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Quasi $\theta(\tau_1, \tau_2)$-continuity for Multifunctions

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Abstract. This paper presents new classes of multifunctions called upper quasi $\theta(\tau_1, \tau_2)$-continuous multifunctions and lower quasi $\theta(\tau_1, \tau_2)$-continuous multifunctions. Furthermore, several characterizations and some properties concerning upper quasi $\theta(\tau_1, \tau_2)$-continuous multifunctions and lower quasi $\theta(\tau_1, \tau_2)$-continuous multifunctions are established.

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1. Introduction

Stronger and weaker forms of open sets in topological spaces such as semi-open sets [42], preopen sets [44], α-open sets [46], β-open sets [35] and θ-open sets [65] play an important role in the research of generalizations of continuity. Using these notions many authors introduced and studied various types of generalizations of continuity for functions and multifunctions. Levine [42] introduced and studied the notion of semi-continuous functions. Arya and Bhamini [1] introduced the concept of θ-semi-continuity as a generalization of semi-continuity. Noiri [47] and Jafari and Noiri [36] have further investigated some characterizations of θ-semi-continuous functions. Marcus [43] introduced and investigated the notion of quasi continuous functions. Popa [51] introduced and studied the notion of almost quasi continuous functions. Neubrunnovaá [45] showed that quasi continuity is equivalent to semi-continuity due to Levine [42]. Popa and Stan [54] introduced and investigated the notion of weakly quasi continuous functions. Weak quasi continuity is implied by quasi continuity and weak continuity [41] which are independent of each other. Viriyapong and Boonpok [67] investigated some characterizations of (Λ, sp)-continuous

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functions by utilizing the notions of (Λ, sp)-open sets and (Λ, sp)-closed sets due to Boonpok and Khampakdee [13]. Dungthaisong et al. [34] introduced and studied the concept of $g_{(m,n)}$-continuous functions. Duangphui et al. [33] introduced and investigated the notion of $(\mu, \mu')^{(m,n)}$-continuous functions. Moreover, several characterizations of almost (Λ, p)-continuous functions, strongly $\theta(\Lambda, p)$-continuous functions, almost strongly $\theta(\Lambda, p)$-continuous functions, $\theta(\Lambda, p)$-continuous functions, weakly (Λ, b)-continuous functions, $\theta(\star)$-precontinuous functions, $(\Lambda, p(\star))$-continuous functions, $\star$-continuous functions, θ-$\mathscr{I}$-continuous functions, almost (g, m)-continuous functions, pairwise almost M-continuous functions, (τ_1, τ_2)-continuous functions, almost (τ_1, τ_2)-continuous functions and weakly (τ_1, τ_2)-continuous functions were presented in [60], [63], [17], [55], [26], [12], [9], [11], [5], [2], [3], [27], [24] and [19], respectively. Srisarakham et al. [61] introduced and studied the concept of faintly (τ_1, τ_2)-continuous functions. Kong-ied et al. [40] introduced and investigated the notion of almost quasi (τ_1, τ_2)-continuous functions. Chiangpradit et al. [32] introduced and studied the concept of weakly quasi (τ_1, τ_2)-continuous functions.

In 1975, Popa [50] extended the concept of quasicontinuous functions to the setting of multifunctions. Furthermore, Popa and Noiri [53] introduced the concept of almost quasi continuous multifunctions and investigated some characterizations of such multifunctions. Noiri and Popa [48] introduced and studied the notion of weakly quasi continuous multifunctions. Popa and Noiri [52] introduced the notion of θ-quasicontinuous multifunctions and investigated several further properties of such multifunctions. Moreover, several characterizations and some properties concerning $(\tau_1, \tau_2)\delta$-semicontinuous multifunctions, almost weakly (τ_1, τ_2)-continuous multifunctions, weakly quasi (Λ, sp)-continuous multifunctions, $\star$-continuous multifunctions, $\beta(\star)$-continuous multifunctions, α-$\star$-continuous multifunctions, almost α-$\star$-continuous multifunctions, almost quasi $\star$-continuous multifunctions, weakly α-$\star$-continuous multifunctions, $s\beta(\star)$-continuous multifunctions, weakly $s\beta(\star)$-continuous multifunctions, $\theta(\star)$-quasi continuous multifunctions, almost $\imath^\star$-continuous multifunctions, weakly (Λ, sp)-continuous multifunctions, $\alpha(\Lambda, sp)$-continuous multifunctions, almost $\alpha(\Lambda, sp)$-continuous multifunctions, weakly $\alpha(\Lambda, sp)$-continuous multifunctions, almost $\beta(\Lambda, sp)$-continuous multifunctions, slightly (Λ, sp)-continuous multifunctions, (τ_1, τ_2)-continuous multifunctions, almost (τ_1, τ_2)-continuous multifunctions, weakly (τ_1, τ_2)-continuous multifunctions, weakly quasi (τ_1, τ_2)-continuous multifunctions, almost quasi (τ_1, τ_2)-continuous multifunctions, c-(τ_1, τ_2)-continuous multifunctions and slightly $(\tau_1, \tau_2)p$-continuous multifunctions were established in [6], [29], [68], [4], [8], [18], [25], [7], [22], [21], [16], [10], [20], [23], [37], [14], [28], [62], [15], [58], [39], [64], [59], [57], [38] and [70], respectively. Noiri and Popa [49] investigated some characterizations of upper and lower θ-quasicontinuous multifunctions. Pue-on et al. [56] introduced and studied the concept of c-quasi (τ_1, τ_2)-continuous multifunctions. Viriyapong et al. [72] introduced and investigated the notion of s-$(\tau_1, \tau_2)p$-continuous multifunctions. Furthermore, Viriyapong et al. [69] introduced and studied the concept of slightly (τ_1, τ_2)-continuous multifunctions. In this paper, we introduce the notions of upper quasi $\theta(\tau_1, \tau_2)$-continuous multifunctions and lower quasi $\theta(\tau_1, \tau_2)$-continuous multifunctions. We also investigate several characterizations of upper quasi $\theta(\tau_1, \tau_2)$-continuous multifunctions and lower quasi $\theta(\tau_1, \tau_2)$-continuous multifunctions.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [30] if $A = \tau_1$-Cl$(\tau_2$-Cl$(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. Let A be a subset of a bitopological space (X, τ_1, τ_2). The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [30] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [30] of A and is denoted by $\tau_1\tau_2$-Int(A). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-*clopen* [30] if A is both $\tau_1\tau_2$-open and $\tau_1\tau_2$-closed. A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)r$-*open* [66] (resp. $(\tau_1, \tau_2)s$-*open* [6], $(\tau_1, \tau_2)p$-*open* [6], $(\tau_1, \tau_2)\beta$-*open* [6]) if $A = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$ (resp. $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A))$, $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$, $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))))$. The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open) set is called $(\tau_1, \tau_2)r$-*closed* (resp. $(\tau_1, \tau_2)s$-*closed*, $(\tau_1, \tau_2)p$-*closed*, $(\tau_1, \tau_2)\beta$-*closed*). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\alpha(\tau_1, \tau_2)$-*open* [71] if $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A)))$. The complement of an $\alpha(\tau_1, \tau_2)$-open set is said to be $\alpha(\tau_1, \tau_2)$-*closed*.

