EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 1, Article Number 5714
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

New Criteria for Guaranteeing Oscillation of Second-Order Differential Equations with Several Delays

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Abstract. The primary objective of this work is to establish new criteria to guarantee the oscillation of solutions for second-order differential equations with p-Laplace type operator. New prerequisites are presented in order to analyze the oscillatory features of the analyzed equations. To support these findings, we employ a range of analysis tools, establishing new conditions to address specific the problems that have hindered previous researches. More specifically, we obtain results that both build upon and extend those discovered in earlier studies by applying the Riccati transformation and the principles of comparison. Several examples are given to illustrate the significance of our results.

2020 Mathematics Subject Classifications: 34C10, 34K11

Key Words and Phrases: Differential equations, Oscillation theorems, Second-order, Delay terms

1. Introduction

In this article, we examine the p-Laplace type operator oscillation problem for second-order differential equations

$$\left(b\left(t\right)\left|\varpi'(t)\right|^{p-2}\varpi'(t)\right)' + \sum_{i=1}^{n} q_i \left|\varkappa^{p-2}\left(\pi_i\left(t\right)\right)\right| \varkappa\left(\pi_i\left(t\right)\right) = 0, t \geq t_0, \tag{1}$$

where $p > 1, \varpi\left(t\right) := \varkappa\left(t\right) + y\left(t\right)\varkappa\left(\zeta\left(t\right)\right)$, $b \in C\left([t, \infty), (0, \infty)\right)$, $y \in C\left([t, \infty), [0, \infty)\right)$, $q_i \in C\left([t, \infty), [0, \infty)\right), \zeta, \pi_i \in C\left([t, \infty), \mathbb{R}\right), \zeta\left(t\right) \leq t,\ \pi_i\left(t\right) \leq t, \lim_{t\to\infty}\zeta\left(t\right) = \lim_{t\to\infty}\pi_i\left(t\right) = \infty,\ q_i\left(t\right)$ does not vanish identically, $i = 1, 2, ..., n,\ y\left(t\right) < 1$ and

$$\int_{t_0}^{\infty} b^{-1/(p-1)}\left(s\right) \mathrm{d}s = \infty. \tag{2}$$

DOI: https://doi.org/10.29020/nybg.ejpam.v18i1.5714

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Definition 1. *By a solution of (1), we mean a function* $\varkappa \in C^1([t,\infty),\mathbb{R})$ *,* $t_\varkappa \geq t_0$*, which has the property* $b(t)(\varpi'(t))^{(p-1)} \in C^1([t_0,\infty),\mathbb{R}), p > 1$*, and satisfies (1) on* $[t_\varkappa,\infty)$*. We consider only those solutions* $\varkappa$ *of (1) which satisfy* $\sup\{|\varkappa(t)| : t \geq t_\varkappa\} > 0$*, for all* $t > t_\varkappa$*.*

Definition 2. $\varkappa$ *is referred to as oscillatory if it is neither finally positive nor eventually negative. Otherwise it is referred to as non-oscillatory. If every solution to the equation oscillates, then the equation is said to be oscillatory.*

In differential equations (DEs), which are the most successful models for studying natural events, each dependant variable represents a quantity in the modeled phenomenon. DEs have helped us understand a variety of complex events in our daily lives and are crucial to many technical applications. These days, they are essential tools in applied sciences and technology, used to study media, conversations, phone signals, and online purchasing data. In a more traditional sense, astronomers used them to describe the motion of stars and the orbits of planets. They also serve a variety of purposes in biology and medicine, see [11].

Neutral differential equations (NDEs), a specific subset of functional differential equations, have derivatives that depend on both the function's derivatives from earlier periods and its current values. This unique characteristic distinguishes NDEs from traditional differential equations and establishes a distinct analytical framework. The relationship between NDEs and FDEs is essential because they often arise in systems where past values and rates of change influence future states. NDEs' significance is particularly evident in fields like control theory and signal processing since they represent systems with memory effects. For instance, in mechanical systems with inertia, acceleration may be affected by both velocity and current position. This association emphasizes the importance of NDEs in accurately modeling and simulating dynamic systems. Moreover, the research of NDEs complements that of DDEs because understanding one usually provides significant insights into the other [2, 7, 9, 14]. These equations find use in a wide range of fields, including problems requiring masses connected to a flexible, shaky rod [10, 13, 16].

The ordinary differential equation (ODE) is a crucial tool for understanding and modeling a wide range of technical and natural systems. The complexity and diversity of real-world events often necessitate the use of sophisticated arguments to obtain more comprehensive and correct solutions, despite the widespread use of ODEs (see [15, 17]). The behavior of many nonlinear systems is not well described by conventional linear differential equations, which emphasizes the importance of including complex arguments into ODEs. Advanced nonlinear dynamics may make these systems more realistically represented, improving insights and predictions. Furthermore, perturbation methods can be used to analyze systems that are subject to small perturbations, providing a means of understanding how complex systems react in different contexts. Furthermore, stability analysis is essential for determining the long-term behavior of ODE solutions, which is important in fields such as control theory and epidemiology (see [12]).

In recent years, there has been a substantial advancement in the study of oscillation conditions for higher-order equations, particularly second-order differential equations with delays [1, 3]. This explains why the qualitative aspects of these equations are so fascinating. Oscillation phenomena are present in many real-world models; for instance, mathematical biology models that use cross-diffusion terms to construct oscillation and/or delay actions are discussed in the publications [5, 6]. This methodology includes a detailed development of the oscillation theory of this type of equation.

