0, 0)\ or\ (0, 0, 1)$. The quasi-frame becomes singular in all cases where T and $\mathbf{M}$ are parallel and in these cases we change the projection. The Quasi formulae are given in [25] by

$$\frac{d}{ds}\begin{bmatrix} t_q(s) \\ n_q(s) \\ b_q(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & \kappa_2 \\ -\kappa_1 & 0 & \kappa_3 \\ -\kappa_2 & -\kappa_3 & 0 \end{bmatrix}\begin{bmatrix} t_q(s) \\ n_q(s) \\ b_q(s) \end{bmatrix}, \tag{2.2}$$

where the functions κ_1, κ_2 and κ_3 are the first, second, and third Quasi-curvatures of the curve, respectively, given by

$$\kappa_1 = \kappa(s)\ cos(\phi), \tag{2.3}$$

$$\kappa_2 = -\kappa(s)\ sin(\phi), \tag{2.4}$$

$$\kappa_3 = \frac{d\phi}{ds} + \tau(s), \tag{2.5}$$

where $\kappa(s)$ and $\tau(s)$ are the Frenet curvature and Frenet torsion, respectively [13, 14, 22, 25].
If ϕ is the angle between the Frenet normal n and the Quasi normal n_q and the relation between the Quasi frame and usual orthonormal Frenet frame $\{t, n, b\}$ given by

$$t_q(s) = t(s), \tag{2.6}$$

$$n_q(s) = cos(\phi)n(s) + sin(\phi)b(s), \tag{2.7}$$

$$b_q(s) = -sin(\phi)n(s) + cos(\phi)b(s). \tag{2.8}$$

A ruled surface W can be defined as a surface formed by the movement of a line L in space. Suppose $\alpha(s)$ represents a regular curve in Euclidean 3−space and $Y(s)$ represents the direction vector of the line L. The parametric representation of the ruled surface W can be given by

$$W(s, u) = \alpha(s) + uY(s),$$

where $\alpha(s)$ denotes the base curve [2]. The striction line and the distribution parameter of the ruled surface W can be given respectively as

$$\alpha^*(s) = \alpha(s) + \frac{< t_q(s), Y'(s) >}{\| Y'(s) \|^2} Y(s), \tag{2.9}$$

and

$$\lambda(s) = \frac{det\ [\ t_q(s),\ Y(s),\ Y'(s)]}{\| Y'(s) \|^2}, \tag{2.10}$$

where $t_q(s)$ is unit tangent vector field of the base curve $\alpha(s)$. The ruled surface W is developable if and only if $\lambda(s) = 0$. If $\| Y'(s) \| = 0$, then the ruled surface does not have any striction curve. In this case, the ruled surface is cylindrical. Thus, the base curve can be taken as a striction curve [3].

The standard unit normal vector field N on a surface W can be defined by

$$N = \frac{W_s \times W_u}{\| W_s \times W_u \|}, \tag{2.11}$$

where W_s and W_u are partial derivatives of $W(s, u)$ with respect to s and u. The 1^{st} F.F, the 2^{nd} F.F and 3^{rd} F.F of the surface $W(s, u)$ are given, respectively, by

$$I = E(ds)^2 + 2F\ ds\ du + G(du)^2, \tag{2.12}$$

$$II = L(ds)^2 + 2M\ ds\ du + N(du)^2, \tag{2.13}$$

$$III = e(ds)^2 + 2f\ ds\ du + g(du)^2, \tag{2.14}$$

where

$$E =< W_s, W_s >, F =< W_s, W_u >, G =< W_u, W_u >,$$

$$L =< W_{ss}, N >, M =< W_{su}, N >, N =< W_{uu}, N >,$$

$$e =< N_s, N_s >, f =< N_s, N_u > \text{ and } G =< N_u, N_u > .$$

Also, the Gaussian curvature K and the Mean curvature H are given, respectively, by

$$K = \frac{LN - M^2}{EG - F^2}, \ H = \frac{EN - 2MF + GL}{2(EG - F^2)}. \tag{2.15}$$

The $G-$ curvature κ_g, the $N-$ curvature κ_n and the $G-$ torsion τ_g which associate the curve $\alpha(s)$ on the surface W can be computed as follows

$$\kappa_g =< N(s) \times t_q(s), t'_q(s) >, \ \kappa_n =< N(s), \alpha''(s) >, \ \tau_g =< N \times N', t'_q(s) > \tag{2.16}$$

where N represents the unit normal vector of the surface along the curve $\alpha(s)$ and t_q denotes the unit tangent vector of $\alpha(s)$.

Definition 2.1. *[16] Let $\alpha(s)$ be a regular curve lying on a surface $W(s,u)$.*
(a) The curve $\alpha(s)$ is said to be geodesic curve if only if $G-$ curvature vanishes.
(b) The curve $\alpha(s)$ is said to be an asymptotic line if and only if $N-$ curvature vanishes.
(c) The curve $\alpha(s)$ is said to be a principal line if and only if $G-$ torsion vanishes.

Definition 2.2. *(a) A regular surface is flat (developable) if and only if its Gaussian curvature vanishes identically.*
(b) A regular surface is a minimal surface if and only if the Mean curvature vanishes identically.

3. Main Result

This section has three subsections that introduce three different types of ruled surfaces according to the Quasi frame called QRT-surfaces, QRN-surfaces, and QRB-surfaces, respectively. Also, discuss their fundamental properties.

3.1. Quasi-Tangent Ruled Surfaces According to the Quasi frame

In this section, we establish the definition of ruled surfaces that arise from a regular curve (referred to as the base curve) and are generated by the Quasi tangent vector t_q, which represents the direction vector. Additionally, we explore the fundamental properties associated with this particular type of ruled surface.