Let A be a subset of a bitopological space (X, τ_1, τ_2). A point $x \in X$ is called a $(\tau_1, \tau_2)\theta$-*cluster point* [66] of A if $\tau_1\tau_2$-Cl$(U) \cap A \neq \emptyset$ for every $\tau_1\tau_2$-open set U containing x. The set of all $(\tau_1, \tau_2)\theta$-cluster points of A is called the $(\tau_1, \tau_2)\theta$-*closure* [66] of A and is denoted by $(\tau_1, \tau_2)\theta$-Cl(A). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)\theta$-*closed* [66] if $(\tau_1, \tau_2)\theta$-Cl$(A) = A$. The complement of a $(\tau_1, \tau_2)\theta$-closed set is said to be $(\tau_1, \tau_2)\theta$-*open*. The union of all $(\tau_1, \tau_2)\theta$-open sets of X contained in A is called the $(\tau_1, \tau_2)\theta$-*interior* [66] of A and is denoted by $(\tau_1, \tau_2)\theta$-Int(A).

Lemma 1. [66] *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) If A is $\tau_1\tau_2$-open in X, then $\tau_1\tau_2$-Cl$(A) = (\tau_1, \tau_2)\theta$-Cl$(A)$.

(2) $(\tau_1, \tau_2)\theta$-Cl(A) is $\tau_1\tau_2$-closed in X.

Let A be a subset of a bitopological space (X, τ_1, τ_2). A point $x \in X$ is called a $\theta(\tau_1, \tau_2)s$-*cluster point* of A if (τ_1, τ_2)-sCl$(U) \cap A \neq \emptyset$ for every $(\tau_1, \tau_2)s$-open set U containing x. The set of all $\theta(\tau_1, \tau_2)s$-cluster points of A is called the $\theta(\tau_1, \tau_2)s$-*closure* of A and is denoted by $\theta(\tau_1, \tau_2)$-sCl(A). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\theta(\tau_1, \tau_2)s$-*closed* if $\theta(\tau_1, \tau_2)$-sCl$(A) = A$. The complement of a $\theta(\tau_1, \tau_2)s$-closed set is said to be $\theta(\tau_1, \tau_2)s$-*open*. The union of all $\theta(\tau_1, \tau_2)s$-open sets of X contained in A is called the $\theta(\tau_1, \tau_2)s$-*interior* of A and is denoted by $\theta(\tau_1, \tau_2)$-sInt(A).

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$.

3. Upper and lower quasi $\theta(\tau_1, \tau_2)$-continuous multifunctions

In this section, we introduce the notions of upper quasi $\theta(\tau_1, \tau_2)$-continuous multifunctions and lower quasi $\theta(\tau_1, \tau_2)$-continuous multifunctions. Moreover, several characterizations of upper quasi $\theta(\tau_1, \tau_2)$-continuous multifunctions and lower quasi $\theta(\tau_1, \tau_2)$-continuous multifunctions are discussed.

Definition 1. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be upper quasi $\theta(\tau_1, \tau_2)$-continuous if for each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y containing $F(x)$, there exists a $(\tau_1, \tau_2)s$-open set U of X containing x such that $F((\tau_1, \tau_2)\text{-}sCl(U)) \subseteq \sigma_1\sigma_2\text{-}Cl(V)$.*

Theorem 1. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper quasi $\theta(\tau_1, \tau_2)$-continuous;

(2) $\theta(\tau_1, \tau_2)\text{-}sCl(F^-(\sigma_1\sigma_2\text{-}Int((\sigma_1, \sigma_2)\theta\text{-}Cl(B)))) \subseteq F^-((\sigma_1, \sigma_2)\theta\text{-}Cl(B))$ for every subset B of Y;

(3) $\theta(\tau_1, \tau_2)\text{-}sCl(F^-(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ for every $\sigma_1\sigma_2$-open set V of Y;

(4) $\theta(\tau_1, \tau_2)\text{-}sCl(F^-(\sigma_1\sigma_2\text{-}Int(K))) \subseteq F^-(K)$ for every $(\sigma_1, \sigma_2)r$-closed set K of Y;

(5) $F^+(V) \subseteq \theta(\tau_1, \tau_2)\text{-}sInt(F^+(\sigma_1\sigma_2\text{-}Cl(V)))$ for every $\sigma_1\sigma_2$-open set V of Y;

(6) $\theta(\tau_1, \tau_2)\text{-}sCl(F^-(\sigma_1\sigma_2\text{-}Int(K))) \subseteq F^-(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;

(7) $\theta(\tau_1, \tau_2)\text{-}sCl(F^-(V)) \subseteq F^-(\sigma_1\sigma_2\text{-}Cl(V))$ for every $\sigma_1\sigma_2$-open set V of Y.