Alqahtani et al. [22] established asymptotic behavior for equations with several delays

$$\left(b\left(t\right)\left(\varkappa'\left(t\right)\right)^{\alpha}\right)' + \sum_{i=1}^{n} q_i\left(t\right)\varkappa^{\alpha}\left(\pi_i\left(t\right)\right) = 0, \alpha > 0.$$

In [8], the authors was able to provide some oscillation conditions for

$$(b(t)|\varkappa'(t)|^{\alpha-1}\varkappa'(t))' + q(t)|\varkappa[\pi(t)]|^{\alpha-1}\varkappa[\pi(t)] = 0. \tag{3}$$

Later contributions include studies by Sahiner and Wang [18, 20], Zhao and Meng [23] and Xu and Weng [21] that focus on oscillation criteria and asymptotic behavior. Baculikova and Dzurina's recent study [4] is significant because it provides crucial new details on oscillation conditions for second-order delay differential equations of type

$$\left(b\left(t\right)\left(\left(\varkappa\left(t\right) + y\left(t\right)\varkappa\left(\zeta\left(t\right)\right)\right)'\right)^{\alpha}\right)' + q\left(t\right)\varkappa^{\alpha}\left(\pi\left(t\right)\right) = 0. \tag{4}$$

Lastly, recent publications by Al-Jaser et al. [3] give additional helpful criteria for assessing the asymptotic and oscillatory behavior of solutions. The first-order differential equations and the second-order (4) differential equations are compared using established comparison theorems.

In this research, we use comparison principles and Riccati transformations to obtain the different conditions for oscillation of (1). Examples are provided to illustrate the main findings. The format of this document is as follows. In the first section (Introduction), we present the studied equation and the general conditions needed to reach the main results of the paper. We also provide an overview of pertinent topics and the goal of this study. The oscillation results discussed in the "Oscillation Results" part will be derived using a few relationships and findings that we present in Section 2. In Section 3, we provide several examples to illustrate the significance of the obtained results. We summarize the main conclusions of the paper in Section 4 and draw attention to an open question that may be of interest to researchers in the considered field.

2. Oscillation Results

We start by listing a number of auxiliary lemmas and conditions that we will employ in order to accomplish the primary findings.
For ease of use, we set the following notation:

$$B_{t_0}\left(t\right) \ := \int_{t_0}^{t} b^{-1/(p-1)}\left(t\right) \mathrm{d}t, p > 1,$$

$$\widetilde{B}_{t_0}(t) \quad : \quad = B_{t_0}(t) + \frac{1}{(p-1)} \int_{t_0}^{t} B_{t_1}(t)\, B_{t_0}^{p-1}(\pi_i(t)) \sum_{i=1}^{n} q_i(t)\,(1 - y(\pi_i(t)))^{(p-1)}\, \mathrm{d}t,$$

and

$$\widehat{B}(t) := \exp\left(-(p-1) \int_{\pi_i(t)}^{t} \frac{\mathrm{d}t}{\widetilde{B}_{t_0}(t) b^{1/(p-1)}(t)} \right).$$

Lemma 1. *[4] If $\varkappa$ be an eventually positive solution of (1), then*

$$\varpi(t) > 0,\ \varpi'(t) > 0, \left(b(t)\,(\varpi'(t))^{(p-1)} \right)' \leq 0, \tag{5}$$

for $t \geq t_1$.

Lemma 2. *[19] Let $G, W > 0$ be constants and*

$$\max_{\varkappa \in b} f = f(\varkappa^*) = \alpha^{\alpha} (\alpha+1)^{-(\alpha+1)} \frac{G^{\alpha+1}}{W^{\alpha}}, \alpha \geq 1, \tag{6}$$

where $\varkappa^* = (\alpha G / ((\alpha+1) W))^{\alpha}$ and $f(\varkappa) = G\varkappa - W \varkappa^{(\alpha+1)/\alpha}$.

Lemma 3. *Let $\varkappa$ be an eventually positive solution of (1). Then*

$$\left(b(t)\,(\varpi'(t))^{(p-1)} \right)' \leq -\sum_{i=1}^{n} q_i(t)\,(1 - y(\pi_i(t)))^{(p-1)}\, \varpi^{(p-1)}(\pi_i(t)), \tag{7}$$

and

$$\varpi(t) \geq \widetilde{B}_{t_1}(t)\, b^{1/(p-1)}(t)\, \varpi'(t), \tag{8}$$

also,

$$\left(b(t)\,(\varpi'(t))^{(p-1)} \right)' \leq -\sum_{i=1}^{n} q_i(t)\,(1 - y(\pi_i(t)))^{(p-1)}\, \widehat{B}(t)\, \varpi^{(p-1)}(t). \tag{9}$$

Proof. Let $\varkappa$ be an eventually positive solution of (1). (5) holds according to Lemma 1. Therefore, using the definition of$\varpi(t)$, we get

$$\begin{aligned} \varkappa(t) &= \varpi(t) - y(t)\, \varkappa(\zeta(t)) \\ &\geq \varpi(t) - y(t)\, \varpi(\zeta(t)) \\ &\geq \varpi(t)\,(1 - y(t)). \end{aligned}$$

This suggests that (1)

$$\left(b(t)\,(\varpi'(t))^{(p-1)} \right)' \leq -\sum_{i=1}^{n} q_i(t)\, \varpi^{(p-1)}(\pi_i(t))\,(1 - y(\pi_i(t)))^{(p-1)}.$$

Since $\varpi'(t) > 0$ and $\frac{\partial}{\partial s}\pi_i(t) > 0$, we obtain $\varpi(\pi_i(t)) > \varpi(\pi_i(t))$ and so

$$\left(b(t)\left(\varpi'(t)\right)^{(p-1)}\right)' \leq -\sum_{i=1}^{n} q_i(t)\left(1 - y(\pi_i(t))\right)^{(p-1)} \varpi^{(p-1)}(\pi_i(t)).$$

Using basic computation and the chain rule, it is evident that

$$\begin{aligned} B_{t_1}(t)\left(b(t)\left(\varpi'(t)\right)^{(p-1)}\right)' &= (p-1)\left(b^{1/(p-1)}(t)\varpi'(t)\right)^{(p-1)-1} B_{t_1}(t)\left(b^{1/(p-1)}(t)\varpi'(t)\right)' \\ &= -(p-1)\left(b^{1/(p-1)}(t)\varpi'(t)\right)^{(p-1)-1} \frac{\mathrm{d}}{\mathrm{d}t}\left(\varpi(t) - B_{t_1}(t)b^{1/(p-1)}(t)\varpi'(t)\right). \end{aligned} \tag{10}$$