Definition 3.1. *Let $\alpha(s)$ be a regular curve with Quasi frame $\{t_q, n_q, b_q\}$, then the parametric representations of the QTR-Surface $W^T(s,u)$ with the ruling u given by*

$$W^T(s,u) = \alpha(s) + u \ t_q(s). \tag{3.1}$$

Theorem 3.1. *The striction curve of the Quasi-Tangent Ruled Surface is also a base function of the surface.*

Proof. Let $W^T(s,u)$ be a QTR-Surface with base curve $\alpha(s)$ from Equation (2.9) the striction curve defined by

$$\alpha^*(s) = \alpha(s) + \frac{< t_q(s), t'_q(s) >}{\| t'_q(s) \|^2} t_q.$$

By the Equations in (2.2)

$$\alpha^*(s) = \alpha(s) + \frac{< t_q(s), (\kappa_1\ n_q(s)\ +\ \kappa_2\ b_q(s)) >}{\| t'_q(s) \|^2} t_q,$$

or

$$\alpha^*(s) = \alpha(s) + \frac{\kappa_1 < t_q(s), n_q(s) >\ + \kappa_2 < t_q(s), b_q(s) >}{\| t'_q(s) \|^2} t_q,$$

Therefore,

$$\alpha^*(s) = \alpha(s).$$

Theorem 3.2. *The first fundamental form of the Quasi-Tangent Ruled Surface $W^T(s,u)$ is given by*

$$I = 1 +\ u^2\ ({\kappa_1}^2 +\ {\kappa_2}^2)(ds)^2 + 2\ dsdu + (du)^2.$$

Proof. Let $W^T(s,u)$ be a QTR-Surface with base curve $\alpha(s)$. Considering Equations (2.2) and (2.6), the first partial derivatives of the QTR-surface with respect to s and u are given by

$$W^T{}_s = t_q(s)\ + u\ (\ \kappa_1\ n_q(s)\ +\ \kappa_2\ b_q(s)), \tag{3.2}$$

$$W^T{}_u = t_q(s). \tag{3.3}$$

From Equations (2.12),(3.2) and (3.3) the coefficients of the 1^{st} F.F are given by

$$\begin{aligned} E &= < W^T{}_s, W^T{}_s > = 1 +\ u^2\ ({\kappa_1}^2 +\ {\kappa_2}^2), \\ F &= < W^T{}_s, W^T{}_u > = 1, \\ G &= < W^T{}_u, W^T{}_u > = 1. \end{aligned} \tag{3.4}$$

Theorem 3.3. *The second fundamental form of the Quasi-Tangent Ruled Surface $W^T(s,u)$ is given by*

$$II = -\frac{u\left(\kappa_2\ \ (\kappa_2\ \kappa_3 - {\kappa_1}') + \kappa_1\ {\kappa_2}' + {\kappa_1}^2\ \kappa_3\right)}{\sqrt{{\kappa_1}^2 + {\kappa_2}^2}}(ds)^2.$$

Proof. Let $W^T(s,u)$ be a QTR-Surface with base curve $\alpha(s)$. The cross product of Equations (3.2) and (3.3) given by

$$W^T{}_s \times W^T{}_u = \ u \ (\kappa_2 \ n_q(s) - \kappa_1 \ b_q(s)), \tag{3.5}$$

taking the norm we get

$$\| \ W^T{}_s \times W^T{}_u \ \| = \ u^2({\kappa_1}^2 + {\kappa_2}^2). \tag{3.6}$$

From Equations (2.11), (3.5) and (3.6), the unit normal vector can be defined by

$$N^T = \frac{W^T{}_s \times W^T{}_u}{\| \ W^T{}_s \times W^T{}_u \ \|} = \frac{u \ (\ \kappa_2 \ n_q(s) \ - \ \kappa_1 \ b_q(s) \ )}{\sqrt{E-1}}, \tag{3.7}$$

where $E = \ 1 + \ u^2 \ ({\kappa_1}^2 + \ {\kappa_2}^2)$. Considering Equations (2.2),(3.2) and (3.3), the second partial derivatives of the QTR-surface with respect to s and u are given by

$$\begin{aligned} W^T{}_{ss} &= -u \ ({\kappa_1}^2 + \ {\kappa_2}^2)t_q(s) + (\kappa_1 + u(-\kappa_2\kappa_3 + {\kappa_1}'))n_q(s) \\ &\quad +(\kappa_2 + u(-\kappa_1\kappa_3 + {\kappa_2}'))b_q(s), \end{aligned} \tag{3.8}$$

$$W^T{}_{su} = \kappa_1 \ n_q(s) \ + \ \kappa_2 \ b_q(s), \qquad W^T{}_{uu} = 0.$$

From Equations (2.13), and (3.8) the coefficients of the 2^{nd} F.F are given by

$$\begin{aligned} L &= < W^T{}_{ss}, N > = \frac{-u^2 \left(\kappa_2 \ \ (\kappa_2 \ \kappa_3 - {\kappa_1}') + \kappa_1 \ {\kappa_2}' + {\kappa_1}^2 \ \kappa_3\right)}{\sqrt{E-1}}, \\ M &= < W^T{}_{su}, N > = 0, \\ N &= < W^T{}_{uu}, N > = 0. \end{aligned} \tag{3.9}$$

Theorem 3.4. *The third fundamental form of the Quasi-Tangent Ruled Surface $W^T(s,u)$ is given by*

$$III = \frac{\left(\kappa_2 \left(\kappa_2\kappa_3 - {\kappa_1}'\right) + \kappa_1{\kappa_2}' + {\kappa_1}^2\kappa_3\right)^2}{\left({\kappa_1}^2 + {\kappa_2}^2\right)^2}(ds)^2.$$

Proof. Let $W^T(s,u)$ be a QTR-Surface with base curve $\alpha(s)$. The first partial derivatives of Equation (3.7) with respect to s and u given by

$$\begin{aligned} (N^T)_s &= \frac{\kappa_1 \left(\kappa_2 \left(\kappa_2\kappa_3 - {\kappa_1}'\right) + \kappa_1{\kappa_2}' + {\kappa_1}^2\kappa_3\right)}{\left({\kappa_1}^2 + {\kappa_2}^2\right)^{3/2}} \ n_q(s) \\ &\quad + \frac{\kappa_2 \left(\kappa_2 \left(\kappa_2\kappa_3 - {\kappa_1}'\right) + \kappa_1{\kappa_2}' + {\kappa_1}^2\kappa_3\right)}{\left({\kappa_1}^2 + {\kappa_2}^2\right)^{3/2}} \ b_q(s), \end{aligned} \tag{3.10}$$

and

$$(N^T)_u = 0. \tag{3.11}$$

From Equations (2.14), (3.10) and (3.11) the coefficients of 3^{rd} F.F are given by

$$\begin{aligned} e &= \; < {N^T}_s, {N^T}_s > \; = \frac{\left(\kappa_2 \left(\kappa_2\kappa_3 - {\kappa_1}'\right) + \kappa_1 {\kappa_2}' + {\kappa_1}^2\kappa_3\right)^2}{\left({\kappa_1}^2 + {\kappa_2}^2\right)^2} = \frac{L}{\sqrt{E-1}}\,, \\ f &= \; < {N^T}_s, {N^T}_u > \; = 0, \\ g &= \; < {N^T}_u, {N^T}_u > \; = 0, \end{aligned}$$

where $E = \; 1 + \; u^2 \; ({\kappa_1}^2 + \; {\kappa_2}^2)$.