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y. Suppose that $x \notin F^-((\sigma_1, \sigma_2)\theta\text{-Cl}(B))$. Then, $x \in X - F^-((\sigma_1, \sigma_2)\theta\text{-Cl}(B))$ and $F(x) \subseteq Y - (\sigma_1, \sigma_2)\theta\text{-Cl}(B)$. Since $(\sigma_1, \sigma_2)\theta\text{-Cl}(B)$ is $\sigma_1\sigma_2$-closed in Y, there exists a $(\tau_1, \tau_2)s$-open set U of X containing x such that $F((\tau_1, \tau_2)\text{-sCl}(U)) \subseteq \sigma_1\sigma_2\text{-Cl}(Y - (\sigma_1, \sigma_2)\theta\text{-Cl}(B)) = Y - \sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B))$. Thus, we have $F((\tau_1, \tau_2)\text{-sCl}(U)) \cap \sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B)) = \emptyset$ and

$$(\tau_1, \tau_2)\text{-sCl}(U) \cap F^-(\sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B))) = \emptyset.$$

This shows that $x \notin \theta(\tau_1, \tau_2)\text{-sCl}(F^-(\sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B))))$. Thus,

$$\theta(\tau_1, \tau_2)\text{-sCl}(F^-(\sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B)))) \subseteq F^-((\sigma_1, \sigma_2)\theta\text{-Cl}(B)).$$

(2) $\Rightarrow$ (3): This is obvious since $\sigma_1\sigma_2\text{-Cl}(V) = (\sigma_1, \sigma_2)\theta\text{-Cl}(V)$ for every $\sigma_1\sigma_2$-open set V of Y.

(3) $\Rightarrow$ (4): Let K be any $(\sigma_1, \sigma_2)r$-closed set of Y. By (3), we have

$$\theta(\tau_1, \tau_2)\text{-sCl}(F^-(\sigma_1\sigma_2\text{-Int}(K))) = \theta(\tau_1, \tau_2)\text{-sCl}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)))))$$

$$\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)))$$
$$= F^-(K).$$

(4) ⇒ (5): Let V be any $\sigma_1\sigma_2$-open set of Y. Then, we have

$$\begin{aligned} X - \theta(\tau_1,\tau_2)\text{-sInt}(F^+(\sigma_1\sigma_2\text{-Cl}(V))) &= \theta(\tau_1,\tau_2)\text{-sCl}(X - F^+(\sigma_1\sigma_2\text{-Cl}(V))) \\ &= \theta(\tau_1,\tau_2)\text{-sCl}(F^-(Y - \sigma_1\sigma_2\text{-Cl}(V))), \end{aligned}$$

$Y - \sigma_1\sigma_2\text{-Cl}(V) = \sigma_1\sigma_2\text{-Int}(Y - \sigma_1\sigma_2\text{-Cl}(V)) \subseteq \sigma_1\sigma_2\text{-Int}(Y - \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$ and $Y - \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$ is $(\sigma_1,\sigma_2)r$-closed in Y. Thus by (4),

$$\begin{aligned} \theta(\tau_1,\tau_2)\text{-sCl}(F^-(\sigma_1\sigma_2\text{-Int}(Y - \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))) &\subseteq F^-(Y - \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))) \\ &= X - F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))) \\ &\subseteq X - F^+(V) \end{aligned}$$

and hence $F^+(V) \subseteq \theta(\tau_1,\tau_2)\text{-sInt}(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$.

(5) ⇒ (6): Let K be any $\sigma_1\sigma_2$-closed set of Y. Then by (5), we have

$$\begin{aligned} X - F^-(K) &= F^+(Y - K) \\ &\subseteq \theta(\tau_1,\tau_2)\text{-sInt}(F^+(\sigma_1\sigma_2\text{-Cl}(Y - K))) \\ &= \theta(\tau_1,\tau_2)\text{-sInt}(F^+(Y - \sigma_1\sigma_2\text{-Int}(K))) \\ &= \theta(\tau_1,\tau_2)\text{-sInt}(X - F^-(\sigma_1\sigma_2\text{-Int}(K))) \\ &= X - \theta(\tau_1,\tau_2)\text{-sCl}(F^-(\sigma_1\sigma_2\text{-Int}(K))). \end{aligned}$$

Thus, $\theta(\tau_1,\tau_2)\text{-sCl}(F^-(\sigma_1\sigma_2\text{-Int}(K))) \subseteq F^-(K)$.

(6) ⇒ (7): Let V be any $\sigma_1\sigma_2$-closed set of Y. Then, we have $\sigma_1\sigma_2\text{-Cl}(V)$ is $\sigma_1\sigma_2$-closed in Y and by (6),

$$\begin{aligned} \theta(\tau_1,\tau_2)\text{-sCl}(F^-(V)) &\subseteq \theta(\tau_1,\tau_2)\text{-sCl}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V)). \end{aligned}$$

(7) ⇒ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. Then, $\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V)) \cap F(x) = \emptyset$ and $x \notin F^-(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V)))$. It follows from (7) that $x \notin \theta(\tau_1,\tau_2)\text{-sCl}(F^-(Y - \sigma_1\sigma_2\text{-Cl}(V)))$. Then, there exists a $(\tau_1,\tau_2)s$-open set U of X containing x such that $(\tau_1,\tau_2)\text{-sCl}(U) \cap F^-(Y - \sigma_1\sigma_2\text{-Cl}(V)) = \emptyset$; hence $F((\tau_1,\tau_2)\text{-sCl}(U)) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. This shows that F is upper quasi $\theta(\tau_1,\tau_2)$-continuous.