Combining (7) and (10), we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\varpi(t) - B_{t_1}(t)b^{1/(p-1)}(t)\varpi'(t)\right) \geq& \left(\frac{1}{(p-1)}B_{t_1}(t)\left(b^{1/(p-1)}(t)\varpi'(t)\right)^{2-p}\right) \\ & \left(\sum_{i=1}^{n} q_i(t)\left(1 - y(\pi_i(t))\right)^{p-1}\varpi^{p-1}(\pi_i(t))\right). \end{aligned}$$

Integrating this inequality from t_1 to t, we have

$$\begin{aligned} \varpi(t) \geq& B_{t_1}(t)b^{1/(p-1)}(t)\varpi'(t) \\ &+\frac{1}{(p-1)}\int_{t_1}^{t} B_{t_1}(t)\sum_{i=1}^{n} q_i(t)\left(1 - y(\pi_i(t))\right)^{(p-1)}\left(b^{1/(p-1)}(t)\varpi'(t)\right)^{2-p}\varpi^{(p-1)}(\pi_i(t))\mathrm{d}t. \end{aligned} \tag{11}$$

From the monotonicity of $b^{1/(p-1)}(t)\varpi'(t)$, we have

$$\varpi(t) = \varpi(t_1) + \int_{t_1}^{t} \frac{1}{b^{1/(p-1)}(t)}\left(b^{1/(p-1)}(t)\varpi'(t)\right)\mathrm{d}t \geq B_{t_1}(t)b^{1/(p-1)}(t)\varpi'(t).$$

So, by $\left(b^{1/(p-1)}(t)\varpi'(t)\right)' \leq 0$, (11) becomes

$$\begin{aligned} \varpi(t) \geq& B_{t_1}(t)b^{1/(p-1)}(t)\varpi'(t) \\ &+\frac{1}{(p-1)}\int_{t_1}^{t}(B_{t_1}(t)\sum_{i=1}^{n} q_i(t)\left(1 - y(\pi_i(t))\right)^{(p-1)}\left(b^{1/(p-1)}(t)\varpi'(t)\right)^{1-(p-1)} B_{t_1}^{(p-1)}(\pi_i(t)) \\ &\left[b(\pi_i(t))\left(\varpi'(\pi_i(t))\right)^{(p-1)}\right])\mathrm{d}t \\ \geq& B_{t_1}(t)b^{1/(p-1)}(t)\varpi'(t) \\ &+\frac{1}{(p-1)}\int_{t_1}^{t}\left(b^{1/(p-1)}(t)\varpi'(t)\right)^{2-p} B_{t_1}(t)B_{t_1}^{(p-1)}\pi_i(t)\sum_{i=1}^{n} q_i(t)\left(1 - y(\pi_i(t))\right)^{(p-1)}\left[b^{1/(p-1)}(t)\varpi'(t)\right. \\ \geq& b^{1/(p-1)}(t)\varpi'(t)\left[B_{t_1}(t) + \frac{1}{(p-1)}\int_{t_1}^{t} B_{t_1}(t)B_{t_1}^{p-1}(\pi_i(t))\sum_{i=1}^{n} q_i(t)\left(1 - y(\pi_i(t))\right)^{p-1}\mathrm{d}t\right]. \end{aligned}$$

$$\geq \widetilde{B}_{t_1}(t) b^{1/(p-1)}(t)\, \varpi'(t),$$

or

$$\frac{\varpi'(t)}{\varpi(t)} \leq \frac{1}{\widetilde{B}_{t_1}(t) b^{1/(p-1)}(t)}.$$

Integrating from $\pi_i(t)$ to t, we find that

$$\frac{\varpi(\pi_i(t))}{\varpi(t)} \geq \exp\left(-\int_{\pi_i(t)}^{t} \frac{\mathrm{d}t}{\widetilde{B}_{t_1}(t) b^{1/(p-1)}(t)}\right),$$

which with (7), gives

$$\begin{aligned} \frac{\left(b(t)(\varpi'(t))^{(p-1)}\right)'}{\varpi^{(p-1)}(t)} &\leq -\sum_{i=1}^{n} q_i(t)(1-y(\pi_i(t)))^{(p-1)} \left(\frac{\varpi(\pi_i(t))}{\varpi(t)}\right)^{(p-1)} \\ &\leq -\sum_{i=1}^{n} q_i(t)(1-y(\pi_i(t)))^{(p-1)} \widehat{B}(t). \end{aligned}$$

The proof is complete.

Lemma 4. *Let (1) have a positive solution. If*

$$\xi(t) = x(t)\, b(t) \left(\frac{\varpi'(t)}{\varpi(t)}\right)^{p-1} > 0, \tag{12}$$

then

$$\xi'(t) \leq \frac{x'_+(t)}{x(t)}\xi(t) - x(t)\sum_{i=1}^{n} q_i(t)(1-y(\pi_i(t)))^{p-1}\, \widehat{B}(t) - \frac{(p-1)}{(x(t)\, b(t))^{1/(p-1)}} \xi^{p/(p-1)}(t). \tag{13}$$

Proof. Let $\varkappa$ be a positive solution of equation (1). From Lemma 3, we have (9) holds. Thus, when we differentiate $\xi(t)$ we get

$$\xi'(t) = \frac{x'(t)}{x(t)}\xi(t) + x(t)\frac{(b(t)\,\varpi'(t))'}{\varpi^{(p-1)}(t)} - (p-1)\, x(t)\, b(t) \left(\frac{\varpi'(t)}{\varpi(t)}\right)^{p}.$$

From (9) and (12), we see that

$$\xi'(t) \leq \frac{x'_+(t)}{x(t)}\xi(t) - 1x(t)\sum_{i=1}^{n} q_i(t)(1-y(\pi_i(t)))^{(p-1)}\, \widehat{B}(t) - \frac{(p-1)}{(x(t)\, b(t))^{1/(p-1)}} \xi^{p/(p-1)}(t).$$

The proof is complete.