Theorem 3.5. *The Gaussian curvature K and the Mean curvature H of the Quasi-Tangent Ruled Surface $W^T(s,u)$ are given, respectively, by*

$$K = \; 0, \qquad H = \frac{\kappa_2 \left({\kappa_1}' - \kappa_2\kappa_3\right) + \kappa_1 \left(-{\kappa_2}'\right) - {\kappa_1}^2\kappa_3}{2u\left({\kappa_1}^2 + {\kappa_2}^2\right)^{3/2}}.$$

Proof. Let $W^T(s,u)$ be a QTR-Surface with base curve $\alpha(s)$. From Equations (2.15), (3.4) and (3.9) we deduce the result.

Theorem 3.6. *The geodesic curvature κ_g, the normal curvature κ_n and the geodesic torsion τ_g which associate the base curve on the Quasi-Tangent Ruled Surface $W^T(s,u)$ are given, respectively, by*

$$\kappa_g = -\sqrt{{\kappa_1}^2 + {\kappa_2}^2}, \quad \kappa_n = 0, \quad \tau_g = 0.$$

Proof. Let $W^T(s,u)$ be a QTR-Surface with base curve $\alpha(s)$. From Equations (2.2) and (2.16) we deduce the result.

Corollary 3.1. *Let $\alpha(s)$ be a regular curve lying on a surface $W^T(s,u)$.*
(a) The curve $\alpha(s)$ is said to be a geodesic curve if only if $\alpha(s)$ is a straight line.
(b) The curve $\alpha(s)$ is always asymptotic line.
(c) The curve $\alpha(s)$ is a always principal line.

Corollary 3.2. *(a) The QTR-surface is always flat (developable) surface.*
(b) For non-straight lines the QTR-surface is a minimal surface if and only if

$$\kappa_3 = \frac{-\kappa_1^2 \frac{d}{du}\left(\frac{\kappa_2}{\kappa_1}\right)}{\kappa_1^2 + \kappa_2^2}.$$

3.2. Quasi-Normal Ruled Surfaces According to the Quasi frame

In this section, we outline the characteristics of QNR-Surfaces. These surfaces are formed by a regular curve (known as the base curve) and are generated using the quasi-normal vector N_q, which represents the direction vector. Furthermore, we will examine the fundamental properties that are inherent to this type of ruled surface.

Definition 3.2. *Let $\eta(s)$ be a regular curve with quasi frame $\{t_q, n_q, b_q\}$, then the parametric representations of the QNR-Surface $W^N(s,u)$ with the ruling u given by*

$$W^N(s,u) = \eta(s) + u\ n_q(s). \tag{3.12}$$

The theories presented in this section are built upon the same foundational evidence as those discussed in the first section.

Theorem 3.7. *The striction curve of the Quasi-Normal Ruled Surface is given by*

$$\eta^*(s) = \eta(s) - \frac{\kappa_1}{{\kappa_1}^2 + {\kappa_3}^2} n_q$$

Theorem 3.8. *The first fundamental form of the Quasi-Normal Ruled Surface $W^T(s,u)$ is given by*

$$I = (u^2{\kappa_3}^2 + (1 - u\ {\kappa_1}^2))(ds)^2 + (du)^2,$$

where

$$\begin{aligned}
E &= < {W^N}_s, {W^N}_s > = u^2{\kappa_3}^2 + (1 - u\ {\kappa_1}^2),\\
F &= < {W^N}_s, {W^N}_u > = 0,\\
G &= < {W^N}_u, {W^N}_u > = 1.
\end{aligned}$$

Theorem 3.9. *The second fundamental form of the Quasi-Normal Ruled Surface $W^N(s,u)$ is given by*

$$\begin{aligned}
II = (&\frac{\kappa_2\left(u^2{\kappa_1}^2 + u^2{\kappa_3}^2 - 2u\kappa_1 + 1\right) + u\left(u\kappa_3{\kappa_1}' + {\kappa_3}'(1 - u\kappa_1)\right)}{\sqrt{u^2{\kappa_1}^2 + u^2{\kappa_3}^2 - 2u\kappa_1 + 1}})(ds)^2\\
&+ \frac{2\kappa_3}{\sqrt{u^2{\kappa_1}^2 + u^2{\kappa_3}^2 - 2u\kappa_1 + 1}} ds\ du,
\end{aligned}$$

where

$$\begin{aligned}
L &= < {W^N}_{ss}, N > = \frac{\kappa_2\left(u^2{\kappa_1}^2 + u^2{\kappa_3}^2 - 2u\kappa_1 + 1\right) + u\left(u\kappa_3{\kappa_1}' + {\kappa_3}'(1 - u\kappa_1)\right)}{\sqrt{u^2{\kappa_1}^2 + u^2{\kappa_3}^2 - 2u\kappa_1 + 1}},\\
M &= < {W^N}_{su}, N > = \frac{\kappa_3}{\sqrt{u^2{\kappa_1}^2 + u^2{\kappa_3}^2 - 2u\kappa_1 + 1}},\\
N &= < {W^N}_{uu}, N > = 0.
\end{aligned}$$

Theorem 3.10. *The third fundamental form of the Quasi-Normal Ruled Surface $W^N(s,u)$ is given by*

$$III = e(ds)^2 + 2f\ dsdu + g(du)^2,$$

where

$$\begin{aligned} e =< {N^N}_s, {N^N}_s >= & \frac{\kappa_3^2\left(u^4{\kappa_1}'^2 + (u\kappa_1 - 1)^2\right) - 2u^3\kappa_3{\kappa_1}'{\kappa_3}'(u\kappa_1 - 1)}{\left(u^2{\kappa_3}^2 + (1 - u\kappa_1)^2\right)^2} \\ & + \frac{2u\kappa_2\left(u^2\kappa_3^2 + (u\kappa_1 - 1)^2\right)\left(u\kappa_3{\kappa_1}' + {\kappa_3}'(1 - u\kappa_1)\right)}{\left(u^2{\kappa_3}^2 + (1 - u\kappa_1)^2\right)^2} \\ & + \frac{\kappa_2^2\left(u^2\kappa_3^2 + (u\kappa_1 - 1)^2\right)^2 + u^2{\kappa_3}'^2(u\kappa_1 - 1)^2 + u^2\kappa_3^4}{\left(u^2{\kappa_3}^2 + (1 - u\kappa_1)^2\right)^2} = \frac{L^2}{E} + M^2, \\ f =< {N^N}_s, {N^N}_u >= & \frac{\kappa_3\left(\kappa_2\left(u^2\kappa_1^2 + u^2\kappa_3^2 - 2u\kappa_1 + 1\right) + u\left(u\kappa_3{\kappa_1}' + {\kappa_3}'(1 - u\kappa_1)\right)\right)}{\left(u^2{\kappa_3}^2 + (1 - u\kappa_1)^2\right)^2} = \frac{LM}{E}, \\ g =< {N^N}_u, {N^N}_u >= & \frac{\kappa_3^2}{\left(u^2{\kappa_3}^2 + (1 - u\kappa_1)^2\right)^2} = \frac{M^2}{E}. \end{aligned}$$