Definition 2. *A multifunction $F : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is said to be lower quasi $\theta(\tau_1,\tau_2)$-continuous if for each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $(\tau_1,\tau_2)s$-open set U of X containing x such that $\sigma_1\sigma_2$-Cl$(V) \cap F(z) \neq \emptyset$ for every $z \in (\tau_1,\tau_2)$-sCl(U).*

Lemma 2. *If $F : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is lower quasi $\theta(\tau_1,\tau_2)$-continuous, then for each $x \in X$ and each subset B of Y with $(\sigma_1,\sigma_2)\theta$-Int$(B) \cap F(x) \neq \emptyset$ there exists a $(\tau_1,\tau_2)s$-open set U of X containing x such that (τ_1,τ_2)-sCl$(U) \subseteq F^-(B)$.*

Proof. Since $(\sigma_1, \sigma_2)\theta\text{-Int}(B) \cap F(x) \neq \emptyset$, there exists a $\sigma_1\sigma_2$-open set V of Y such that $V \subseteq \sigma_1\sigma_2\text{-Cl}(V) \subseteq B$ and $F(x) \cap V \neq \emptyset$. Since F is lower quasi $\theta(\tau_1, \tau_2)$-continuous, there exists a $(\tau_1, \tau_2)s$-open set U of X containing x such that $\sigma_1\sigma_2\text{-Cl}(V) \cap F(z) \neq \emptyset$ for every $z \in (\tau_1, \tau_2)\text{-sCl}(U)$ and hence $(\tau_1, \tau_2)\text{-sCl}(U) \subseteq F^-(B)$.

Theorem 2. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower quasi $\theta(\tau_1, \tau_2)$-continuous;

(2) $\theta(\tau_1, \tau_2)\text{-sCl}(F^+(B)) \subseteq F^+((\sigma_1, \sigma_2)\theta\text{-Cl}(B))$ for every subset B of Y;

(3) $\theta(\tau_1, \tau_2)\text{-sCl}(F^+(V)) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V))$ for every $\sigma_1\sigma_2$-open set V of Y;

(4) $F^-(V) \subseteq \theta(\tau_1, \tau_2)\text{-sInt}(F^-(\sigma_1\sigma_2\text{-Cl}(V)))$ for every $\sigma_1\sigma_2$-open set V of Y;

(5) $F(\theta(\tau_1, \tau_2)\text{-sCl}(A)) \subseteq (\sigma_1, \sigma_2)\theta\text{-Cl}(F(A))$ for every subset A of X;

(6) $\theta(\tau_1, \tau_2)\text{-sCl}(F^+(\sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B)))) \subseteq F^+((\sigma_1, \sigma_2)\theta\text{-Cl}(B))$ for every subset B of Y;

(7) $\theta(\tau_1, \tau_2)\text{-sCl}(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq F^+(\sigma_1\sigma_2\text{-Cl}(V))$ for every $\sigma_1\sigma_2$-open set V of Y;

(8) $\theta(\tau_1, \tau_2)\text{-sCl}(F^+(\sigma_1\sigma_2\text{-Int}(K))) \subseteq F^+(K)$ for every $(\sigma_1, \sigma_2)r$-closed set K of Y;

(9) $\theta(\tau_1, \tau_2)\text{-sCl}(F^+(\sigma_1\sigma_2\text{-Int}(K))) \subseteq F^+(K)$ for every $\sigma_1\sigma_2$-closed set K of Y.

Proof. (1) ⇒ (2): Let B be any subset of Y. Suppose that $x \notin F^+((\sigma_1, \sigma_2)\theta\text{-Cl}(B))$. Then, $x \in F^-(Y - (\sigma_1, \sigma_2)\theta\text{-Cl}(B)) = F^-((\sigma_1, \sigma_2)\theta\text{-Int}(Y - B))$. Since F is lower quasi $\theta(\tau_1, \tau_2)$-continuous, by Lemma 2 there exists a $(\tau_1, \tau_2)s$-open set U of X containing x such that $(\tau_1, \tau_2)\text{-sCl}(U) \subseteq F^-(Y - B) = X - F^+(B)$. Thus, we have

$$(\tau_1, \tau_2)\text{-sCl}(U) \cap F^+(B) = \emptyset$$

and hence $x \notin \theta(\tau_1, \tau_2)\text{-sCl}(F^+(B))$.

(2) ⇒ (3): This is obvious since $\sigma_1\sigma_2\text{-Cl}(V) = (\sigma_1, \sigma_2)\theta\text{-Cl}(V)$ for every $\sigma_1\sigma_2$-open set V of Y.

(3) ⇒ (4): Let V be any $\sigma_1\sigma_2$-open set of Y. Then by (3), we have

$$\begin{aligned} X - \theta(\tau_1, \tau_2)\text{-sInt}(F^-(\sigma_1\sigma_2\text{-Cl}(V))) &= \theta(\tau_1, \tau_2)\text{-sCl}(X - F^-(\sigma_1\sigma_2\text{-Cl}(V))) \\ &= \theta(\tau_1, \tau_2)\text{-sCl}(F^+(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\ &\subseteq F^+(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\ &\subseteq F^+(\sigma_1\sigma_2\text{-Cl}(Y - V)) \\ &= F^+(Y - V) \\ &= X - F^-(V) \end{aligned}$$

and hence $F^-(V) \subseteq \theta(\tau_1, \tau_2)$-sInt$(F^-(\sigma_1\sigma_2\text{-Cl}(V)))$.