Theorem 1. *If the equation*

$$\omega'(t) + \widetilde{B}_{t_1}^{(p-1)}(\pi_i(t)) \sum_{i=1}^{n} q_i(t)\left(1 - y(\pi_i(t))\right)^{(p-1)} \omega(\pi_i(t)) = 0, \tag{14}$$

is oscillatory, then (1) is oscillatory.

Proof. Let $\varkappa(t) > 0$, that is $\varkappa(\zeta(t)) > 0$ and $\varkappa(\pi_i(t)) > 0$. From Lemma 3, we have (7) and (8) hold. Using (7) and (8), we find $\omega(t) = b(t)(\varpi'(t))^{(p-1)}$ is a positive solution of

$$\omega'(t) + \widetilde{B}_{t_1}^{(p-1)}(\pi_i(t)) \sum_{i=1}^{n} q_i(t)\left(1 - y(\pi_i(t))\right)^{(p-1)} \omega(\pi_i(t)) \leq 0.$$

By [12, Theorem 1], then also, the solution of the associated equation (14) is a positive, and this a contradiction. The proof is complete.

Corollary 1. *Let*

$$\limsup_{t\to\infty} \int_{\pi_i(t)}^{t} \widetilde{B}_{t_1}^{(p-1)}(\pi_i(t)) \sum_{i=1}^{n} q_i(t)\left(1 - y(\pi_i(t))\right)^{(p-1)} \mathrm{d}t > 1, \ \frac{\partial}{\partial t}\pi_i(t) \geq 0, \tag{15}$$

or

$$\liminf_{t\to\infty} \int_{\pi_i(t)}^{t} \widetilde{B}_{t_1}^{(p-1)}(\pi_i(t)) \sum_{i=1}^{n} q_i(t)\left(1 - y(\pi_i(t))\right)^{p-1} \mathrm{d}t > \frac{1}{\mathrm{e}}, \tag{16}$$

then all solutions of (1) is oscillatory.

Proof. As may be shown from [10, Theorem 2.1.1], (15) or (16) guarantee oscillation of (14).

Lemma 5. *Suppose π_i is strictly growing in relation to t and*

$$\liminf_{t\to\infty} \int_{\pi_i(t)}^{t} \widetilde{B}_{t_1}^{(p-1)}(\pi_i(t)) \sum_{i=1}^{n} q_i(t)\left(1 - y(\pi_i(t))\right)^{p-1} \mathrm{d}t \geq \delta, \tag{17}$$

for some $\delta > 0$, and (1) has an eventually positive solution $\varkappa$. Then,

$$\frac{H(\pi_i(t))}{H(t)} \geq z_n(\delta), n \geq 0, \tag{18}$$

where $H(t) := b(t)(\varpi'(t))^{(p-1)}$, and

$$z_0(t) := 1 \text{ and } z_n(t) := \exp(\rho z_{n-1}(t)). \tag{19}$$

Proof. Let $\varkappa(t) > 0$, $\varkappa(\zeta(t)) > 0$ and $\varkappa(\pi_i(t)) > 0$ for $t \geq t_1$. We conclude that ω is a positive solution of (14) by following the same procedure as in the proof of Theorem 1. We can demonstrate that (18) holds in a manner akin to that used in the proof of Lemma 1 in [19].

Lemma 6. *Let (1) have a positive solution. If*

$$\sigma(t) := \varsigma(t)b(t)\left(\frac{\varpi'(t)}{\varpi\left(\pi_i\left(t\right)\right)}\right)^{p-1} > 0, \tag{20}$$

then

$$\sigma'\left(t\right) \leq -\varsigma\left(t\right)\sum_{i=1}^{n} q_i\left(t\right)\left(1-y\left(\pi_i\left(t\right)\right)\right)^{p-1} + \frac{\varsigma'_+\left(t\right)}{\varsigma(t)}\sigma\left(t\right) - \frac{\left(p-1\right)z_n^{1/(p-1)}\left(\delta\right)\pi'_i\left(t\right)}{\left(\varsigma\left(t\right)b\left(\pi_i\left(t\right)\right)\right)^{1/(p-1)}}\sigma^{p/(p-1)}\left(t\right). \tag{21}$$

Proof. Let $\varkappa$ be a positive solution of equation (1). From Lemma 3, we obtain (7) holds. By Lemma 5, we find

$$\frac{\varpi'\left(\pi_i\left(t\right)\right)}{\varpi'\left(t\right)} \geq \left(\frac{z_n\left(\delta\right)b\left(t\right)}{b\left(\pi_i\left(t\right)\right)}\right)^{1/(p-1)}. \tag{22}$$

Now, we differentiate $\sigma\left(t\right)$, we get

$$\sigma'(t) = \frac{\varsigma'(t)}{\varsigma(t)}\sigma(t) + \varsigma(t)\frac{(b(t)(\varpi'(t))^{p-1})'}{\varpi^{(p-1)}\left(\pi_i\left(t\right)\right)} - (p-1)\varsigma(t)b(t)\left(\frac{\varpi'(t)}{\varpi\left(\pi_i\left(t\right)\right)}\right)^{(p-1)}\left(\frac{\varpi'\left(\pi_i\left(t\right)\right)}{\varpi\left(\pi_i\left(t\right)\right)}\right)\pi'_i\left(t\right).$$

From (7), (20) and (22), we obtain

$$\sigma'\left(t\right) \leq -\varsigma\left(t\right)\sum_{i=1}^{n} q_i\left(t\right)\left(1-y\left(\pi_i\left(t\right)\right)\right)^{p-1} + \frac{\varsigma'_+\left(t\right)}{\varsigma(t)}\sigma\left(t\right) - \frac{\left(p-1\right)z_n^{1/(p-1)}\left(\delta\right)\pi'_i\left(t\right)}{\left(\varsigma\left(t\right)b\left(\pi_i\left(t\right)\right)\right)^{1/(p-1)}}\sigma^{p/(p-1)}\left(t\right).$$

The proof is complete.