Theorem 3.11. *The Gaussian curvature K and the Mean curvature H of the Quasi-Normal Ruled Surface $W^N(s,u)$ are given, respectively, by*

$$K = \frac{-{\kappa_3}^2}{\left(u^2{\kappa_3}^2 + (1 - u\kappa_1)^2\right)^2} = -\frac{M^2}{E},$$

$$H = \frac{\kappa_2\left(u^2\kappa_3^2 + (u\kappa_1 - 1)^2\right) + u\left(u\kappa_3{\kappa_1}' + {\kappa_3}'(1 - u\kappa_1)\right)}{2\left(u^2\kappa_3^2 + (u\kappa_1 - 1)^2\right)^{3/2}} = \frac{L}{2E}.$$

Theorem 3.12. *the geodesic curvature κ_g, the normal curvature κ_n and the geodesic torsion τ_g which associate the base curve on the Quasi-Normal Ruled Surface $W^N(s,u)$ are given, respectively, by*

$$\kappa_g = \frac{\kappa_1 - u\kappa_1^2}{\sqrt{u^2\kappa_3^2 + (u\kappa_1 - 1)^2}},$$

$$\kappa_n = \frac{\kappa_2(1 - u\kappa_1)}{\sqrt{u^2\kappa_3^2 + (1 - u\kappa_1)^2}},$$

$$\tau_g = \frac{-\kappa_1\left(\kappa_2\left(u^2\kappa_3^2 + 1\right) + u\left({\kappa_3}' + u\kappa_3{\kappa_1}'\right)\right) - u^2\kappa_1^3\kappa_2 + u\kappa_1^2\left(2\kappa_2 + u{\kappa_3}'\right) + u\kappa_2\kappa_3^2}{u^2\kappa_3^2 + (u\kappa_1 - 1)^2}.$$

Corollary 3.3. *Let $\eta(s)$ be a regular curve lying on a surface $W^N(s,u)$.*
(a) The curve $\eta(s)$ is a geodesic curve if only if $\kappa_1 = 0$ or $\kappa_1 = \frac{1}{u}$.

(b) The curve $\eta(s)$ is an asymptotic line if only if $\kappa_2 = 0$ or $\kappa_1 = \frac{1}{u}$.
(c) The curve $\eta(s)$ is a principal line if and only if

$$\kappa_2(s) = \frac{u\kappa_1(s)\left(\frac{d\kappa_3(s)}{dt} + u\kappa_3(s)\frac{d\kappa_1(s)}{dt}\right) - u^2\kappa_1(s)^2\frac{d\kappa_3(s)}{dt}}{-\kappa_1(s)\left(u^2\kappa_3^2(s)+1\right) - u^2\kappa_1(s)^3 + 2u\kappa_1^2(s) + u\kappa_3^2(s)}.$$

Corollary 3.4. *(a) The QNR-surface is a flat (developable)surface if and only if $\kappa_3 = 0$.*
(b) A QNR-surface is a minimal surface if and only if

$$\kappa_2(s) = -\frac{u\left(u\kappa_3(s)\frac{d\kappa_1(s)}{dt} + \frac{d\kappa_3(s)}{dt}(1-u\kappa_1(s))\right)}{u^2\kappa_1(s)^2 + u^2\kappa_3(s)^2 - 2u\kappa_1(s) + 1}.$$

3.3. Quasi-Binormal Ruled Surfaces According to the Quasi frame

In this section, we define the QBR-Surfaces that are created by combining a regular curve (referred to as the base curve) with the Quasi binormal vector b_q, which determines the direction. Additionally, we explore and discuss the fundamental properties associated with this particular type of ruled surface.

Definition 3.3. *Let $\gamma(s)$ be a regular curve with Quasi frame $\{t_q, n_q, b_q\}$. Then the parametric representations of the QBR-Surface $W^B(s,u)$ with the ruling u is given by*

$$W^B(s,u) = \gamma(s) + u\ b_q(s). \tag{3.13}$$

The theories presented in this section are built upon the same foundational evidence as those discussed in the first section.

Theorem 3.13. *The striction curve of the Quasi-Binormal Ruled Surface is given by*

$$\gamma^*(s) = \gamma(s) - \frac{\kappa_2}{{\kappa_2}^2 + {\kappa_3}^2} b_q.$$

Theorem 3.14. *The first fundamental form of the Quasi-Binormal Ruled Surface $W^B(s,u)$ is given by*

$$I = (u^2{\kappa_3}^2 + (1 - u\ {\kappa_2}^2))(ds)^2 + (du)^2,$$

Theorem 3.15. *The second fundamental form of the Quasi-Binormal Ruled Surface $W^B(s,u)$ is given by*

$$II = \left(\frac{-\kappa_1\left(u^2{\kappa_3}^2 + (1-u\ {\kappa_2}^2\right) + u\left(u\kappa_3{\kappa_2}' + {\kappa_3}'(1-u\kappa_2)\right)}{\sqrt{u^2{\kappa_3}^2 + (1-u\ \kappa_2)^2}}\right)(ds)^2$$
$$+ \frac{2\kappa_3}{\sqrt{u^2{\kappa_3}^2 + (1-u\ \kappa_2)^2}} ds\ du,$$