(4) $\Rightarrow$ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y such that $F(x) \cap V \neq \emptyset$. By (4), $x \in F^-(V) \subseteq \theta(\tau_1, \tau_2)$-sInt$(F^-(\sigma_1\sigma_2\text{-Cl}(V)))$. Then, there exists a $(\tau_1, \tau_2)s$-open set U of X containing x such that (τ_1, τ_2)-sCl$(U) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V))$; hence

$$\sigma_1\sigma_2\text{-Cl}(V) \cap F(z) \neq \emptyset$$

for every $z \in (\tau_1, \tau_2)$-sCl(U). This shows that F is lower quasi $\theta(\tau_1, \tau_2)$-continuous.

(2) $\Rightarrow$ (5): Let A be any subset of X. By replacing B in (2) by $F(A)$, we have $\theta(\tau_1, \tau_2)$-sCl$(A) \subseteq \theta(\tau_1, \tau_2)$-sCl$(F^+(F(A))) \subseteq F^+((\sigma_1, \sigma_2)\theta\text{-Cl}(F(A)))$. Thus,

$$F(\theta(\tau_1, \tau_2)\text{-sCl}(A)) \subseteq (\sigma_1, \sigma_2)\theta\text{-Cl}(F(A)).$$

(5) $\Rightarrow$ (2): Let B be any subset of Y. Replacing A in (5) by $F^+(B)$, we have $F(\theta(\tau_1, \tau_2)\text{-sCl}(F^+(B))) \subseteq (\sigma_1, \sigma_2)\theta\text{-Cl}(F(F^+(B))) \subseteq (\sigma_1, \sigma_2)\theta\text{-Cl}(B)$ and hence

$$\theta(\tau_1, \tau_2)\text{-sCl}(F^+(B)) \subseteq F^+((\sigma_1, \sigma_2)\theta\text{-Cl}(B)).$$

(3) $\Rightarrow$ (6): Let B be any subset of Y. Put $V = \sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B))$ in (3). Then, since $(\sigma_1, \sigma_2)\theta$-Cl$(B)$ is $\sigma_1\sigma_2$-closed in Y, we have

$$\begin{aligned}\theta(\tau_1, \tau_2)\text{-sCl}(F^+(\sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B)))) &\subseteq F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B)))) \\ &\subseteq F^+((\sigma_1, \sigma_2)\theta\text{-Cl}(B)).\end{aligned}$$

(6) $\Rightarrow$ (7): This is obvious since $\sigma_1\sigma_2$-Cl$(V) = (\sigma_1, \sigma_2)\theta$-Cl$(V)$ for every $\sigma_1\sigma_2$-open set V of Y.

(7) $\Rightarrow$ (8): Let K be any $(\sigma_1, \sigma_2)r$-closed set of Y. Then by (7), we have

$$\begin{aligned}\theta(\tau_1, \tau_2)\text{-sCl}(F^+(\sigma_1\sigma_2\text{-Int}(K))) &= \theta(\tau_1, \tau_2)\text{-sCl}(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))))) \\ &\subseteq F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) \\ &= F^+(K).\end{aligned}$$

(8) $\Rightarrow$ (9): Let K be any $\sigma_1\sigma_2$-closed set of Y. Then, $\sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2$-Int$(K))$ is $(\sigma_1, \sigma_2)r$-closed in Y and by (8),

$$\begin{aligned}\theta(\tau_1, \tau_2)\text{-sCl}(F^+(\sigma_1\sigma_2\text{-Int}(K))) &= \theta(\tau_1, \tau_2)\text{-sCl}(F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))))) \\ &\subseteq F^+(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) \\ &\subseteq F^+(K).\end{aligned}$$

(9) $\Rightarrow$ (4): Let V be any $\sigma_1\sigma_2$-open set of Y. Then, $Y - V$ is $\sigma_1\sigma_2$-closed in Y and by (9), $\theta(\tau_1, \tau_2)$-sCl$(F^+(\sigma_1\sigma_2\text{-Int}(Y - V))) \subseteq F^+(Y - V) = X - F^-(V)$. Moreover, we have

$$\begin{aligned}\theta(\tau_1, \tau_2)\text{-sCl}(F^+(\sigma_1\sigma_2\text{-Int}(Y - V))) &= \theta(\tau_1, \tau_2)\text{-sCl}(F^+(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\ &= \theta(\tau_1, \tau_2)\text{-sCl}(X - F^-(\sigma_1\sigma_2\text{-Cl}(V))) \\ &= X - \theta(\tau_1, \tau_2)\text{-sInt}(F^-(\sigma_1\sigma_2\text{-Cl}(V))).\end{aligned}$$

Thus, $F^-(V) \subseteq \theta(\tau_1, \tau_2)$-sInt$(F^-(\sigma_1\sigma_2\text{-Cl}(V)))$.

Theorem 3. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper quasi $\theta(\tau_1, \tau_2)$-continuous;

(2) $\theta(\tau_1, \tau_2)$-sCl($F^-(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))) \subseteq F^-(\sigma_1\sigma_2$-Cl$(V))$ for every $(\sigma_1, \sigma_2)\beta$-open set V of Y;

(3) $\theta(\tau_1, \tau_2)$-sCl($F^-(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))) \subseteq F^-(\sigma_1\sigma_2$-Cl$(V))$ for every $(\sigma_1, \sigma_2)s$-open set V of Y.

Proof. (1) ⇒ (2): Let V be any $(\sigma_1, \sigma_2)\beta$-open set of Y. Then,

$$V \subseteq \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$$

and hence $\sigma_1\sigma_2$-Cl$(V) = \sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))$. Since $\sigma_1\sigma_2$-Cl(V) is $(\sigma_1, \sigma_2)r$-closed in Y, by Theorem 1 we have

$$\theta(\tau_1, \tau_2)\text{-sCl}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V)).$$

(2) ⇒ (3): The proof is obvious.