Theorem 2. *Suppose π_i is strictly growing in relation to t and (17) holds. If $\varsigma \in C^1(I,(0,\infty))$ such that*

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left(\varsigma\left(t\right)\sum_{i=1}^{n} q_i\left(t\right)\left(1-y\left(\pi_i\left(t\right)\right)\right)^{p-1} - \frac{\left(\varsigma'_+\left(t\right)\right)^p b\left(\pi_i\left(t\right)\right)}{p^p z_n\left(\delta\right)\varsigma^{(p-1)}\left(t\right)\left(\pi'_i(t)\right)^{p-1}}\right) = \infty, \tag{23}$$

for some $\delta < 0$ and $n \geq 0$, where $\varsigma'_+(t) = \max\{0, \varsigma'(t)\}$ and $z_n(\delta)$ is defined as (19), then all solutions of (1) is oscillatory.

Proof. Suppose $\varkappa\left(t\right) > 0$, $\varkappa\left(\zeta\left(t\right)\right) > 0$ and $\varkappa\left(\pi_i\left(t\right)\right) > 0$. From Lemma 6, we have (21) holds. Using Lemma 2 with $W = \left(p-1\right)z_n^{1/(p-1)}\left(\delta\right)/\left(\varsigma\left(t\right)b\left(\pi_i\left(t\right)\right)\right)^{-1/(p-1)}$ and $G = \varsigma'_+\left(t\right)/\varsigma\left(t\right)$, (21) yield

$$\sigma'\left(t\right) \leq -\varsigma\left(t\right)\sum_{i=1}^{n} q_i\left(t\right)\left(1-y\left(\pi_i\left(t\right)\right)\right)^{(p-1)} + \frac{\varsigma'_+\left(t\right)^p b\left(\pi_i\left(t\right)\right)}{p^p z_n\left(\delta\right)\varsigma^{p-1}\left(t\right)\left(\pi'_i\left(t\right)\right)^{p-1}}.$$

Integrating this inequality from t_1 to t, we find

$$\int_{t_1}^{t}\left(\varsigma(t)\sum_{i=1}^{n}q_i(t)(1-y(\pi_i(t)))^{(p-1)}-\frac{\left(\varsigma'_+(t)\right)^p b(\pi_i(t))}{p^p z_n(\delta)\varsigma^{p-1}(t)(\pi'_i(t))^{p-1}}\right)\mathrm{d}t\leq\sigma(t).$$

A contradiction with condition (23) is then discovered. The proof is finished.

Theorem 3. *If*

$$\lim_{t\to\infty}\sup\int_{t_1}^{t}\left(x(t)\sum_{i=1}^{n}q_i(t)(1-y(\pi_i(t)))^{p-1}\widehat{B}(t)-\frac{b(t)\left(x'_+(t)\right)^p}{p^p x^{p-1}(t)}\right)\mathrm{d}t=\infty, \tag{24}$$

where $x\in C^1(I,(0,\infty))$ *and* $x'_+(t)=\max\{0,\psi'(t)\}$, *then (1) is oscillatory.*

Proof. Let $\varkappa(t)>0$, that is $\varkappa(\zeta(t))$ and $\varkappa(\pi_i(t))$ are positive on $[t_0,\infty)$. From Lemma 3, we have (7)-(9) hold. Next, we arrive at (13) using Lemma 2 with $G=x'_+(t)/x(t)$ and $W=(p-1)(x(t)b(t))^{-1/(p-1)}$ (Lemma 4), the inequality(13) becomes

$$\xi'(t)\leq -x(t)\sum_{i=1}^{n}q_i(t)(1-y(\pi_i(t)))^{p-1}\widehat{B}(t)+\frac{b(t)\left(x'_+(t)\right)^p}{p^p x^{p-1}(t)}.$$

Integrating this inequality from t_1 to t, we have

$$\int_{t_1}^{t}\left(x(t)\sum_{i=1}^{n}q_i(t)(1-y(\pi_i(t)))^{p-1}\widehat{B}(t)-\frac{b(t)\left(x'_+(t)^p\right)}{p^p x^{p-1}(t)}\right)\mathrm{d}t\leq\xi(t).$$

This contradicts the condition (24). The proof is finished.

Now, we obtain some oscillation results for equation (1) using other methods.

Theorem 4. *Let*

$$\int_{t_0}^{\infty}\sum_{i=1}^{n}q_i(t)(1-y(\pi_i(t)))^{(p-1)}\widehat{B}(t)\,\mathrm{d}t=\infty, \tag{25}$$

then, equation (1) is oscillatory.

Proof. Suppose $\varkappa(t)>0$, $\varkappa(\zeta(t))>0$ and $\varkappa(\pi_i(t))>0$, we can infer from Lemma 4 that (13) holds. if we set $x(t):=1$, then (13) becomes

$$\xi'(t)+\sum_{i=1}^{n}q_i(t)(1-y(\pi_i(t)))^{(p-1)}\widehat{B}(t)+(p-1)/(b(t))^{1/(p-1)}\xi^{\frac{p}{(p-1)}}(t)\leq 0, \tag{26}$$

or

$$\xi'(t)+\sum_{i=1}^{n}q_i(t)(1-y(\pi_i(t)))^{(p-1)}\widehat{B}(t)\leq 0. \tag{27}$$

Integrating (27) from t_3 to t and using (25), we arrive at

$$\xi\left(t\right)\leq\xi\left(t_3\right)-\int_{t_3}^{t}\sum_{i=1}^{n}q_i\left(t\right)\left(1-y\left(\pi_i\left(t\right)\right)\right)^{\left(p-1\right)}\widehat{B}\left(t\right)\mathrm{d}s\rightarrow\infty\quad\text{as}\quad t\rightarrow\infty.$$

This contradicts the conclusion that the evidence is complete because $\xi\left(t\right)>0$.