Theorem 3.16. *The third fundamental form of the Quasi-Binormal Ruled Surface* $W^B(s,u)$ *is given by*

$$III = e(ds)^2 + 2f\ dsdu + g(du)^2.$$

where

$$\begin{aligned} e =< {N^N}_s, {N^N}_s >= & \frac{\kappa_3^2\left(u^4{\kappa_2}'^2+(1-u\kappa_2)^2\right)+2u^3\kappa_3{\kappa_2}'{\kappa_3}'(1-u\kappa_2)}{\left(u^2{\kappa_3}^2+(1-u\kappa_2)^2\right)^2} \\ & - \frac{2u\kappa_1\left(u^2\kappa_3^2+(1-u\kappa_2)^2\right)\left(u\kappa_3{\kappa_2}'+{\kappa_3}'(1-u\kappa_2)\right)}{\left(u^2{\kappa_3}^2+(1-u\kappa_2)^2\right)^2} \\ & + \frac{\kappa_1^2\left(u^2\kappa_3^2+(1-u\kappa_2)^2\right)^2+u^2{\kappa_3}'^2(1-u\kappa_2)^2+u^2\kappa_3^4}{\left(u^2{\kappa_3}^2+(1-u\kappa_2)^2\right)^2} = \frac{L^2}{E} + M^2, \\ f =< {N^N}_s, {N^N}_u >= & \frac{\kappa_3\left(\kappa_1\left(u^2\kappa_3^2+(1-u\kappa_2)^2\right)+u\left(u\kappa_3{\kappa_2}'+{\kappa_3}'(1-u\kappa_2)\right)\right)}{\left(u^2{\kappa_3}^2+(1-u\kappa_2)^2\right)^2} = \frac{LM}{E}, \\ g =< {N^N}_u, {N^N}_u >= & \frac{\kappa_3^2}{\left(u^2{\kappa_3}^2+(1-u\kappa_2)^2\right)^2} = \frac{M^2}{E}. \end{aligned}$$

Theorem 3.17. *The Gaussian curvature* K *and the Mean curvature* H *of the Quasi-Binormal Ruled Surface* $W^B(s,u)$ *are given, respectively, by*

$$K = -\frac{M^2}{E} = \frac{-{\kappa_3}^2}{\left(u^2{\kappa_3}^2+(1-u\kappa_2)^2\right)^2},$$

$$H = \frac{L}{2E} = \frac{-\kappa_1\left(u^2\kappa_3^2+(1-u\kappa_2)^2\right)+u\left(u\kappa_3{\kappa_2}'+{\kappa_3}'(1-u\kappa_2)\right)}{2\left(u^2\kappa_3^2+(1-u\kappa_2)^2\right)^{3/2}}.$$

Theorem 3.18. *the geodesic curvature* κ_g*, the normal curvature* κ_n *and the geodesic torsion* τ_g *which associate the base curve on the Quasi-Binormal Ruled Surface* $W^B(s,u)$ *are given, respectively, by*

$$\begin{aligned} \kappa_g &= \frac{\kappa_2(1-u\kappa_2)}{\sqrt{u^2\kappa_3^2+(1-u\kappa_2)^2}}, \\ \kappa_n &= \frac{-\kappa_1(1-u\kappa_2)}{\sqrt{u^2\kappa_3^2+(1-u\kappa_2)^2}}, \\ \tau_g &= \frac{-\kappa_2\left(\kappa_1\left(u^2\kappa_3^2+1\right)+u\left({\kappa_3}'+u\kappa_3{\kappa_2}'\right)\right)-u^2\kappa_2^3\kappa_1+u\kappa_2^2\left(2\kappa_1+u{\kappa_3}'\right)+u\kappa_1\kappa_3^2}{u^2\kappa_3^2+(1-u\kappa_2)^2}. \end{aligned}$$

Corollary 3.5. *Let $\gamma(s)$ be a regular curve lying on a surface $W^N(s,u)$.*
(a) The curve $\gamma(s)$ is a geodesic curve if only if $\kappa_2 = 0$ or $\kappa_2 = \frac{1}{u}$.
(b) The curve $\gamma(s)$ is an asymptotic line if only if $\kappa_1 = 0$ or $\kappa_2 = \frac{1}{u}$.
(c) The curve $\gamma(s)$ is a principal line if and only if

$$\kappa_1(s) = -\frac{u\kappa_2(s)\left(-\kappa_2(s) + u\kappa_2(s)^2 + u\kappa_3(s)^2\right)\left(\kappa_3'(s)(u\kappa_2(s)-1) - u\kappa_3(s)\kappa_2'(s)\right)}{u\kappa_2(s)-1}$$

.

Corollary 3.6. *(a) The QBR-surface is a flat (developable)surface if and only if $\kappa_3 = 0$.*
(b) A QBR-surface is a minimal surface if and only if

$$\kappa_1(s) = -\frac{u\left(u\kappa_3(s)\frac{d\kappa_2(s)}{ds} + \frac{d\kappa_3(s)}{ds}(1-u\kappa_2(s))\right)}{-u^2\kappa_3(s)^2 - (u\kappa_2(s)-1)^2}.$$

Example 3.1. *Let $\zeta(s)$ be a general helix curve given by the parametrization*

$$\zeta(s) = \left(4\cos\left(\frac{s}{5}\right), 4\sin\left(\frac{s}{5}\right), \frac{3s}{5}\right).$$

By Equation (2.1) where choose $\mathbf{M} = (0,1,0)$ the Quasi-frame obtained by

$$\begin{aligned}
t =& \left(\frac{-4}{5}\sin\left(\frac{s}{5}\right), \frac{4}{5}\cos\left(\frac{s}{5}\right), \frac{3}{5}\right),\\
n_q =& \left(0, \frac{3}{\sqrt{8\cos\left(\frac{2s}{5}\right)+17}}, -\frac{4\cos\left(\frac{s}{5}\right)}{\sqrt{8\cos\left(\frac{2s}{5}\right)+17}}\right),\\
b_q =& \left(\frac{-8\cos\left(\frac{2s}{5}\right)-17}{5\sqrt{8\cos\left(\frac{2s}{5}\right)+17}}, -\frac{8\sin\left(\frac{2s}{5}\right)}{5\sqrt{8\cos\left(\frac{2s}{5}\right)+17}}, -\frac{12\sin\left(\frac{s}{5}\right)}{5\sqrt{8\cos\left(\frac{2s}{5}\right)+17}}\right).
\end{aligned}$$

Also by Equation (2.3) the Quasi-Curvatures along $\zeta(s)$ are obtained by

$$\begin{aligned}
\kappa_1 =& -\frac{12\sin\left(\frac{s}{5}\right)}{25\sqrt{8\cos\left(\frac{2s}{5}\right)+17}},\\
\kappa_2 =& \frac{4\cos\left(\frac{s}{5}\right)}{5\sqrt{8\cos\left(\frac{2s}{5}\right)+17}},\\
\kappa_3 =& -\frac{48\sin^2\left(\frac{s}{5}\right)}{25\left(8\cos\left(\frac{2s}{5}\right)+17\right)}.
\end{aligned}$$