(3) ⇒ (1): Let V be any $\sigma_1\sigma_2$-open set of Y. Then, V is $(\sigma_1, \sigma_2)s$-open in Y and by (3), $\theta(\tau_1, \tau_2)$-sCl$(F^-(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))) \subseteq F^-(\sigma_1\sigma_2$-Cl$(V))$. Thus by Theorem 1, F is upper quasi $\theta(\tau_1, \tau_2)$-continuous.

Theorem 4. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower quasi $\theta(\tau_1, \tau_2)$-continuous;

(2) $\theta(\tau_1, \tau_2)$-sCl($F^+(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))) \subseteq F^+(\sigma_1\sigma_2$-Cl$(V))$ for every $(\sigma_1, \sigma_2)\beta$-open set V of Y;

(3) $\theta(\tau_1, \tau_2)$-sCl($F^+(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))) \subseteq F^+(\sigma_1\sigma_2$-Cl$(V))$ for every $(\sigma_1, \sigma_2)s$-open set V of Y.

Proof. The proof is similar to that of Theorem 3.

Theorem 5. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper quasi $\theta(\tau_1, \tau_2)$-continuous;

(2) $\theta(\tau_1, \tau_2)$-sCl($F^-(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))) \subseteq F^-(\sigma_1\sigma_2$-Cl$(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;

(3) $\theta(\tau_1, \tau_2)$-sCl$(F^-(V)) \subseteq F^-(\sigma_1\sigma_2$-Cl$(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;

(4) $F^+(V) \subseteq \theta(\tau_1, \tau_2)$*-sInt*$(F^+(\sigma_1\sigma_2$*-Cl*$(V)))$ *for every* $(\sigma_1, \sigma_2)p$*-open set* V *of* Y*.*

Proof. (1) ⇒ (2): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. Since $\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$ is a $\sigma_1\sigma_2$-open set of Y, by Theorem 3 we have

$$\begin{aligned}\theta(\tau_1, \tau_2)\text{-sCl}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \\ &= F^-(\sigma_1\sigma_2\text{-Cl}(V)).\end{aligned}$$

(2) ⇒ (3): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. Then, $V \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$ and by (2),

$$\begin{aligned}\theta(\tau_1, \tau_2)\text{-sCl}(F^-(V)) &\subseteq \theta(\tau_1, \tau_2)\text{-sCl}(F^-(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V)).\end{aligned}$$

(3) ⇒ (4): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. Then by (3), we have

$$\begin{aligned}X - \theta(\tau_1, \tau_2)\text{-sInt}(F^+(\sigma_1\sigma_2\text{-Cl}(V))) &= \theta(\tau_1, \tau_2)\text{-sCl}(X - F^+(\sigma_1\sigma_2\text{-Cl}(V))) \\ &= \theta(\tau_1, \tau_2)\text{-sCl}(F^-(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\ &\subseteq F^-(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\ &= X - F^+(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))) \\ &\subseteq X - F^+(V)\end{aligned}$$

and hence $F^+(V) \subseteq \theta(\tau_1, \tau_2)\text{-sInt}(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$.

(4) ⇒ (1): Let V be any $\sigma_1\sigma_2$-open set of Y. Then, V is $(\sigma_1, \sigma_2)p$-open in Y and by (4), we have $F^+(V) \subseteq \theta(\tau_1, \tau_2)\text{-sInt}(F^+(\sigma_1\sigma_2\text{-Cl}(V)))$. By Theorem 1, F is upper quasi $\theta(\tau_1, \tau_2)$-continuous.

Theorem 6. *For a multifunction* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) F *is lower quasi* $\theta(\tau_1, \tau_2)$*-continuous;*

(2) $\theta(\tau_1, \tau_2)$*-sCl*$(F^+(\sigma_1\sigma_2$*-Int*$(\sigma_1\sigma_2$*-Cl*$(V)))) \subseteq F^+(\sigma_1\sigma_2$*-Cl*$(V))$ *for every* $(\sigma_1, \sigma_2)p$*-open set* V *of* Y*;*

(3) $\theta(\tau_1, \tau_2)$*-sCl*$(F^+(V)) \subseteq F^+(\sigma_1\sigma_2$*-Cl*$(V))$ *for every* $(\sigma_1, \sigma_2)p$*-open set* V *of* Y*;*

(4) $F^-(V) \subseteq \theta(\tau_1, \tau_2)$*-sInt*$(F^-(\sigma_1\sigma_2$*-Cl*$(V)))$ *for every* $(\sigma_1, \sigma_2)p$*-open set* V *of* Y*.*

Proof. The proof is similar to that of Theorem 5.

Recall that a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$*-compact* [30] if every cover of X by $\tau_1\tau_2$-open sets of X has a finite subcover. A bitopological space (X, τ_1, τ_2) is said to be *quasi* (τ_1, τ_2)*-*$\mathscr{H}$*-closed* [64] if every $\tau_1\tau_2$-open cover $\{U_\gamma \mid \gamma \in \Gamma\}$, there exists a finite subset Γ_0 of Γ such that $X = \cup\{\tau_1\tau_2\text{-Cl}(U_\gamma) \mid \gamma \in \Gamma_0\}$.