Definition 3. *Assume that the series of functions* $\left\{\vartheta_n\left(t\right)\right\}_{n=0}^{\infty}$ *is defined as*

$$\vartheta_n\left(t\right)=\int_{t}^{\infty}\left(p-1\right)/\left(b\left(s\right)\right)^{1/(p-1)}\vartheta_{n-1}^{\frac{p}{(p-1)}}\left(s\right)\mathrm{d}s+\vartheta_0\left(t\right),\qquad t\geq t_0,\qquad n=1,2,3,...,\tag{28}$$

and

$$\vartheta_0\left(t\right)=\int_{t}^{\infty}\sum_{i=1}^{n}q_i\left(t\right)\left(1-y\left(\pi_i\left(t\right)\right)\right)^{\left(p-1\right)}\widehat{B}\left(t\right)\mathrm{d}t,\qquad t\geq t_0,$$

where $\vartheta_n\left(t\right)\leq\vartheta_{n+1}\left(t\right),\ t\geq t_0$.

Lemma 7. *Let* $\varkappa$ *be a solution of equation (1) that becomes positive for suffciently large* t*. Then* $\xi\left(t\right)\geq\vartheta_n\left(t\right)$ *where* $\lim_{n\rightarrow\infty}\vartheta_n\left(t\right)=\vartheta\left(t\right)$ *for* $t\geq T\geq t_0$ *when* $\vartheta\left(t\right)$ *on* $[T,\infty)$ *and*

$$\vartheta\left(t\right)=\int_{t}^{\infty}\left(p-1\right)/\left(b\left(s\right)\right)^{1/(p-1)}\vartheta^{\frac{p}{(p-1)}}\left(s\right)\mathrm{d}s+\vartheta_0\left(t\right),\ t\geq T.\tag{29}$$

Proof. Let $\varkappa$ be a solution of equation (1) that becomes positive for suffciently large t. We get to (26) by using the same steps as in the proof of Theorem 4. The result of integrating (26) from t to t' is

$$\xi\left(t'\right)-\xi\left(t\right)+\int_{t}^{t'}\sum_{i=1}^{n}q_i\left(s\right)\left(1-y\left(\pi_i\left(s\right)\right)\right)^{\left(p-1\right)}\widehat{B}\left(s\right)ds+\int_{t}^{t'}\xi^{\frac{p}{(p-1)}}\left(s\right)\left(p-1\right)/\left(b\left(s\right)\right)^{1/(p-1)}\mathrm{d}s\leq0.$$

This implies

$$\xi\left(t'\right)-\xi\left(t\right)+\int_{t}^{t'}\xi^{\frac{p}{(p-1)}}\left(s\right)\left(p-1\right)/\left(b\left(s\right)\right)^{1/(p-1)}\mathrm{d}s\leq0.$$

Then, we conclude that

$$\int_{t}^{\infty}\xi^{\frac{p}{(p-1)}}\left(s\right)\left(p-1\right)/\left(b\left(s\right)\right)^{1/(p-1)}\mathrm{d}s<\infty\text{ for }t\geq T,\tag{30}$$

Otherwise, when $t'\rightarrow\infty$, $\xi(t')\leq\xi(t)-\int_t^{t'}\xi^{\frac{p}{(p-1)}}\left(s\right)\left(p-1\right)/\left(b\left(s\right)\right)^{1/(p-1)}\mathrm{d}s\rightarrow-\infty$, which contradicts $\xi(t)>0$. Given that $\xi(t)>0$ and $\xi'(t)>0$, (26) indicates that

$$\xi\left(t\right)\quad\geq\quad\int_{t}^{\infty}\sum_{i=1}^{n}q_i\left(t\right)\left(1-y\left(\pi_i\left(t\right)\right)\right)^{\left(p-1\right)}\widehat{B}\left(t\right)\mathrm{d}t+\int_{t}^{\infty}\xi^{\frac{p}{(p-1)}}\left(s\right)\left(p-1\right)/\left(b\left(s\right)\right)^{1/(p-1)}\mathrm{d}s$$

$$= \quad \vartheta_0(t) + \int_t^\infty \xi^{\frac{p}{(p-1)}}(s)(p-1)/(b(s))^{1/(p-1)} \mathrm{d}s, \tag{31}$$

or

$$\xi(t) \geq \int_t^\infty \sum_{i=1}^n q_i(t)(1-y(\pi_i(t)))^{(p-1)} \widehat{B}(t) \mathrm{d}t := \vartheta_0(t).$$

Consequently, $\xi(t) \geq \vartheta_n(t)$, where $n = 1, 2, 3, ...$ We obtain that $\vartheta_n \to \vartheta$ as $n \to \infty$ since $\{\vartheta_n(t)\}_{n=0}^\infty$ is growing and bounded above. The monotone convergence theorem of Lebesgue shows that when $n \to \infty$, (28) becomes (29).

Theorem 5. *If*

$$\liminf_{t\to\infty} \frac{1}{\vartheta_0(t)} \int_t^\infty \vartheta_0^{\frac{p}{(p-1)}}(s)(p-1)/(b(s))^{1/(p-1)} \mathrm{d}s > \frac{p-1}{p^{\frac{p}{p-1}}}, \tag{32}$$

then all solutions (1) are oscillatory.

Proof. Assume that $\varkappa(t) > 0$, meaning that both $\varkappa(\zeta(t))$ and $\varkappa(\pi_i(t))$ are positive. Following the same steps as in the Lemma 7 proof, we get (31). Using (31), we discover

$$\frac{\xi(t)}{\vartheta_0(t)} \geq 1 + \frac{1}{\vartheta_0(t)} \int_t^\infty \vartheta_0^{\frac{p-}{p-1}}(s)(p-1)/(b(s))^{1/(p-1)} \left(\frac{\xi(s)}{\vartheta_0(s)}\right)^{\frac{p}{p-1}} \mathrm{d}s. \tag{33}$$

If we consider $\mu = inf_{t\geq T}(\xi(t)/\vartheta_0(t))$, then $\mu \geq 1$ of course. We can observe using(32) and (33) that

$$\mu \geq p\left(\frac{\mu}{p}\right)^{\frac{p}{p-1}},$$

or

$$\frac{p-1}{p}\left(\frac{\mu}{p}\right)^{\frac{p}{p-1}} + \frac{1}{p} \leq \frac{\mu}{p}.$$

It defies the predicted value of μ and p, hence, the proof is finished.