The QTR-Surface, the QNR-Surface, and the QBR-Surface are, respectively, given in Figure 3.1 by

$$W^T(s,u) = \left(4\cos\left(\frac{s}{5}\right) - \frac{4}{5}u\sin\left(\frac{s}{5}\right), \frac{4}{5}u\cos\left(\frac{s}{5}\right) + 4\sin\left(\frac{s}{5}\right), \quad \frac{3s}{5} + \frac{3u}{5}\right),$$

$$W^N(s,u) = \left(4\cos\left(\frac{s}{5}\right), \frac{3u}{\sqrt{8\cos\left(\frac{2s}{5}\right)+17}} + 4\sin\left(\frac{s}{5}\right), \frac{3s}{5} - \frac{4u\cos\left(\frac{s}{5}\right)}{\sqrt{8\cos\left(\frac{2s}{5}\right)+17}}\right),$$

$$W^B(s,u) = \left(4\cos\frac{s}{5} - \frac{1}{5}u\sqrt{8\cos\frac{2s}{5}+17}, \quad 4\sin\left(\frac{s}{5}\right) - \frac{8u\sin\left(\frac{2s}{5}\right)}{5\sqrt{8\cos\left(\frac{2s}{5}\right)+17}}\right.$$
$$\left., \frac{3}{5}(s - \frac{4u\sin\left(\frac{s}{5}\right)}{\sqrt{8\cos\left(\frac{2s}{5}\right)+17}})\right).$$

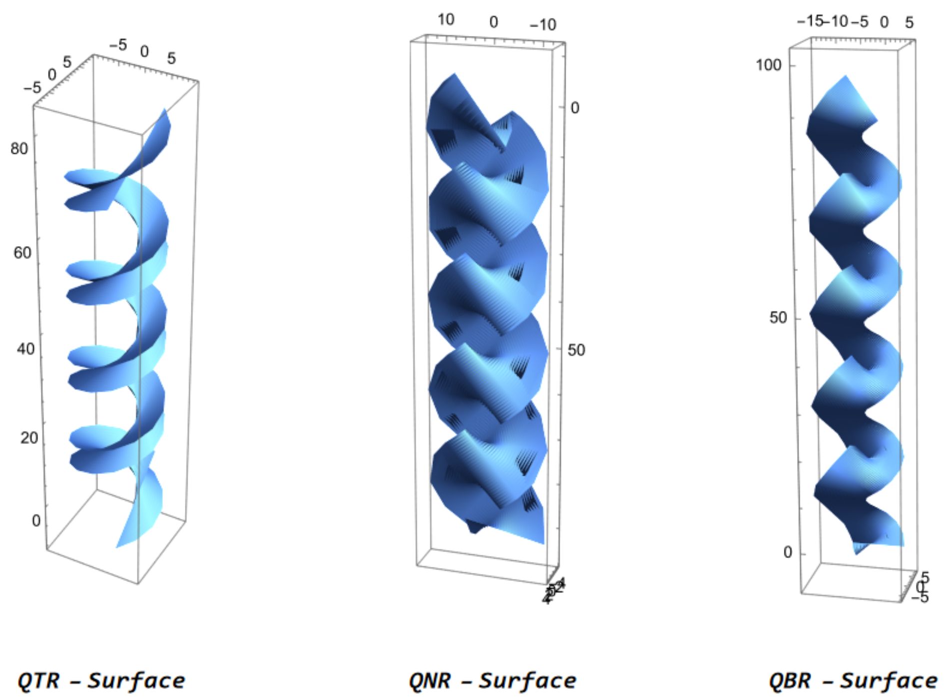


Figure 1: Ruled Surfaces generated by a general helix

Example 3.2. *Let $\xi(s)$ be a regular curve parameterized by*

$$\xi = \left(\frac{3}{2}\cos\left(\frac{s}{2}\right) + \frac{1}{6}\cos\left(\frac{3s}{2}\right), \frac{3}{2}\sin\left(\frac{s}{2}\right) + \frac{1}{6}\sin\left(\frac{3s}{2}\right), \sqrt{3}\cos\left(\frac{s}{2}\right)\right).$$

By Equation (2.1) where choose $\mathbf{M} = (0, 0, 1)$ *the Quasi-frame obtained by*

$$\begin{aligned}
t =&\left(\frac{1}{4}\left(-3\sin\left(\frac{s}{2}\right)-\sin\left(\frac{3s}{2}\right)\right), \cos^3\left(\frac{s}{2}\right), -\frac{1}{2}\sqrt{3}\sin\left(\frac{s}{2}\right)\right),\\
n_q =&\left(\frac{2\sqrt{2}\cos^3\left(\frac{s}{2}\right)}{\sqrt{3\cos(s)+5}}, \frac{3\sin\left(\frac{s}{2}\right)+\sin\left(\frac{3s}{2}\right)}{\sqrt{6\cos(s)+10}}, 0\right),\\
b_q =&\left(\frac{\sin^2\left(\frac{s}{2}\right)(\cos(s)+2)}{\sqrt{2\cos(s)+\frac{10}{3}}}, -\frac{\sin(s)(\cos(s)+1)}{2\sqrt{2\cos(s)+\frac{10}{3}}}, -\frac{\sqrt{3\cos(s)+5}}{2\sqrt{2}}\right).
\end{aligned}$$

Also by Equation (2.3) the Quasi-Curvatures along $\xi(s)$ *are obtained by*

$$\begin{aligned}
\kappa_1 =& -\frac{3\cos^2\left(\frac{s}{2}\right)}{\sqrt{6\cos(s)+10}},\\
\kappa_2 =& \frac{\cos\left(\frac{s}{2}\right)}{\sqrt{2\cos(s)+\frac{10}{3}}},\\
\kappa_3 =& -\frac{3\sqrt{3}\sin^2(s)\csc\left(\frac{s}{2}\right)}{12\cos(s)+20}.
\end{aligned}$$