Definition 3. *A bitopological space (X, τ_1, τ_2) is called s-(τ_1, τ_2)-closed if every $(\tau_1, \tau_2)s$-open cover $\{U_\gamma \mid \gamma \in \Gamma\}$, there exists a finite subset Γ_0 of Γ such that*

$$X = \cup\{(\tau_1, \tau_2)\text{-}sCl(U_\gamma) \mid \gamma \in \Gamma_0\}.$$

Theorem 7. *Let $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be an upper quasi $\theta(\tau_1, \tau_2)$-continuous surjective multifunction such that $F(x)$ is $\sigma_1\sigma_2$-compact for each $x \in X$. If (X, τ_1, τ_2) is s-(τ_1, τ_2)-closed, then (Y, σ_1, σ_2) is quasi (σ_1, σ_2)-$\mathscr{H}$-closed.*

Proof. Let $\{V_\gamma \mid \gamma \in \Gamma\}$ be any $\sigma_1\sigma_2$-open cover of Y. For each $x \in X$, $F(x)$ is $\sigma_1\sigma_2$-compact and there exists a finite subset $\Gamma(x)$ of Γ such that $F(x) \subseteq \cup\{V_\gamma \mid \gamma \in \Gamma(x)\}$. Put $V(x) = \cup\{V_\gamma \mid \gamma \in \Gamma(x)\}$. Then, $F(x) \subseteq V(x)$ and $V(x)$ is $\sigma_1\sigma_2$-open in Y. Since F is upper quasi $\theta(\tau_1, \tau_2)$-continuous, there exists a $(\tau_1, \tau_2)s$-open set $U(x)$ of X containing x such that $F((\tau_1, \tau_2)\text{-sCl}(U(x))) \subseteq \sigma_1\sigma_2\text{-Cl}(V(x))$. The family $\{U(x) \mid x \in X\}$ is a $(\tau_1, \tau_2)s$-open cover of X. Since (X, τ_1, τ_2) is s-(τ_1, τ_2)-closed, there exists a finite number of points, says, $x_1, x_2, ..., x_n$ in X such that $X = \cup\{(\tau_1, \tau_2)\text{-sCl}(U(x_i)) \mid i = 1, 2, ..., n\}$. Since F is surjective,

$$\begin{aligned} Y = F(X) &= F(\overset{n}{\underset{i=1}{\cup}}(\tau_1, \tau_2)\text{-sCl}(U(x_i))) \\ &= \overset{n}{\underset{i=1}{\cup}} F((\tau_1, \tau_2)\text{-sCl}(U(x_i))) \\ &\subseteq \overset{n}{\underset{i=1}{\cup}} \sigma_1\sigma_2\text{-Cl}(V(x_i)) \\ &= \overset{n}{\underset{i=1}{\cup}} \cup_{\gamma \in \Gamma(x_i)} \sigma_1\sigma_2\text{-Cl}(V_\gamma). \end{aligned}$$

This shows that (Y, σ_1, σ_2) is quasi (σ_1, σ_2)-$\mathscr{H}$-closed.

For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, a multifunction

$$\text{sCl}F_{\circledast} : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$$

is defined in [31] as follows: $\text{sCl}F_{\circledast}(x) = (\sigma_1, \sigma_2)\text{-sCl}(F(x))$ for each $x \in X$.

Lemma 3. *Let $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a multifunction. Then, $sClF^-_{\circledast}(V) = F^-(V)$ for ever $(\sigma_1, \sigma_2)s$-open set V of Y.*

Proof. Let V be any $(\sigma_1, \sigma_2)s$-open set of Y. Let $x \in \text{sCl}F^-_{\circledast}(V)$. Then,

$$(\sigma_1, \sigma_2)\text{-sCl}(F(x)) \cap V = \text{sCl}F_{\circledast}(x) \cap V \neq \emptyset.$$

Since V is $(\sigma_1, \sigma_2)s$-open in Y, we have $V \cap F(x) \neq \emptyset$ and hence $x \in F^-(V)$. Thus, $\text{sCl}F^-_{\circledast}(V) \subseteq F^-(V)$. On the other hand, let $x \in F^-(V)$. Then,

$$\emptyset \neq F(x) \cap V \subseteq (\sigma_1, \sigma_2)\text{-sCl}(F(x)) \cap V$$

and so $x \in \text{sCl}F^-_{\circledast}(V)$. Consequently, we obtain $\text{sCl}F^-_{\circledast}(V) = F^-(V)$.

Theorem 8. *A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is lower quasi $\theta(\tau_1, \tau_2)$-continuous if and only if sCl$F_{\circledast} : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is lower quasi $\theta(\tau_1, \tau_2)$-continuous.*

Proof. Suppose that F is lower quasi $\theta(\tau_1, \tau_2)$-continuous. Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y such that sCl$F_{\circledast}(x) \cap V \neq \emptyset$. By Lemma 3, we have $F(x) \cap V \neq \emptyset$. Since F is lower quasi $\theta(\tau_1, \tau_2)$-continuous, there exists a $(\tau_1, \tau_2)s$-open set of X containing x such that $\sigma_1\sigma_2\text{-Cl}(V) \cap F(z) \neq \emptyset$ for every $z \in (\tau_1, \tau_2)\text{-sCl}(U)$. Since $\sigma_1\sigma_2\text{-Cl}(V)$ is $(\sigma_1, \sigma_2)s$-open in Y, by Lemma 3 we have $(\tau_1, \tau_2)\text{-sCl}(U) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(V)) = \text{sCl}F^-_{\circledast}(\sigma_1\sigma_2\text{-Cl}(V))$ and hence $\text{sCl}F_{\circledast}(z) \cap \sigma_1\sigma_2\text{-Cl}(V) \neq \emptyset$ for every $z \in (\tau_1, \tau_2)\text{-sCl}(U)$. This shows that sCl$F_{\circledast}$ is lower quasi $\theta(\tau_1, \tau_2)$-continuous.