Theorem 6. *Let*

$$\limsup_{t\to\infty} \vartheta_n(t) \left(\int_{t_0}^t b^{-\frac{1}{(p-1)}}(s) \mathrm{d}s\right)^{(p-1)} > 1, \tag{34}$$

then every solutions of (1) are oscillatory.

Proof. Assume that $\varkappa(t) > 0$, meaning that both $\varkappa(\zeta(t))$ and $\varkappa(\pi_i(t))$ are positive. From (12), we obtain

$$\frac{1}{\xi(t)} \quad = \quad \frac{1}{b(t)}\left(\frac{\varpi(t)}{\varpi'(t)}\right)^{(p-1)} = \frac{1}{b(t)}\left(\frac{\varpi(T) + \int_T^t b^{-1/(p-1)}(s)\, b^{1/(p-1)}(s)\, \varpi'(s) \mathrm{d}s}{\varpi'(t)}\right)^{(p-1)}$$

$$\begin{aligned} &\geq \frac{1}{b(t)}\left(\frac{b^{1/(p-1)}(t)\,\varpi'(t)\int_T^t b^{-1/(p-1)}(s)\,\mathrm{d}s}{\varpi'(t)}\right)^{(p-1)} \\ &= \left(\int_T^t b^{-1/(p-1)}(s)\,\mathrm{d}s\right)^{(p-1)}, \end{aligned} \tag{35}$$

for $t \geq T$. So, from (35) we find

$$\xi(t)\left(\int_{t_0}^t b^{-1/(p-1)}(s)\,\mathrm{d}s\right)^{(p-1)} \leq \left(\frac{\int_{t_0}^t b^{-1/(p-1)}(s)\,\mathrm{d}s}{\int_T^t b^{-1/(p-1)}(s)\,\mathrm{d}s}\right)^{(p-1)},$$

and so

$$\limsup_{t\to\infty} \xi(t)\left(\int_{t_0}^t b^{-\frac{1}{(p-1)}}(s)\mathrm{d}s\right)^{(p-1)} \leq 1,$$

which contradicts (34). Hence, the proof is finished.

Corollary 2. *If*

$$\int_{t_0}^{\infty}\sum_{i=1}^{n} q_i(t)\,(1-y(\pi_i(t)))^{(p-1)}\,\widehat{B}(t)\exp\left(\int_{t_0}^t \vartheta_n^{\frac{1}{(p-1)}}(s)\,(p-1)/(b(s))^{1/(p-1)}\,\mathrm{d}s\right)\mathrm{d}t = \infty, \tag{36}$$

or

$$\int_{t_0}^{\infty}(p-1)/(b(t))^{1/(p-1)}\,\vartheta_n^{\frac{1}{(p-1)}}(t)\vartheta_0(t)\exp\left(\int_{t_0}^t (p-1)/(b(s))^{1/(p-1)}\,\vartheta_n^{\frac{1}{(p-1)}}(s)\mathrm{d}s\right)\mathrm{d}t = \infty, \tag{37}$$

then all solutions (1) are oscillatory.

Proof. Suppose that $\varkappa(t) > 0$, meaning that both $\varkappa(\zeta(t))$ and $\varkappa(\pi_i(t))$ are positive on $[t_0, \infty)$. (29) holds according to Lemma 7. (29) gives us

$$\begin{aligned} \vartheta'(t) &= -(p-1)/(b(t))^{1/(p-1)}\,\vartheta^{\frac{p-}{p-1}}(t) - \sum_{i=1}^{n} q_i(t)\,(1-y(\pi_i(t)))^{p-1}\,\widehat{B}(t) \\ &\leq -(p-1)/(b(t))^{1/(p-1)}\,\vartheta_n^{\frac{1}{p-1}}(t)\,\vartheta(t) - \sum_{i=1}^{n} q_i(t)\,(1-y(\pi_i(t)))^{p-1}\,\widehat{B}(t). \end{aligned} \tag{38}$$

Hence,

$$\int_T^t\sum_{i=1}^{n} q_i(s)\,(1-y(\pi_i(s)))^{p-1}\,\widehat{B}(s)\exp\left(\int_T^s \vartheta_n^{\frac{1}{(p-1)}}(t)\,(p-1)/(b(t))^{1/(p-1)}\,\mathrm{d}t\right)\mathrm{d}s \leq \vartheta(T) < \infty,$$

which contradicts (36).

Next, let $M(t) = \int_t^\infty (p-1)/(b(s))^{1/(p-1)}\,\vartheta^{\frac{p}{p-1}}(s)\,\mathrm{d}s$. Then, we obtain

$$M'(t) = -(p-1)/(b(t))^{1/(p-1)}\,\vartheta^{\frac{p}{p-1}}(t)$$

$$\begin{aligned} &\leq \ -\left(p-1\right)/\left(b\left(t\right)\right)^{1/(p-1)}\vartheta_n^{\frac{1}{p-1}}\left(t\right)\vartheta\left(t\right) \\ &= \ -\left(p-1\right)/\left(b\left(t\right)\right)^{1/(p-1)}\vartheta_n^{\frac{1}{p-1}}\left(t\right)\left(M\left(t\right)+\vartheta_0\left(t\right)\right). \end{aligned}$$

Consequently, we discover

$$\int_T^\infty \left(p-1\right)/\left(b\left(t\right)\right)^{1/(p-1)}\vartheta_n^{\frac{1}{p-1}}(t)\vartheta_0(t)\exp\left(\int_T^t \left(p-1\right)/\left(b\left(s\right)\right)^{1/(p-1)}\vartheta_n^{\frac{1}{p-1}}(s)\mathrm{d}s\right)\mathrm{d}t<\infty.$$

This runs counter to (37). The proof is finished.