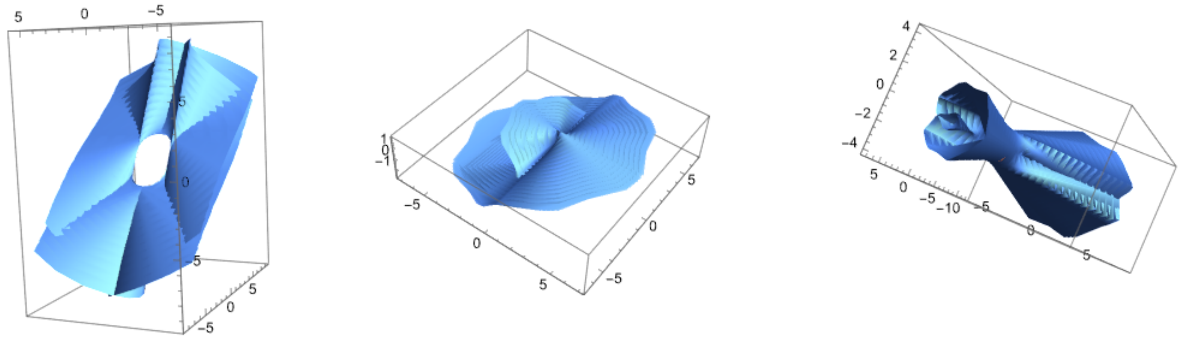
The QTR-Surface, the QNR-Surface, and the QBR-Surface are, respectively, given in Figure 3.2 by

$$\begin{aligned}
W^T(s,u) =&\left(\frac{1}{12}\left(-3u\left(3\sin\left(\frac{s}{2}\right)+\sin\left(\frac{3s}{2}\right)\right)+18\cos\left(\frac{s}{2}\right)+2\cos\left(\frac{3s}{2}\right)\right),\right.\\
&\frac{1}{12}\left(9u\cos\left(\frac{s}{2}\right)+3u\cos\left(\frac{3s}{2}\right)+2\left(9\sin\left(\frac{s}{2}\right)+\sin\left(\frac{3s}{2}\right)\right)\right),\\
&\left.-\frac{1}{2}\sqrt{3}\left(u\sin\left(\frac{s}{2}\right)-2\cos\left(\frac{s}{2}\right)\right)\right),
\end{aligned}$$

$$\begin{aligned}
W^N(s,u) =&\left(\frac{1}{6}\left(\frac{12\sqrt{2}u\cos^3\left(\frac{s}{2}\right)}{\sqrt{3\cos(s)+5}}+9\cos\left(\frac{s}{2}\right)+\cos\left(\frac{3s}{2}\right)\right),\right.\\
&\left.\frac{u\left(3\sin\left(\frac{s}{2}\right)+\sin\left(\frac{3s}{2}\right)\right)}{\sqrt{6\cos(s)+10}}+\frac{3}{2}\sin\left(\frac{s}{2}\right)+\frac{1}{6}\sin\left(\frac{3s}{2}\right), \sqrt{3}\cos\left(\frac{s}{2}\right)\right),
\end{aligned}$$

$$\begin{aligned}
W^B(s,u) =&\left(\frac{u\sin^2\left(\frac{s}{2}\right)(\cos(s)+2)}{\sqrt{2\cos(s)+\frac{10}{3}}}+\frac{3}{2}\cos\left(\frac{s}{2}\right)+\frac{1}{6}\cos\left(\frac{3s}{2}\right),\right.\\
&-\frac{u\sin(s)(\cos(s)+1)}{2\sqrt{2\cos(s)+\frac{10}{3}}}+\frac{3}{2}\sin\left(\frac{s}{2}\right)+\frac{1}{6}\sin\left(\frac{3s}{2}\right),
\end{aligned}$$

$$\sqrt{3}\cos\left(\frac{s}{2}\right) - \frac{u\sqrt{3\cos(s)+5}}{2\sqrt{2}}\Bigg).$$



QTR −Surface ***QNR −Surface*** ***QBR −Surface***

Figure 2: Ruled Surfaces generated by $\xi(s)$

Abbreviation	Full Form
QRT	Quasi-Tangent Ruled Surface
QRN	Quasi-Normal Ruled Surface
QRB	Quasi-Binormal Ruled Surface
F.F	Fundamental Form
G- curvature	geodesic curvature
N- curvature	normal curvature
G- torsion	geodesic torsion

Table 1: List of Abbreviations

4. Conclusion

This paper introduced three distinct types of ruled surfaces based on the quasi-frame: quasi-tangent (QRT), quasi-normal (QRN), and quasi-binormal (QRB) surfaces. These surfaces are generated by the motion of a straight line (ruling) along a base curve, with the direction of the ruling determined by the quasi-tangent, quasi-normal, and quasi-binormal vectors, respectively. We have thoroughly investigated their first, second, and third fundamental forms, as well as their Gaussian and mean curvatures. In addition, we have also explored the geodesic curvature, normal curvature, and geodesic torsion associated with the base curve for each type of surface. Furthermore, we have established the conditions under which the base curve can be classified as a geodesic, an asymptotic line, or a principal line on each type of surface. We have also derived the conditions for these surfaces to be developable or minimal. To illustrate the theoretical results, we provided two detailed examples. All the results derived in this paper can be specialized to the Frenet frame by setting $\kappa_2 = 0$. This connection between the quasi-frame and the Frenet frame allows for a broader interpretation of the results and provides a bridge between different geometric frameworks.

Future research could explore the application of quasi-frames in higher-dimensional spaces, such as Minkowski and Galilean spaces. Additionally, the study of quasi-ruled surfaces in the context of computer-aided design and architecture could lead to new practical applications. Further investigations into the singularities of the quasi-frame and their implications for geometric modeling are also warranted.

References

[1] N. H. Abdel-All, R. A. Abdel-Baky, and F. M. Hamdoon. Ruled surfaces with timelike rulings. *Applied Mathematics and Computation*, 147:241–253, 2004.

[2] P. Alegre, K. Arslan, A. Carriazo, C. Murathan, and G. Ozturk. Some special types of developable ruled surface. *Hacettepe Journal of Mathematics and Statistics*, 39(3):319–325, 2010.

[3] A. T. Ali, H. S. A. Aziz, and A. H. Sorour. Ruled surfaces generated by some special curves in Euclidean 3-space. *Journal of the Egyptian Mathematical Society*, 21(3):285–294, 2013.

[4] A. A. Almoneef and R. A. Abdel-Baky. Timelike constant axis ruled surface family in Minkowski 3-space. *Symmetry*, 16(6):677, 2024.

[5] C. Andradas, T. Recio, L. F. Tabera, J. R. Sendra, and C. Villarino. Proper real reparametrization of rational ruled surfaces. *Computer Aided Geometric Design*, 28(2):102–113, 2011.

[6] L. Buse, M. Elkadi, and A. Galligo. A computational study of ruled surfaces. *Journal of Symbolic Computation*, 44(3):232–241, 2009.

[7] Y. Chen, L. Y. Shen, and C. M. Yuan. Collision and intersection detection of two ruled surfaces using bracket method. *Computer Aided Geometric Design*, 28(2):114–126, 2011.

[8] T. H. Colding and C. D. Lellis. The min-max construction of minimal surfaces. *Surveys in Differential Geometry*, 8(1):75–107, 2003.