Conversely, suppose that sCl$F_{\circledast}$ is lower quasi $\theta(\tau_1, \tau_2)$-continuous. Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y such that $F(x) \cap V \neq \emptyset$. Then, $(\sigma_1, \sigma_2)\text{-sCl}(F(x)) \cap V \neq \emptyset$. Since sCl$F_{\circledast}$ is lower quasi $\theta(\tau_1, \tau_2)$-continuous, there exists a $(\tau_1, \tau_2)s$-open set of X containing x such that $\text{sCl}F_{\circledast}(z) \cap \sigma_1\sigma_2\text{-Cl}(V) \neq \emptyset$ for every $z \in (\tau_1, \tau_2)\text{-sCl}(U)$. Since $\sigma_1\sigma_2\text{-Cl}(V)$ is $(\sigma_1, \sigma_2)s$-open in Y, by Lemma 3

$$(\tau_1, \tau_2)\text{-sCl}(U) \subseteq \text{sCl}F^-_{\circledast}(\sigma_1\sigma_2\text{-Cl}(V)) = F^-(\sigma_1\sigma_2\text{-Cl}(V))$$

and hence $\sigma_1\sigma_2\text{-Cl}(V) \cap F(z) \neq \emptyset$ for every $z \in (\tau_1, \tau_2)\text{-sCl}(U)$. Thus, F is lower quasi $\theta(\tau_1, \tau_2)$-continuous.

Definition 4. [30] *A subset A of a bitopological space (X, τ_1, τ_2) is said to be:*

(1) $\tau_1\tau_2$-paracompact if every cover of A by $\tau_1\tau_2$-open sets of X is refined by a cover of A which consists of $\tau_1\tau_2$-open sets of X and is $\tau_1\tau_2$-locally finite in X;

(2) $\tau_1\tau_2$-regular if for each $x \in A$ and each $\tau_1\tau_2$-open set U of X containing x, there exists a $\tau_1\tau_2$-open set V of X such that $x \in V \subseteq \tau_1\tau_2\text{-}Cl(V) \subseteq U$.

Lemma 4. [30] *If A is a $\tau_1\tau_2$-regular $\tau_1\tau_2$-paracompact set of a bitopological space (X, τ_1, τ_2) and U is a $\tau_1\tau_2$-open neighborhood of A, then there exists a $\tau_1\tau_2$-open set V of X such that $A \subseteq V \subseteq \tau_1\tau_2\text{-}Cl(V) \subseteq U$.*

Lemma 5. *If $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a multifunction such that $F(x)$ is $\tau_1\tau_2$-regular and $\tau_1\tau_2$-paracompact for each $x \in X$, then sCl$F^+_{\circledast}(V) = F^+(V)$ for each $\sigma_1\sigma_2$-open set V of Y.*

Proof. Let V be any $\sigma_1\sigma_2$-open set of Y and $x \in \text{sCl}F^+_{\circledast}(V)$. Then, $\text{sCl}F^+_{\circledast}(x) \subseteq V$ and $F(x) \subseteq (\sigma_1, \sigma_2)\text{-sCl}(F(x)) = \text{sCl}F^+_{\circledast}(x) \subseteq V$. Thus, $x \in F^+(V)$ and hence

$$\text{sCl}F^+_{\circledast}(V) \subseteq F^+(V).$$

On the other hand, let $x \in F^+(V)$. Then, $F(x) \subseteq V$ and by Lemma 5, there exists a $\sigma_1\sigma_2$-open set W of Y such that $F(x) \subseteq W \subseteq \sigma_1\sigma_2\text{-Cl}(W) \subseteq V$; hence

$$\text{sCl}F^+_{\circledast}(x) = (\sigma_1, \sigma_2)\text{-sCl}(F(x)) \subseteq \sigma_1\sigma_2\text{-Cl}(W) \subseteq V.$$

Thus, $x \in \text{sCl}F^+_{\circledast}(V)$ and so $F^+(V) \subseteq \text{sCl}F^+_{\circledast}(V)$. Therefore, $F^+(V) = \text{sCl}F^+_{\circledast}(V)$.

Theorem 9. *Let $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a multifunction such that $F(x)$ is $\sigma_1\sigma_2$-paracompact and $\sigma_1\sigma_2$-regular for each $x \in X$. Then, F is upper quasi $\theta(\tau_1, \tau_2)$-continuous if and only if $sClF_{\circledast} : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is upper quasi $\theta(\tau_1, \tau_2)$-continuous.*

Proof. Suppose that F is upper quasi $\theta(\tau_1, \tau_2)$-continuous. It follows from Theorem 1 and Lemma 5 that for every $\sigma_1\sigma_2$-open set V of Y,

$$\begin{aligned} \mathrm{sCl}F^+_{\circledast}(V) = F^+(V) &\subseteq \theta(\tau_1, \tau_2)\text{-sInt}(F^+(\sigma_1\sigma_2\text{-Cl}(V))) \\ &= \theta(\tau_1, \tau_2)\text{-sInt}(\mathrm{sCl}F^+_{\circledast}(\sigma_1\sigma_2\text{-Cl}(V))). \end{aligned}$$

By Theorem 1, $\mathrm{sCl}F_{\circledast}$ is upper quasi $\theta(\tau_1, \tau_2)$-continuous.

Conversely, suppose that $\mathrm{sCl}F_{\circledast}$ is upper quasi $\theta(\tau_1, \tau_2)$-continuous. It follows from Theorem 1 and Lemma 5 that for every $\sigma_1\sigma_2$-open set V of Y,

$$\begin{aligned} F^+(V) = \mathrm{sCl}F^+_{\circledast}(V) &\subseteq \theta(\tau_1, \tau_2)\text{-sInt}(\mathrm{sCl}F^+_{\circledast}(\sigma_1\sigma_2\text{-Cl}(V))) \\ &= \theta(\tau_1, \tau_2)\text{-sInt}(F^+(\sigma_1\sigma_2\text{-Cl}(V))). \end{aligned}$$

Thus by Theorem 1, F is upper quasi $\theta(\tau_1, \tau_2)$-continuous.

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