3. Examples and Discussion

Example 1. *Let the equation*

$$\left(\left(\left(\varkappa\left(t\right)+y_0\varkappa\left(\zeta_0 t\right)\right)'\right)^{(p-1)}\right)'+\frac{q_0t^{p-1}-pt^p}{t^{2p-1}}\varkappa^{(p-1)}\left(\varepsilon t\right)+\frac{p}{t^{p-1}}\varkappa^{(p-1)}\left(\varepsilon t\right)=0, \tag{39}$$

Let $b\left(t\right)=1, y\left(t\right)=y_0, \zeta\left(t\right)=\zeta_0 t, \sum_{i=1}^n q_i\left(t\right)=\frac{q_0t^{p-1}-pt^p}{t^{2p-1}}+\frac{p}{t^{p-1}}$ *and* $\pi_i\left(t\right)=\varepsilon t,$*where* $\varepsilon,\zeta_0\in\left(0,1\right).$
It is easy to verify that

$$\sum_{i=1}^n q_i\left(t\right)\left(1-y\left(\pi_i\left(t\right)\right)\right)^{p-1}=\frac{q_0}{t^p}\left(1-\varepsilon\right)\left[1-y_0\right]^{p-1},$$

$$B_{t_0}\left(t\right)=t \text{ and } \widetilde{B}_{t_0}(t)=Mt,$$

where

$$M:=1+\varepsilon^{(p-1)}\frac{q_0}{(p-1)}\left(1-\varepsilon\right)\left[1-y_0\right]^{(p-1)}.$$

By Corollary 1, we find (39) is oscillatory if

$$\left(M^{(p-1)}\varepsilon^{(p-1)}q_0\left(1-\varepsilon\right)\left[1-y_0\right]^{(p-1)}\right)\ln\frac{1}{\varepsilon}>\frac{1}{\mathrm{e}},$$

or

$$\left(p-1\right)\left(M-1\right)M^{(p-1)}\ln\frac{1}{\varepsilon}>\frac{1}{\mathrm{e}}. \tag{40}$$

Also, we find that

$$\widehat{B}_{t_1}\left(t\right)=\varepsilon^{1/M}, \ \sum_{i=1}^n q_i\left(t\right)\left(1-y\left(\pi_i\left(t\right)\right)\right)^{(p-1)}\widehat{B}\left(t\right)=\frac{q_0\left(1-y_0\right)^{(p-1)}\left(1-\varepsilon\right)}{t^p}\varepsilon^{(p-1)/M},$$

and

$$\int_t^\infty\sum_{i=1}^n q_i\left(s\right)\left(1-y\left(\pi_i\left(s\right)\right)\right)^{p-1}\widehat{B}\left(s\right)\mathrm{d}s=q_0\left(1-y_0\right)^{p-1}\left(1-\varepsilon\right)\varepsilon^{(p-1)/M}\left(p-1\right)^{-1}t^{-p}.$$

From Theorem 5, (39) is oscillatory if

$$\left(\frac{q_0\left(1-y_0\right)^{p-1}\left(1-\varepsilon\right)}{p-1}\varepsilon^{(p-1)/M}\right)^{1/(p-1)}>\left(p-1\right)p^{-p/(p-1)}.$$

Example 2. *Consider the differential equation*

$$\left(\varkappa\left(t\right)+\frac{1}{2}\varkappa\left(\zeta_0 t\right)\right)''+\frac{q_0 t-2}{t^3}\varkappa\left(1/3t\right)+\frac{2}{t^3}\varkappa\left(1/3t\right)=0, \tag{41}$$

where $\zeta_0\in(0,1)$. *Let* $b\left(t\right)=1, y\left(t\right)=\frac{1}{2}, \zeta\left(t\right)=\zeta_0 t, q_i\left(t\right)=\frac{q_0 t-2}{t^3}+\frac{2}{t^3}$ *and* $\pi_i\left(t\right)=1/3t$. *From Theorem 5, we find the equation (41) is oscillatory if*

$$\frac{q_0}{6}\left(1+\frac{1}{6}q_0\right)\ln 3>\frac{1}{\mathrm{e}}, \tag{42}$$

thus, he equation (41) is oscillatory if $q_0>1.588$.

Example 3. *Let equation*

$$\left(\varkappa\left(t\right)+\frac{1}{2}\varkappa\left(\frac{t}{3}\right)\right)''+\frac{q_0}{t^2}\varkappa\left(\frac{t}{2}\right)=0, q_0>0. \tag{43}$$

Let $b\left(t\right)=1, y\left(t\right)=\frac{1}{2}, \zeta\left(t\right)=\frac{t}{3}, q_i\left(t\right)=\frac{q_0}{t^2}$ *and* $\pi_i\left(t\right)=\frac{t}{2}$.
It is easy to verify that

$$B_{t_0}\left(t\right)=t,$$

and

$$\widetilde{B}_{t_0}=t+\frac{q_0}{4}\int_{t_0}^{t}d\varkappa=t\left(1+\frac{q_{i0}}{4}\right).$$

Using Corollary 1, if

$$q_0+\frac{q_0^2}{4}>\frac{4}{\ln 2e},$$

then (43) is oscillatory.

4. Conclusion

The oscillatory properties of neutral second-order differential equations with p-Laplace type operator is thoroughly examined in this work. The analytical process is greatly simplified by this modification. Under certain limits, we have defined some conditions that effectively exclude the existence of positive solutions. Building on these discoveries, we created new standards that ensure all solutions to the examined equations oscillate. This contribution offers a strong basis for upcoming investigations and is essential for developing the theoretical framework of neutral differential equations. Furthermore, we included illustrated instances that show the theoretical significance and practical implementation of our criteria. These illustrations demonstrate how well our method works to solve challenging neutral differential equation situations. The study's findings broaden the field's current theoretical frameworks and create new research opportunities. We suggest that future research investigate the use of our techniques for higher-order equations, especially odd-order ones.

$$\left(b(t)\left(\left(\varkappa(t)+\sum_{i=1}^{n} y_i(t)\,\varkappa(\zeta_i(t))\right)^{(n-1)}\right)^{(p-1)}\right)' + \sum_{i=1}^{n} q_i(t)\,\varkappa^{(p-1)}(\pi_i(t)) = 0.$$

Acknowledgements

The author would like to thank the deanship of scientific research at Shagra University for supporting this work.

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