[9] F. Dillen and W. Sodsiri. Ruled surfaces of weingarten type in Minkowski 3-space. *Journal of Geometry*, 83:10–21, 2005.

[10] H. K. Elsayied, A. A. Altaha, and A. Elsharkawy. Bertrand curves with the modified orthogonal frame in Minkowski 3-space E_1^3. *Revista de Educacion*, 392(6):43–55, 2022.

[11] H. K. Elsayied, M. Elzawy, and A. Elsharkawy. Equiform timelike normal curves in Minkowski space E_1^3. *Far East Journal of Mathematical Sciences*, 101:1619–1629, 2017.

[12] H. K. Elsayied, M. Elzawy, and A. Elsharkawy. Equiform spacelike normal curves according to equiform-Bishop frame in E_1^3. *Mathematical Methods in the Applied Sciences*, 41(15):5754–5760, 2018.

[13] H. K. Elsayied, A. M. Tawfiq, and A. Elsharkawy. Special smarandach curves according to the quasi frame in 4-dimensional Euclidean space E^4. *Houstoun Journal of Mathematics*, 74:467–482, 2021.

[14] H. K. Elsayied, A. M. Tawfiq, and A. Elsharkawy. The quasi frame and equations of non-lightlike curves in Minkowski E_1^3 and E_1^4. *Italian Journal of Pure and Applied Mathematics*, 225, 2023.

[15] A. Elsharkawy. Generalized involute and evolute curves of equiform spacelike curves with a timelike equiform principal normal in E_1^3. *Journal of the Egyptian Mathematical Society*, 28(1):26, 2020.

[16] A. Elsharkawy. Exploring Hasimoto surfaces within equiform geometry in Minkowski space. *Physica Scripta*, 100(1):016101, 2024.

[17] A. Elsharkawy, C. Cesarano, and H. Alhazmi. On the jerk and snap in motion along non-lightlike curves in Minkowski 3-space. *Mathematical Methods in the Applied Sciences*, pages 1–13, 2024.

[18] A. Elsharkawy, C. Cesarano, A. Tawfiq, and A. A. Ismail. The non-linear schrodinger equation associated with the soliton surfaces in Minkowski 3-space. *AIMS Mathematics*, 7:17879–17893, 2022.

[19] A. Elsharkawy and A. M. Elshenhab. Mannheim curves and their partner curves in Minkowski 3-space E_1^3. *Demonstratio Mathematica*, 55(1):798–811, 2022.

[20] A. Elsharkawy, Y. Tashkandy, W. Emam, C. Cesarano, and N. Elsharkawy. On some quasi-curves in Galilean three-space. *Axioms*, 12(9):823, 2023.

[21] N. Elsharkawy, C. Cesarano, R. Dmytryshyn, and A. Elsharkawy. Timelike spherical curves according to equiform Bishop frame in 3-dimensional Minkowski space. *Carpathian Mathematical Publications*, 15(2):88–95, 2023.

[22] A. M. Elshenhab, O. Moaaz, I. Dassios, and A. Elsharkawy. Motion along a space curve with a quasi-frame in Euclidean 3-space: Acceleration and jerk. *Symmetry*, 14:1610, 2022.

[23] R. Evans. *The Projective Cast: Architecture and Its Three Geometries*. MIT Press, 2000.

[24] E. Hamouda, C. Cesarano, S. Askar, and A. Elsharkawy. Resolutions of the jerk and snap vectors for a quasi curve in Euclidean 3-space. *Mathematics*, 9(23):3128, 2021.

[25] E. Hamouda, O. Moaaz, C. Cesarano, S. Askar, and A. Elsharkawy. Geometry of solutions of the quasi-vortex filament equation in Euclidean 3-space E^3. *Mathematics*, 10:891, 2022.

[26] G. Hu, H. Cao, J. Wu, and G. Wei. Construction of developable surfaces using generalized c-bezier bases with shape parameters. *Computational and Applied Mathematics*, 39(3), 2020.

[27] A. Karger and J. Novak. *Space kinematics and Lie groups*. STNL Publishers of Technical Literature, Prague, 1978.

[28] Y. H. Kim and W. D. Yoon. Classification of ruled surfaces in Minkowski 3-space.

Journal of Geometry and Physics, 49:89–100, 2004.

[29] Y. Lam, Wai. Minimal surfaces from infinitesimal deformations of circle packings. *Advances in Mathematics*, 362:106939, 2020.

[30] Y. Li, H.S. Abdel-Aziz, H.M. Serry, F.M. El-Adawy, and M.K. Saad. Geometric visualization of evolved ruled surfaces via alternative frame in lorentz-Minkowski 3-space. *Space*, 3:1, 2024.

[31] Y. Li, H. Pottmann, J. Wallner, Y. Yang, and W. Wang. Geometric modeling with conical meshes and developable surfaces. *ACM Transactions on Graphics*, 25(3):681–689, 2006.

[32] T. G. Nelson, T. Zimmerman, S. P. Magleby, R. J. Lang, and L. L. Howell. Developable mechanisms on developable surfaces. *Science Robotics*, 4(27), 2019.

[33] M. Peternel, H. Pottmann, and B. Ravani. On the computational geometry of ruled surfaces. *Computer Aided Geometric Design*, 31:17–32, 1999.

[34] H. Pottmann, W. Lu, and B. Ravani. Rational ruled surfaces and their offsets. *Graphical Models and Image Processing*, 58(6):544–552, 1996.

[35] E. Solouma, I. Al-Dayel, M. A. Khan, and M. Abdelkawy. Investigation of special type-Î smarandache ruled surfaces due to rotation minimizing darboux frame in E^3. *Symmetry*, 15(12):2207, 2023.

[36] E. Solouma, I. Al-Dayel, M. A. Khan, and Y. A. Lazer. Characterization of imbricate-ruled surfaces via rotation-minimizing darboux frame in Minkowski 3-space E_1^3. *AIMS Mathematics*, 9(5):13028–13042, 2024.

[37] Y. Tashkandy, W. Emam, C. Cesarano, M. A. El-Raouf, and A. Elsharkawy. Generalized spacelike normal curves in Minkowski three-space. *Mathematics*, 10(21):4145, 2022.

[38] A. M. Tawfiq, C. Cesarano, and A. Elsharkawy. A new method for resolving the jerk and jounce vectors in Euclidean 3-space. *Mathematical Methods in the Applied Sciences*, 46(8):8779–8792, 2023.

[39] A. Turgut and H. H. Hacisalihoglu. Timelike ruled surfaces in the Minkowski 3-space. *Far East Journal of Mathematical Sciences*, 5(1):83–90, 1997.